Lecture – 05

Order parameter, Free energy functional, Ginzburg-Landau (GL) Theory, GL equations

So in this session we are going to learn,

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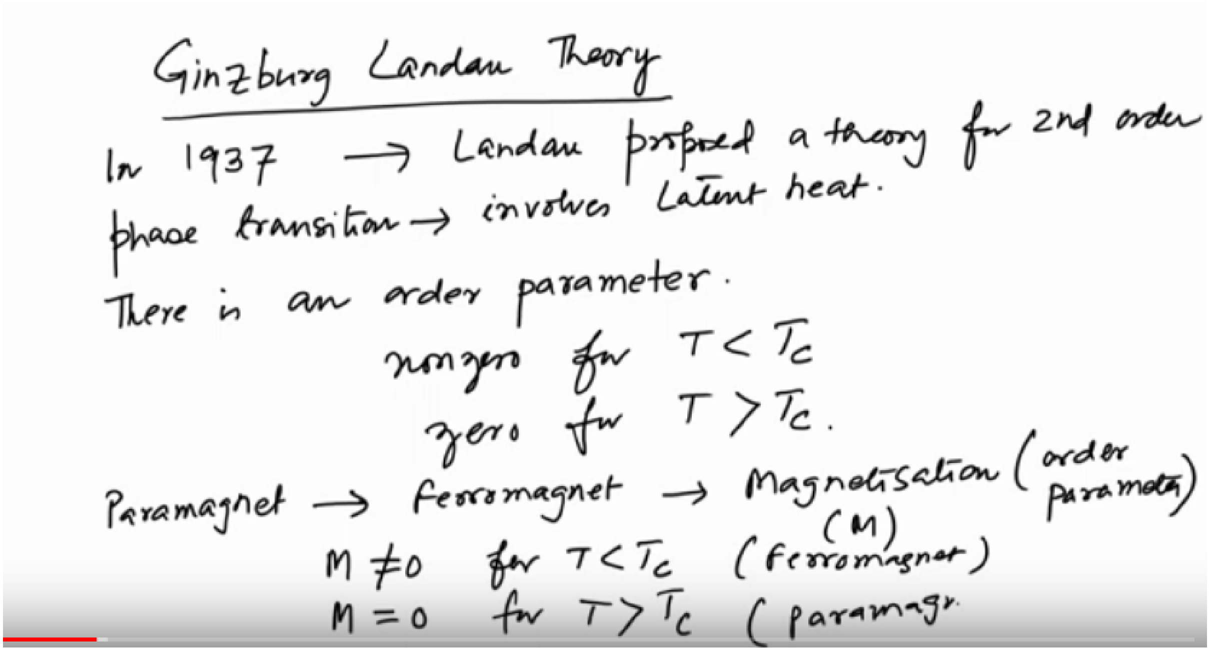

The Ginsberg Landau, theory, of superconductivity. Remember this, was evolved, before the microscopic theory, that is BCS theory, was put forward. And it's a phenomenological theory, of second order phase transition. Which was extremely successful, in describing, a few properties, of these superconductors, such as, the penetration depth and the coherence length. And thereby, a ratio of them, could be formed, which helps in distinguishing between type 1 and type 2 superconductors. So in 1937, Landau proposed, a theory for, for second order, phase transition. Now by second order phase transition, what we mean is that, that involves, latent heat, this is one of the, modern definitions, of phase transition. In this theory, there is an order Parameter, order parameter, that, continuously vanishes across, the phase transition. So this order parameter is, nonzero, for T, less than TC and its equal to zero, for T greater than TC. So this is an indicator of the phase transition. Say for example, in a paramagnet, to a ferromagnet transition, the magnetization can be the order parameter. So the magnetization, let's call it by, M. So magnetization is not 0, for T less Than, TC which, talks about a, ferromagnetic state and magnetization being 0, for T greater than, TC. So this is a ferromagnet and this is a, paramagnet.

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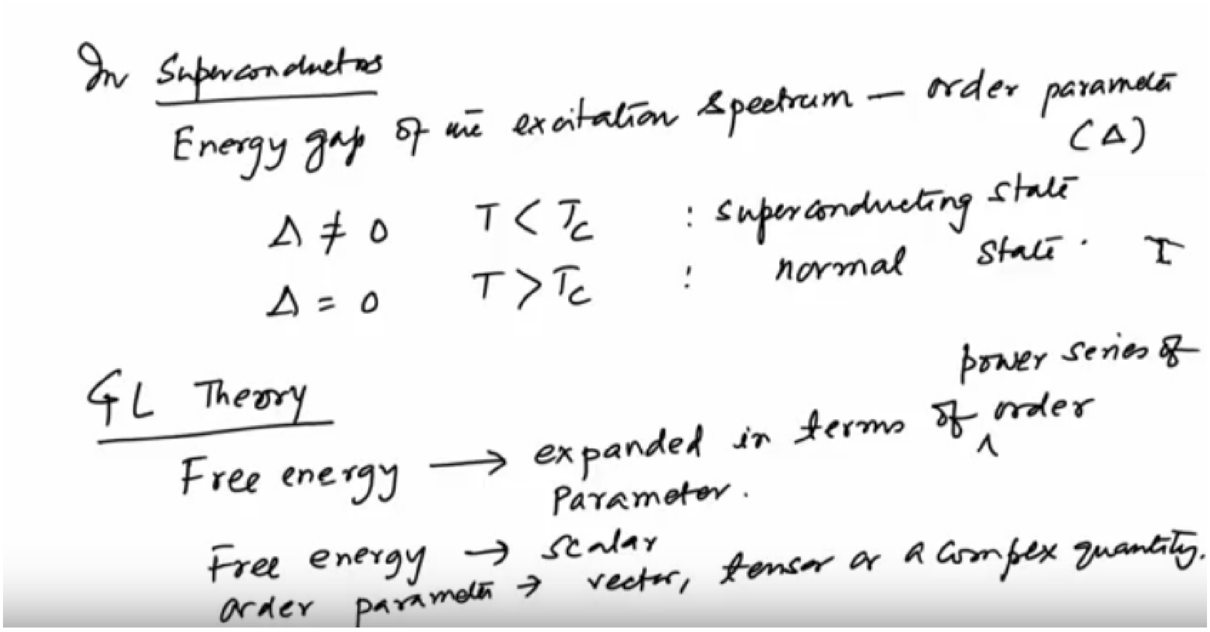

Similarly, in superconductors, we can talk about the, energy gap, of the Excitation Spectrum. This can be taken as the, order parameter, call it as, Delta and Delta is not equal to 0, for T less than TC, where TC is the, transition temperature, for a superconductor, which is a characteristic of a material. So this corresponds to a superconducting state and Delta equal to 0, for T greater than, TC, this is called as, the normal state or a non superconducting state. Now, since, the order parameter continuously evolves, across the boundary, of the phase transition, it should be possible to write down the free energy, as, a power series in the order parameter. So this is the whole idea, of GL Theory, Ginsburg Landau Theory, I'm abbreviating as GL Theory. So it says that, free energy, should be expanded, in terms of, order parameter, powers of, in terms of, power series, of order parameter. Now, it can be noted that, free energy is a scalar, however, the order parameter can be, higher dimensional quantity. Such as, a vector or a tensor or a complex quantity. So this is what we are going to do, that is write down the free energy functional, the meaning of the word functional, is function of a function. So we'll write this free energy functional, which is a function of the order parameter and order parameter could be a function of, say the, special parameter, such as, R or the, or, or any other parameter, that is relevant to the problem. It is usually the space variable, which is taken as that, that the order parameter being a function of that.

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order parameter

Wave function of the "Super" electrons, $\psi(\vec{r})$

density of the Superelectrons, $n_s(\vec{r}) = |\psi(\vec{r})|^2$.

(1) Absence of an external magnetic field

(2) Presence of an external magnetic field.

(A) Coherence length

(B) Penetration depth

Now we'll study the GL theory. So the order parameter in GL theory, is the wave function, wave function of the, so-called, Super Electrons. Okay? Call that as, Psi of R. Such that, the density of the super Electrons, what I mean by super electrons is that, they're superconducting electrons, which is equal to, mod Psi, R square. So the plan is like this, that we will do two cases. One is absence of an external field, magnetic field. And then also we'll do it, in presence of an external magnetic field. And finally we would compute two different quantities, one is called as, The Coherence Length and (B) called as, The Penetration Depth.

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Free energy ($H_{ext} = 0$)

$$F_s = F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$$

F_s : Free energy of a superconductor

F_n : Free energy of a normal metal.

α, β : phenomenological expansion Coefficient.

Minimum of F_s is obtained by

$$\frac{dF_s}{d|\psi|} = 0 = 2\alpha|\psi| + 2\beta|\psi|^3 = 2|\psi|(\alpha+\beta)$$

$$|\psi_0|^2 = -\frac{\alpha}{\beta} \Rightarrow F_n - F_s = \alpha\left(\frac{\alpha}{\beta}\right) - \frac{\beta}{2}\left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2}{2\beta}.$$

So the free-energy, in H external equal to, zero, so there's no external field. And we write down the superconducting free energy, which is equal to free energy, of the normal state, plus alpha psi, mod square, plus beta by 2, Psi to the power 4. Note that we are keeping the expansion, only up to the, biquadratic that is quatic terms, of Psi. And the other thing is that, there are no odd powers of Psi involved, because that would be, anti symmetric, with respect to the change, in the order parameter. So Fs is free energy, of a superconductor. F n, free energy, of normal metal and alpha and beta, are phenomenological coefficients, expansion coefficients. So minimum of Fs, so we want to minimize the free energy, with respect to the order parameter and find out where the, the optimum value, which minimizes the Fs lies. Minimum of FS, is obtained by, taking a dFS, d Psi, equal to 0 and this is equal to a, 2 alpha Psi, so and plus, 2 beta, Psi q, so I can take a Psi common and this becomes, equal to a 2 Psi common, and this will be like, alpha plus beta, so Psi mod square, that does minimize this, let's write it with a psi 0, mod square, equal to, equal to, minus alpha over beta. Thus we can write down, thus we can write Down, F, n minus f s, it's equal to, alpha into alpha, by beta and minus, beta by 2, alpha by beta, whole square, this becomes equal to, alpha square by, 2 beta. And it is almost obvious, that alpha and beta, both could be temperature dependent. Now in the first term, so let's, let's write this once more, that Fs, Fn minus, Fs is alpha square, over 2 beta,

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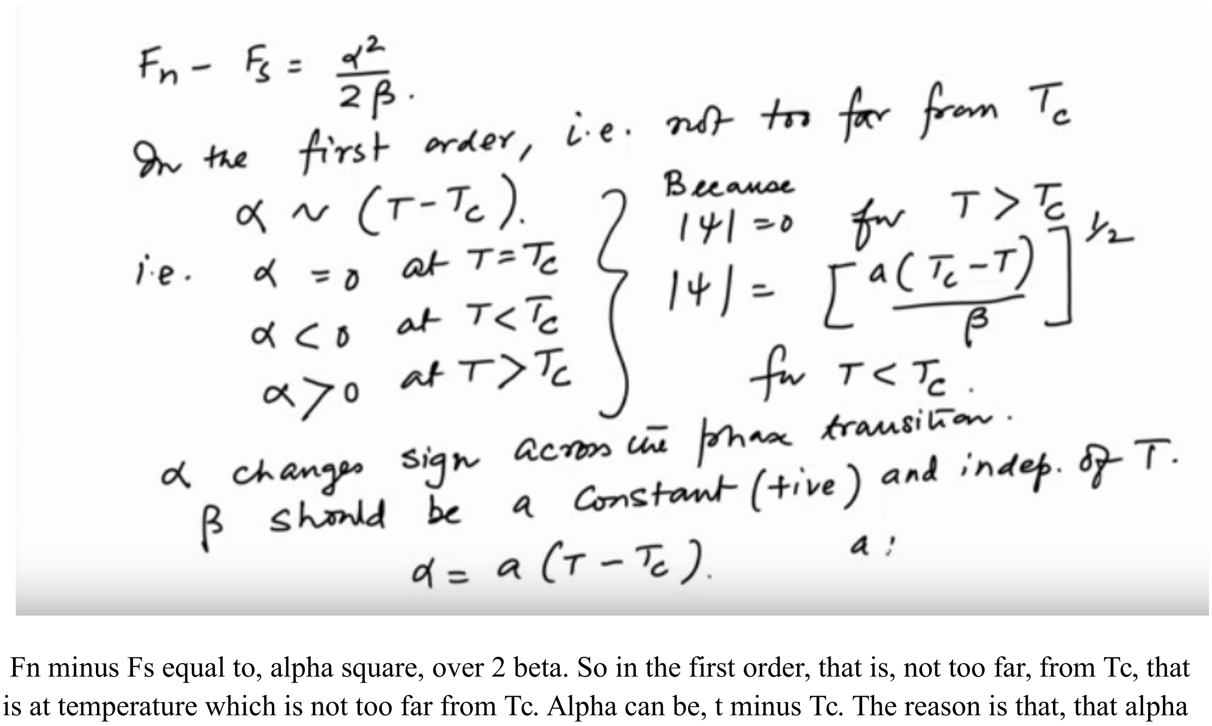

Fn minus Fs equal to, alpha square, over 2 beta. So in the first order, that is, not too far, from Tc, that is at temperature which is not too far from Tc. Alpha can be, t minus Tc. The reason is that, that alpha equal to 0, at t equal to Tc, alpha is less than 0, at T greater than, Tc and alpha is greater than zero, at, sorry, this is T less than Tc and this is T greater than Tc. Okay? Now this happens, because, Psi is equal to 0, for T greater than Tc. So Psi must be going as, a, TC minus T, whole to the power beta and whole to the power, half here, for T less than Tc. So the important thing for us to understand is that, alpha changes sign, across, the phase transition. However, beta should not change sign, Because, if both of them change sign, then of course will not have this, equation valid, which is Psi mod square, equal to minus alpha, by beta. And moreover, beta also should not have any temperature dependence,

so that, we’ll have some Pathological. So if beta has to be, helped, if beta does not have to change sign, across the transition, then it has to be like t minus, Tc whole square, because a whole square doesn't sign. But in that case, your alpha by beta will be like 1 over T, which would be, divergent as T goes to 0 and that should not happen. So beta should be, a constant. So basically it's a positive constant and independent of temperature. Of course, it means that, being constant means that, it is, independent of temperature. So at least for small deviations, from T equal to Tc, this is what should happen.
So alpha has a form, which is like aT, minus Tc, beta is a constant and hence we have The, Psi mod, equal to 0, for T greater than Tc and Psi mod equal, going as, Tc minus T whole to the power half, with A's and B's, as constant, a, is a constant here.

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Consider a superconductor in an external $\vec{H}_{ext}$

$$\vec{p} \to \vec{p} - \frac{e\vec{A}}{c} \qquad \vec{H}_{ext}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

$$\vec{\nabla} \to \vec{\nabla} - \frac{ieA}{\hbar c}.$$

$$F_s = F_n + \alpha|\psi(\vec{r})|^2 + \frac{1}{2}\beta|\psi(\vec{r})|^4 + \frac{\hbar^2}{2m}\left|\left[\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}(\vec{r})\right]\psi(\vec{r})\right|^2 + \frac{H_{ext}^2}{8\pi}$$

$$\text{Free energy, } F = \mathcal{F}\left[\psi(\vec{r}), \psi^*(\vec{r}), \vec{\nabla}\psi(\vec{r}), \vec{\nabla}\psi^*(\vec{r}), A(\vec{r})\right]$$

Now consider, a superconductor, in an external field, magnetic field that is. So the way it is taken care of, is that, the momentum, is changed by, p minus e a by C. This you must have seen in your, classical mechanics course, when we talked about a charged particle, in a magnetic field. Now this A, is the vector potential, which is related to the field as, H of r, external, equal to curl of a. So it is always derivable, from a vector potential, by this relation. This tells that, the del operator, should be replaced in presence of a magnetic field, by this. Now the free energies have to be written down, the same free energies that we have written down, earlier. So Fs equal to a Fn, plus alpha Psi, because of the magnetic field now, which is a function of r, we have an inhomogeneous order parameters, now Psi starts depending upon r. And also, because of the magnetic field, we have an additional term, which is given by, ie by H cross C, a of r and then, a Psi of r, mod square and plus a magnetic energy, because of the external field, which is given by, H external square, by 8 Phi. Now free energy, F is a functional of, Psi of r, psi * of r, del Psi of r, del psi * of r and A of r.

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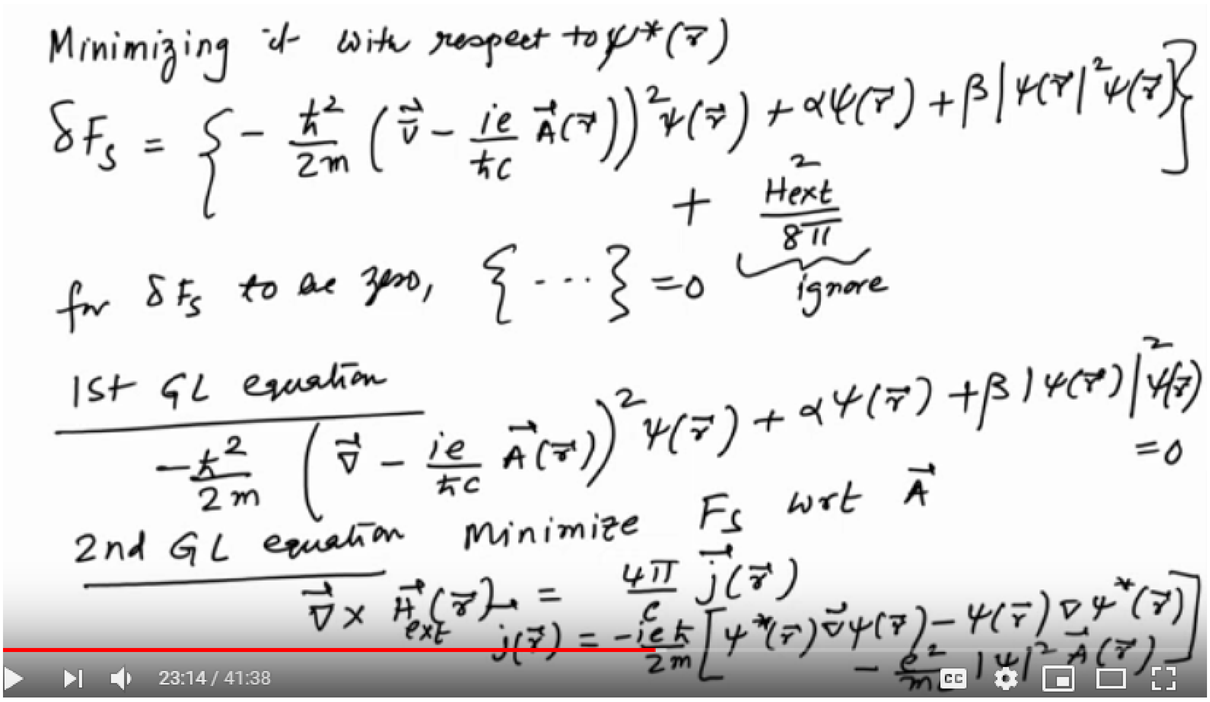

So minimizing it, Psi * r, we have del F s, equal to, minus of, H square by 2 m, del minus, ie by, h cross, C a of r, square Ps of r, plus alpha Psi of r, plus beta, mod Psi r square, Psi of r. And we can still write this as, This, the other term, which is H external, which we don't need to worry about, so let's just write it as H external square, over 8 pi. Right now ignore this term, it is not that, it is not important or it is small, because we don't need it for our discussion now.

So we ignore that and this, when we put, this quantity del Fs equal to 0, that would tell that, these bracket has to vanish, this curly bracket has to vanish, for, Delta Fs, to be 0, this bracket will be equal to 0. And hence we get, the first, Ginsberg Landau equation, as this. So note the difference between, the earlier case, when we do not have magnetic field is that, Psi was, homogeneous, we could, mention that here as well, that Psi is and does not depend upon r, basically. So this is called as a, first GL equation and in which we have neglected the next, the second term there. Now to get the second GL equation; Minimize F s, with respect to A, the vector potential and that gives a very familiar equation, which is called as the, ‘Amperes Law’, which is like this. So this is the second GL equation, where your jr has to be identified as. So that's the, second GL equations. And let us now talk about,

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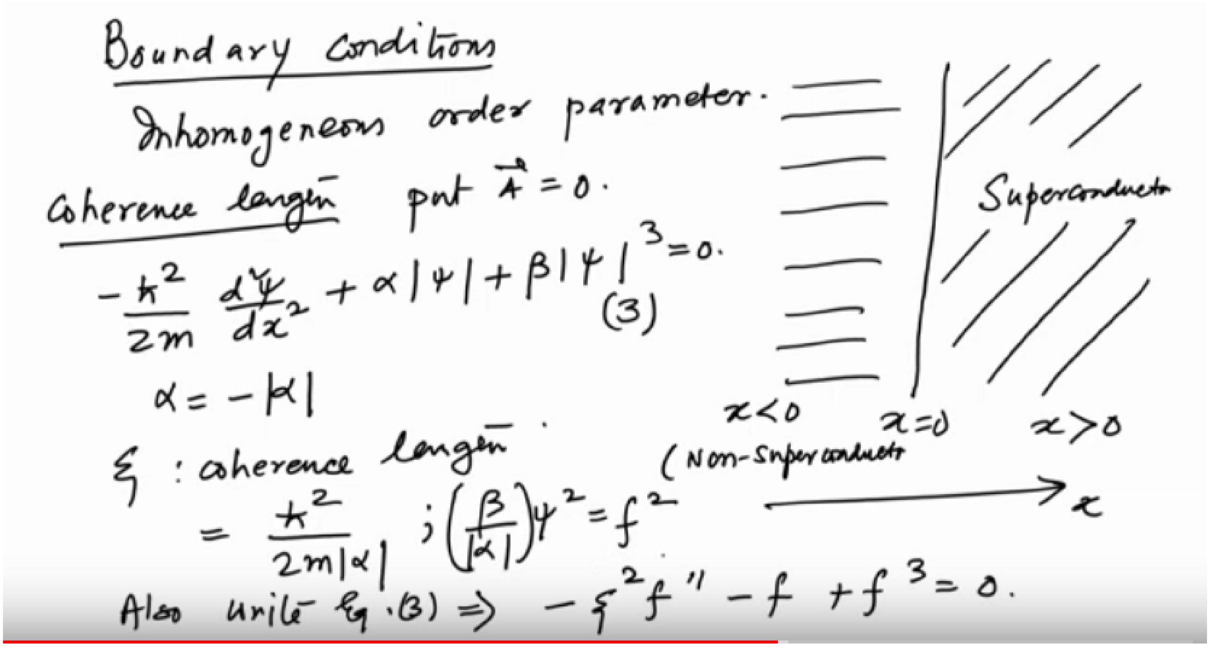

The Boundary Conditions. And in doing so, we would get or we would, sort of, obtain, the relations or the expressions, for the penetration depth and the coherence length. So, let us talk about, an inhomogeneous order parameter, which we have been, talking about, since we have introduced the magnetic field. So let us consider, 1/2 region, of, so this is X equal to 0, so this is where the, superconductor exists. So superconductor exists for x greater than 0 and x less than 0, we don't have a superconductor, it could be a normal metal or it could be a magnetic metal, for example. So there is a non superconductor, will just simply write it as, non superconductor.

So these are Junction systems, which are routinely studied, in experimental physics, experimental condensed matter physics and one can easily make a junction of, a superconductor and a non superconductor. Now for deriving the expression, for coherence length, put A equal to zero, so we don't need the external field there. And in that case, I get from the first GL equation and the variation is taken to be, in one dimension, so this is the x axis. Alpha Psi, plus a beta Psi cube, this is equal to zero. So we have said that earlier, that, alpha is negative, in the superconducting state. So take alpha to be equal to minus alpha and define a quantity called as, Psi, which is called as the, 'Coherence Length', and this is equal to, H cross Square, over 2 m alpha. Also, this is simple algebra, also write, equation, let us call this as, the first one as, equation 1, first GL equation, the second GL equation, to be equation 2 and this to be equation 3. So now write equation 3, in addition define, that, beta over alpha, Psi square, is equal to F square. So with these definitions, one can write down equation 3 as, minus Z square, f double prime, minus F plus, F cube equal to zero. So I have cast this first GL equation, in terms of a scaled variable, which is called as a f.

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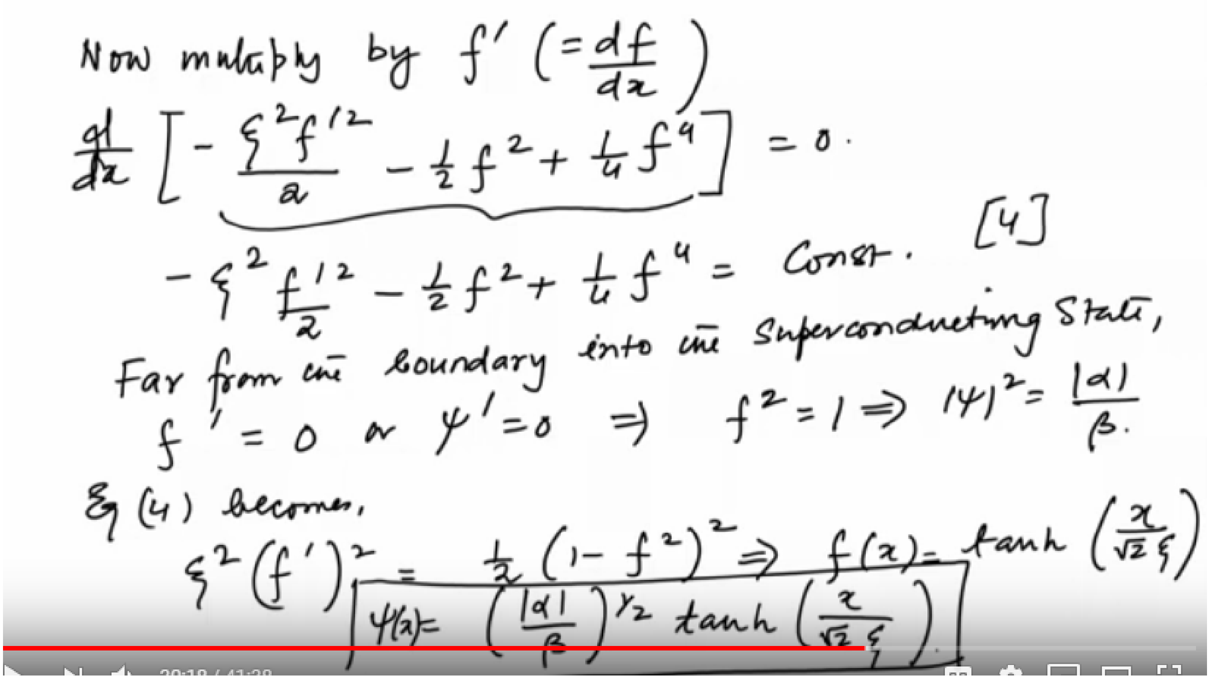

Now multiply, by f prime, which is equal to df dx and then, one should be able to write this as, ddx of minus Psi square, F prime square over 2 minus ½, f square plus 1/4 f 4, this is equal to 0. So if ddx of this equal to 0, which means, that this should be equal to a constant. Now far from the boundary, that is into the superconducting state, so this is the boundary and if you are too much into the superconducting state. f prime should be equal to zero? Or Psi prime should be equal to zero? So the variation of the order parameter, with, as a function of x, should be equal to zero. And hence this gives, that f square, should be equal to 1, which tells that Psi square, should be equal to alpha, over beta, which is what we have obtained. Then this equation, which is equation (4), Becomes, as Psi square, f prime square, equal To, half 1 minus, F square, square. So this is the equation, for Psi or f, which needs to be solved and the solution, is of this form, that f of x, equal to, tan hyperbolic, x by root 2, Psi. So that gives that, Psi, equal to, alpha by beta, whole to the power half and hyperbolic x by, root 2 Psi.

So that tells, that Psi as a function of x, has a variation like this, so that gives the, extent of the wave function for the, for the Cooper pair or the super electrons and there is a characteristic length, that emerges, which is equal to Zi.

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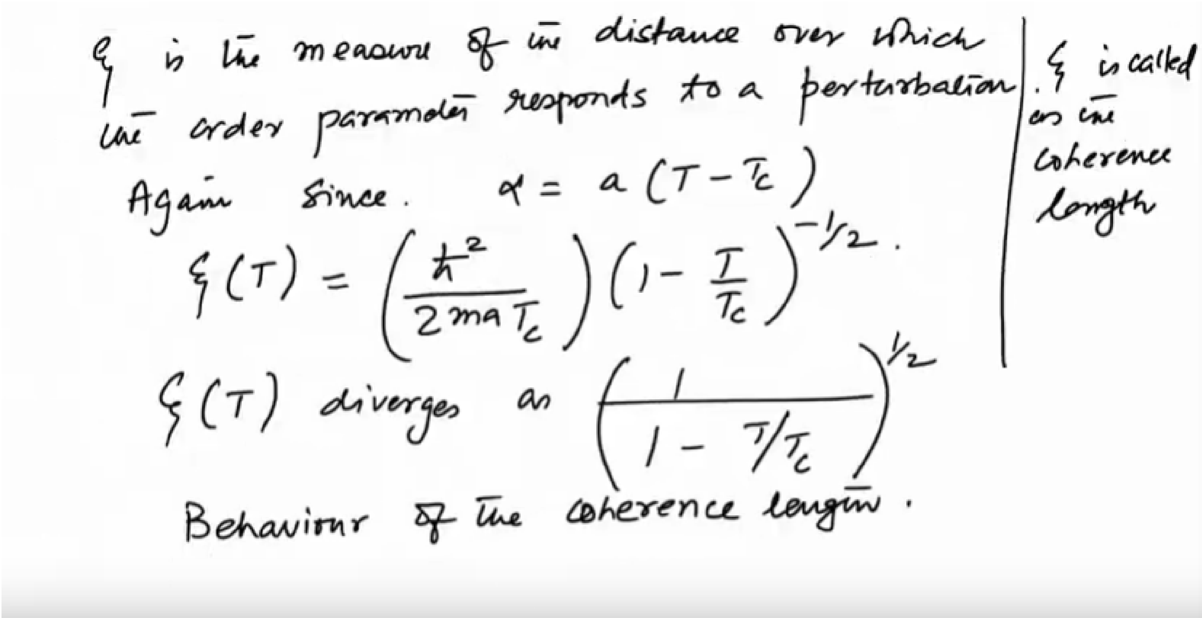

So Zi is the measure, of the distance, over which, the order parameter, responds to a perturbation, in this case the perturbation is the, presence of a boundary, that lies between the, superconductor and maybe a normal metal. Again, since alpha equal to a T, minus Tc, the temperature dependence of the coherence length, so this is called as the coherence length. So Zi of T, is equal to, h cross square, over 2m a, Tc and one minus T by Tc whole to the power minus half, so Zi T diverges, as 1 divided by 1 minus T by Tc, whole to the power, whole to the power half. So as T goes to Tc, this diverges in this fashion. You can see clearly, that the divergence, is like a square root divergence. So this is the behaviour, of the coherence length.

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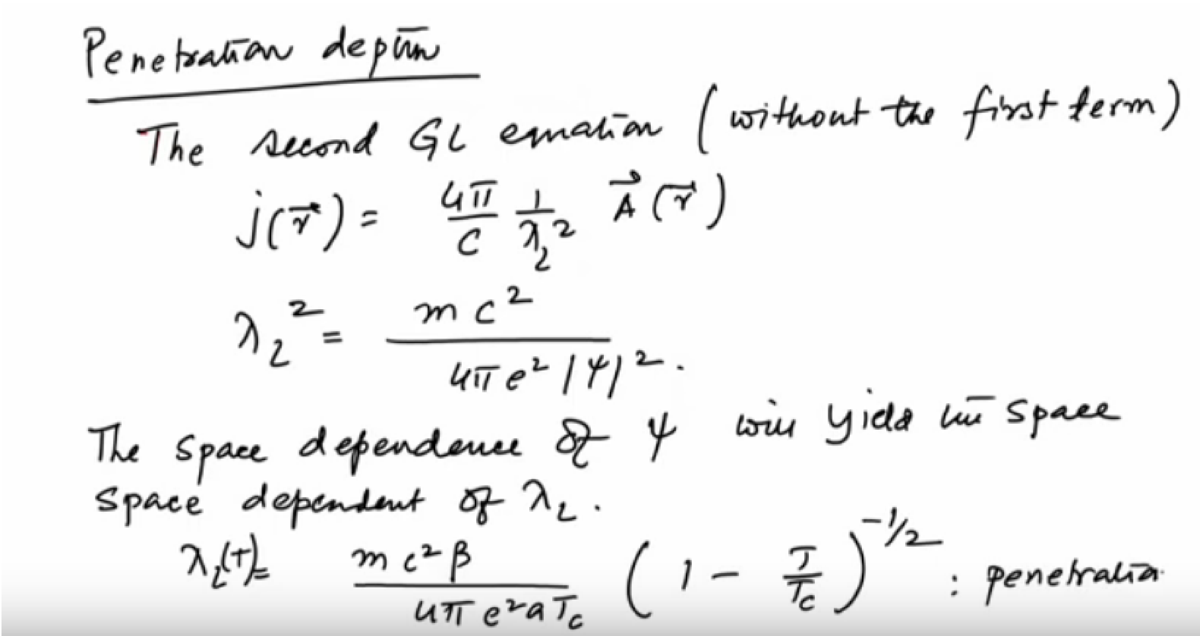

Let us look at the other quantity which is the, penetration depth. Look at this expression, the second GL Equation, if you drop the first term and only look at the second term, that is this term, let me just mark it in red. So if you look at this term, then this exactly looks like, the London equation. This, without the first term, that is a usual current term, so we are purely looking at the current due to the external field and then it looks like that j of r, is simply, 4 phi by C 1 by, lambda L square, a of r and immediately, the lambda L can be, read off as, mc square, divided by 4 phi e square and mod psi square and this, the, so this is the, vanishing or rather, the, the dependence, of the, of Psi, so the space dependence of Psi. Will yield the space dependence of, of lambda.
Remember this sigh actually falls off as, tan hyperbolic, x over, root 2 Zi, so that'll determine that, the how lambda L falls off as a distance. And to look at the temperature dependence, the lambda L, which is equal to, ms square beta, divided by 4 pi, e square, a Tc, 1 minus T by Tc, whole to the power ,minus half. So this is lambda L, as a function of T, this is the penetration depth.

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$\lambda_L(T)$ diverges again $\left(1-\frac{T}{T_c}\right)^{-1/2}$

Define

$$\kappa = \frac{\lambda_L}{\xi} = \frac{mc}{e\hbar}\left(\frac{\beta}{2\pi}\right)^{1/2}$$

$\kappa > \frac{1}{\sqrt{2}}$ type – II Superconductors

$< \frac{1}{\sqrt{2}}$ type – I Superconductors

So lambda LT, diverges, again as, 1 by T my, Tc whole to the power, minus half, exactly like sigh of T. And, one can also define, a dimensionless parameter called as, lambda by Zi, which now becomes equal to, mc over e h cross, beta over, 2 pi, whole to the power half, which has, only one unknown parameter which is beta, which appears in the Ginsberg Landau theory. Now, we know that, Kappa, greater than, 1 by root 2, is termed as the type 2 superconductors and less than 1 by root 2, the type 1 superconductors. So to summarize, that, without doing explicitly a microscopic theory, which we have done, for the BCS case. Here simply writing down, the free energy functional, as, in powers of, the order parameter and minimizing it with respect to the order parameter, one can actually get, the two energy scales, that are relevant, for these superconductors. Namely the penetration depth and the coherence length. And not only that, there, how they diverge, for T close to Tc, is also obtained, which are, of this order, as 1 by t minus Tc, whole to the power, minus half.

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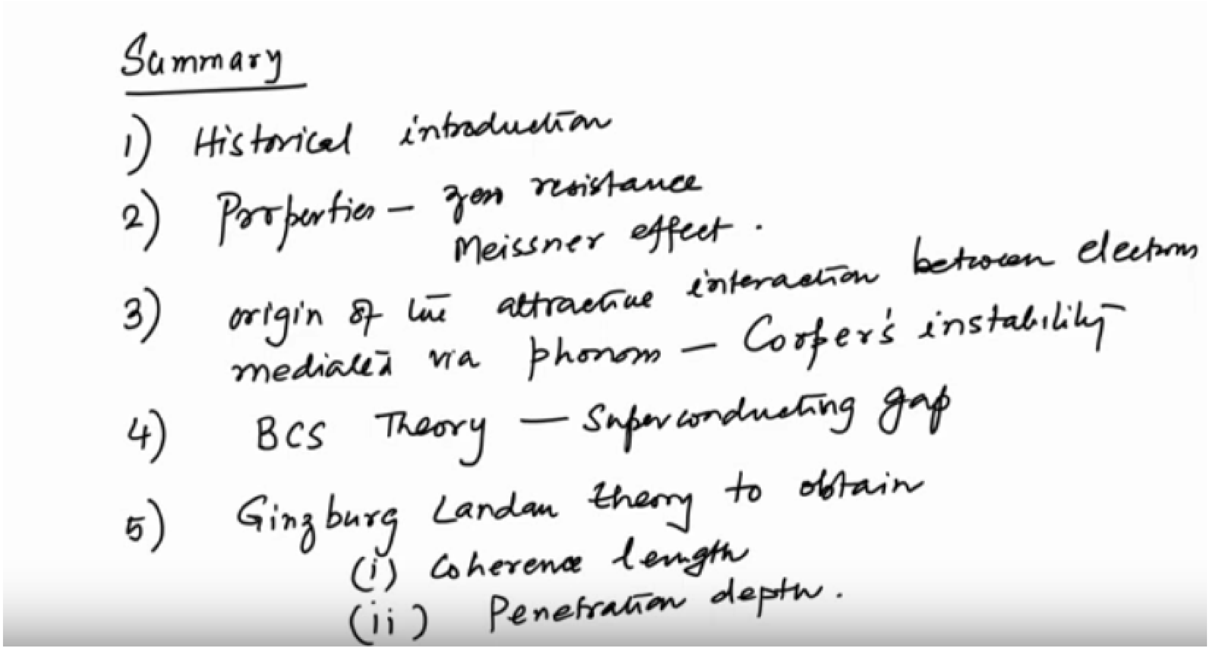

So to conclude the chapter on Superconductivity, we have, given one a historical introduction and then we have, said about properties, such as, such as zero resistance, Meissner effect, etc,. And then we have talked about the origin, of the, attractive interaction, between electrons, mediated via phonons. This is known as, 'Cooper's instability'. And then of course we have done, BCS theory and in which, the gap, superconducting gap, is obtained. And the superconducting gap, is seen to be, non analytic, in powers of the, strength of the attractive, electron-electron interaction. So, one cannot do a, pertamative theory, in order to get this result.

Then we have done a variational theory and also, gotten the, behaviour of the superconducting gap, as a function of temperature. And then finally we have done a Ginsberg Landau theory, to obtain, one coherence length and to penetration depth. And these are all required for you, to learn, because this, story of super conductivity, is fairly old, now you understand it's more than, it's about 110 years old, since its first discovered. And then also, it was, a new class of superconductors, were discovered in 1986. And a large amount of work had gone in. Since then, however we haven't touched, that part, because of poor knowledge or still you know evolving knowledge, in that particular area, but we have done these studies of the weak coupling superconductors, so-called the BCS superconductors, in somewhat details.