Lecture – 3

Electron Pairing, Basics of BCS theory

Let us, look at BCS theory. The Bardeen, Cooper, Schrieffer theory, the first microscopic theory of superconductivity, which was extremely ,successful in describing, the behaviour and the properties of the conventional superconductors, which are also known as weak coupling superconductors.

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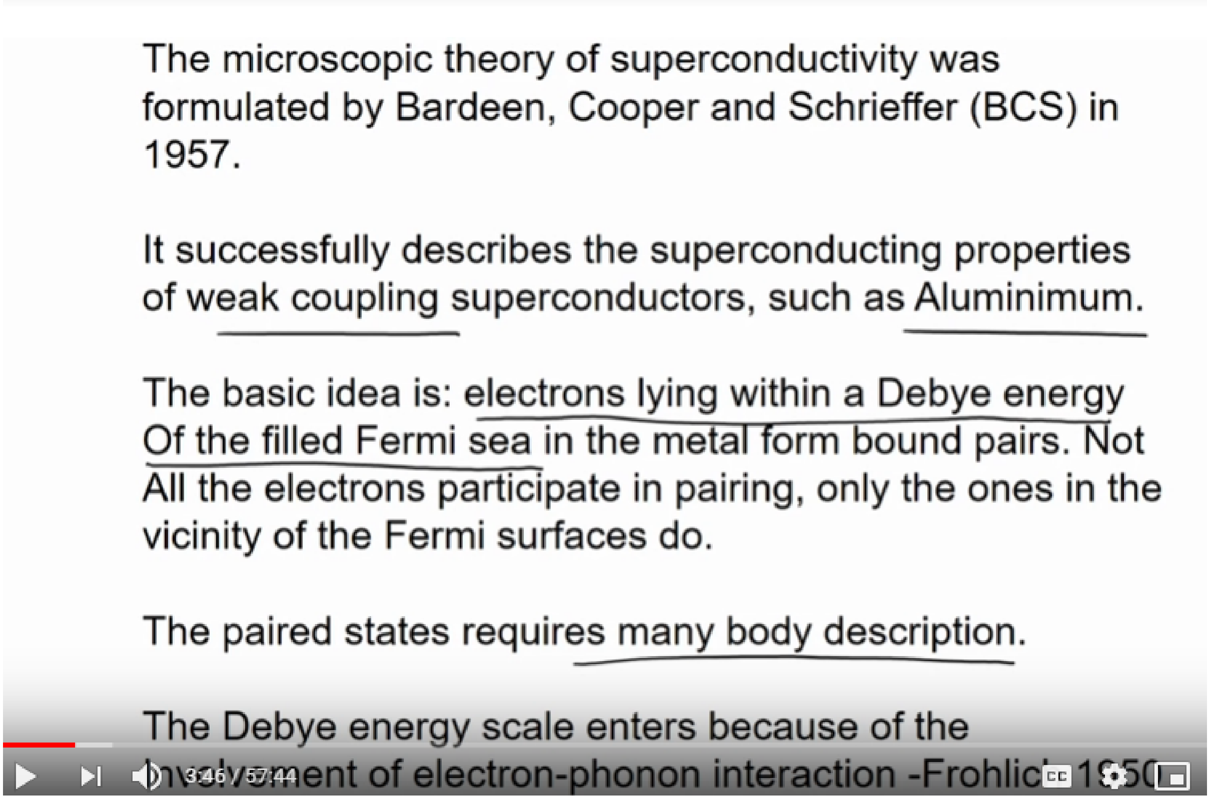
The microscopic theory of superconductivity was formulated by Bardeen, Cooper and Schrieffer (BCS) in 1957.

It successfully describes the superconducting properties of weak coupling superconductors, such as Aluminimum.

The basic idea is: electrons lying within a Debye energy Of the filled Fermi sea in the metal form bound pairs. Not All the electrons participate in pairing, only the ones in the vicinity of the Fermi surfaces do.

The paired states requires many body description.

The Debye energy scale enters because of the involvement of electron-phonon interaction -Frohlich 1950

So, I said that it was the first microscopic theory and it was formulated by Bardeen, cooper and Schrieffer and that's where the name BSC came, in 1957. Which was approximately 50 years, after the discovery of superconductivity experimentally, which as we have seen before, that it was done by K modelling owns in 1908? This theory is successfully describes, the superconducting properties, of weak coupling superconductors, will try to make it clear that, what we coupling? The word weak coupling means, such as, aluminium and other materials, mostly they are metals and not all metals from or give rise to superconductivity. But, there are certain metals which give, this also will discuss, which the potential candidates for superconductivity are. So, the basic idea is, that the electrons lying within Debye energy of a filled Fermi sea. .Okay? So, those two electrons are needed and they are lying within the D by a frequency or Debye energy of a filled Fermi sea in a metal, they can form bound pairs. So, this is something that has to be contrasted from the usual behaviour of electrons, which would repel each other, however, if you can create a situation in which they are in a very close proximity of a filled Fermi sea, the electrons don't interact with the pharmacy, other than by exclusion principle and then these two electrons, actually can have an attractive interaction between them and can form bound pairs and it is important to say that ,not all electrons participate in pairing and only the electrons, which are in the vicinity of the Fermi surfaces do, not in the say the, near the ,centre of the Fermi surface ,they don't take part in pairing. But, it's only which lie in the vicinity of the Fermi surface or close to the Fermi energy they do, this paired state requires a many body description and this is what? Is the central focus of our discussion, at least for, this one and a discussion, that's going to be held after This now, this hint that the Debye energy scale is important for this problem, it actually signals, that there is an involvement of electron-phonon interaction and this was proposed by Frohlich , as early as 1950.

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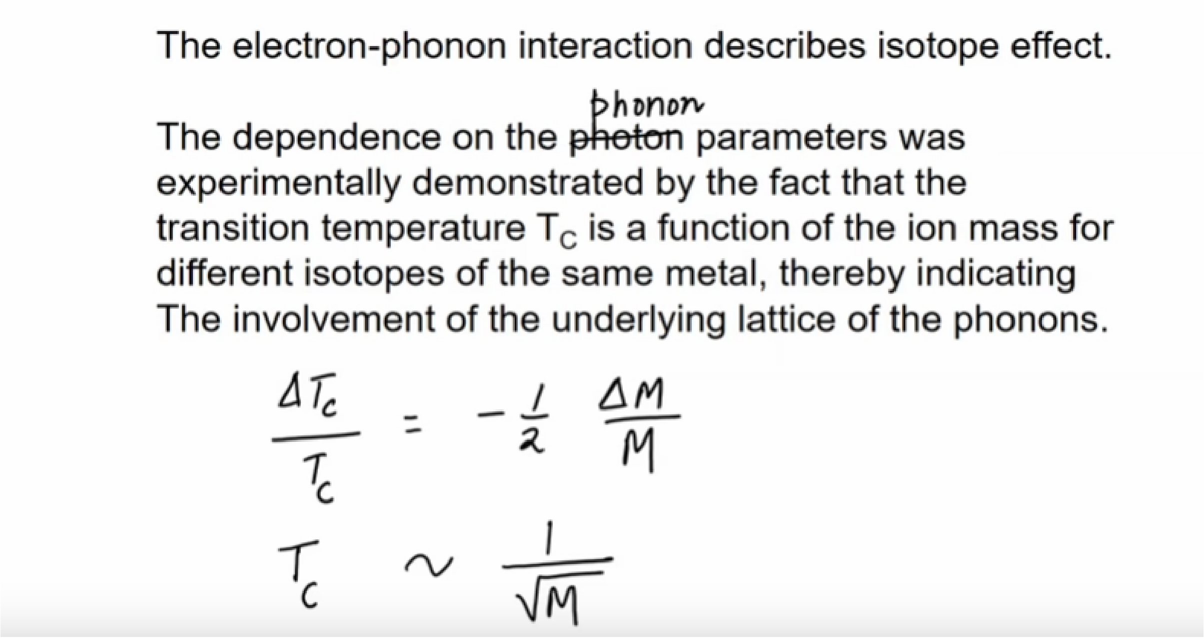

The electron-phonon interaction describes isotope effect.

The dependence on the ~~photon~~ phonon parameters was experimentally demonstrated by the fact that the transition temperature T_C is a function of the ion mass for different isotopes of the same metal, thereby indicating The involvement of the underlying lattice of the phonons.

$$\frac{\Delta T_c}{T_c} = -\frac{1}{2}\frac{\Delta M}{M}$$

$$T_c \sim \frac{1}{\sqrt{M}}$$

So before this theory came up. Now, this electron- phonon interaction describes isotope effect. Just give me a minute to say, what that is? So, the dependence on the sorry, this should be phonon on, the phonon parameters were experimentally demonstrated by the fact, that the transition temperature, we have seen this concept of transition temperature, a temperature that divides superconducting state, with that of a non superconducting one or rather a metallic state, the temperature at which this transition occurs is called as a superconducting transition temperature. So this, transition temperature T C, becomes a function of the ionic mass, of our different isotopes of the same metal. Now, that tells that if that happens then the lattice, because the ions are involved and hence the lattice is involved and then the, the phonons are involved as well and this is an experimental demonstration of an electron phonon interaction or the rather the phonons are playing an important role in the whole process. So, if you want to write this, this goes as a delta TC over TC, this is equal to a minus half Delta M over m, where m is an ionic mass and Delta M is the difference in ionic masses of two isotopes or very simply this is written as TC to be a function of you know, 1 by square root of M, where m being the ionic mass. So, these are experimental demonstration, that the phonons are involved and which is also central to the discussion that we are going to have,

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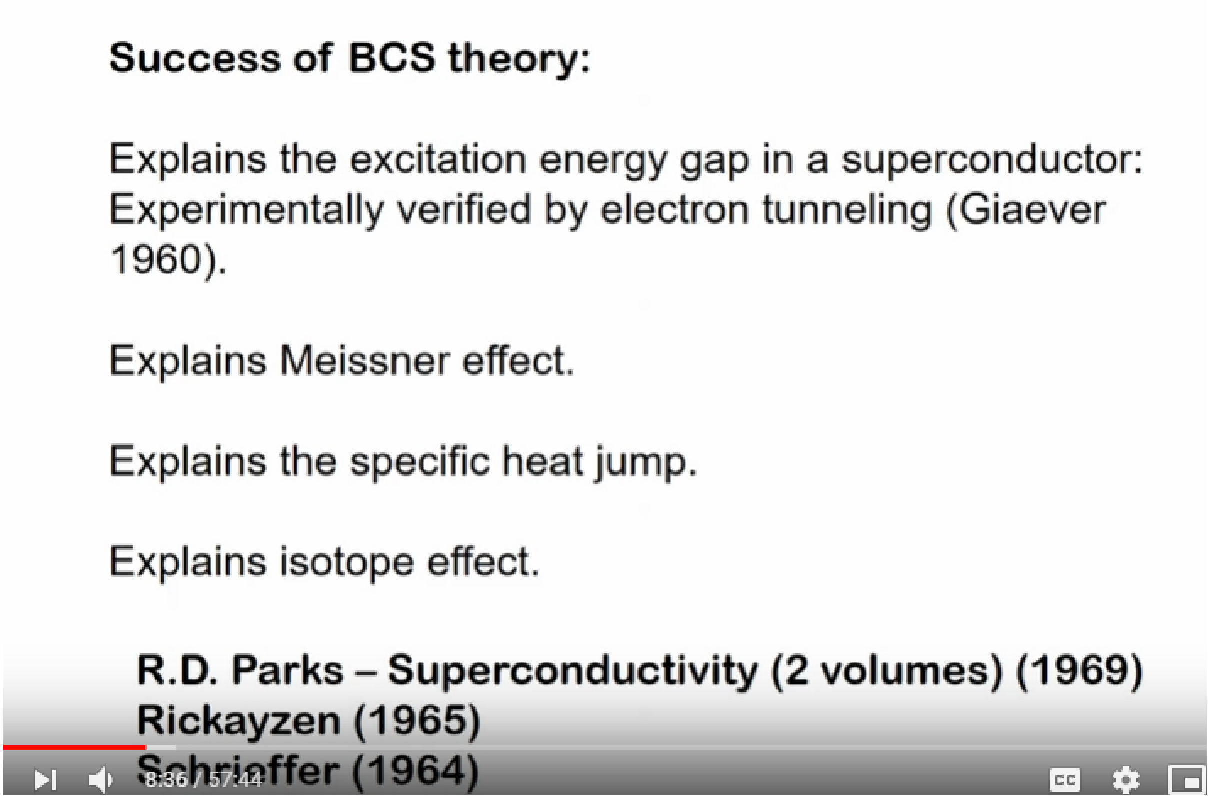

Success of BCS theory:

Explains the excitation energy gap in a superconductor: Experimentally verified by electron tunneling (Giaever 1960).

Explains Meissner effect.

Explains the specific heat jump.

Explains isotope effect.

R.D. Parks – Superconductivity (2 volumes) (1969)
Rickayzen (1965)
Schrieffer (1964)

So, coming to the success of the BCS theory, a priori, of course we haven't talked about BCS theory yet but, it's important to know that why should we be doing it in a course like this there are many successes in fact it's, it gives the excitation energy gap, the energy gap so-called in a superconductor there is certain amount of energy that is required to be supplied, to the superconductor in order to break, the or rather the break the Cooper pairs and promote them to the excited state so this is done experimentally are done by, the electron tunnelling which was done by giver in 1960, so when an electron is sent across a metal in, metal superconductor Junction the superconducting energy gap to Delta, it should not have been able to pass through because there is a additional barrier that is, is there which the electron of which the metallic side does not have but, however there is a process called as a Andreev, reflection which we are not going to discuss here and because of that an three reflection there is a pair, if not the electron there is a pair that can propagate into the superconducting region and this shows up in a peak which is called as a Andreev peak and this happens just below the energy, of the excitation energy of the of the gap of a superconductor and hence its it gives these electron tunnelling measurements they idle the magnitude of the gap and there are other experiments which actually probe, the symmetry of the gap, the symmetry of the gap in the momentum space so, to say it explains Meissner effect, we have seen extensively what Meissner effect is it explains the specific heat jump, it explains isotope effect as we have seen and the important books in this context are, parks are deep ops he had two volumes of super conductivity which were published in 1969 and then there's a Rickayzen, which there there's a book by Rickayzen, which was in 1965 and then this book by, Schrieffer in 1964, one of the proposals of BCS theory.

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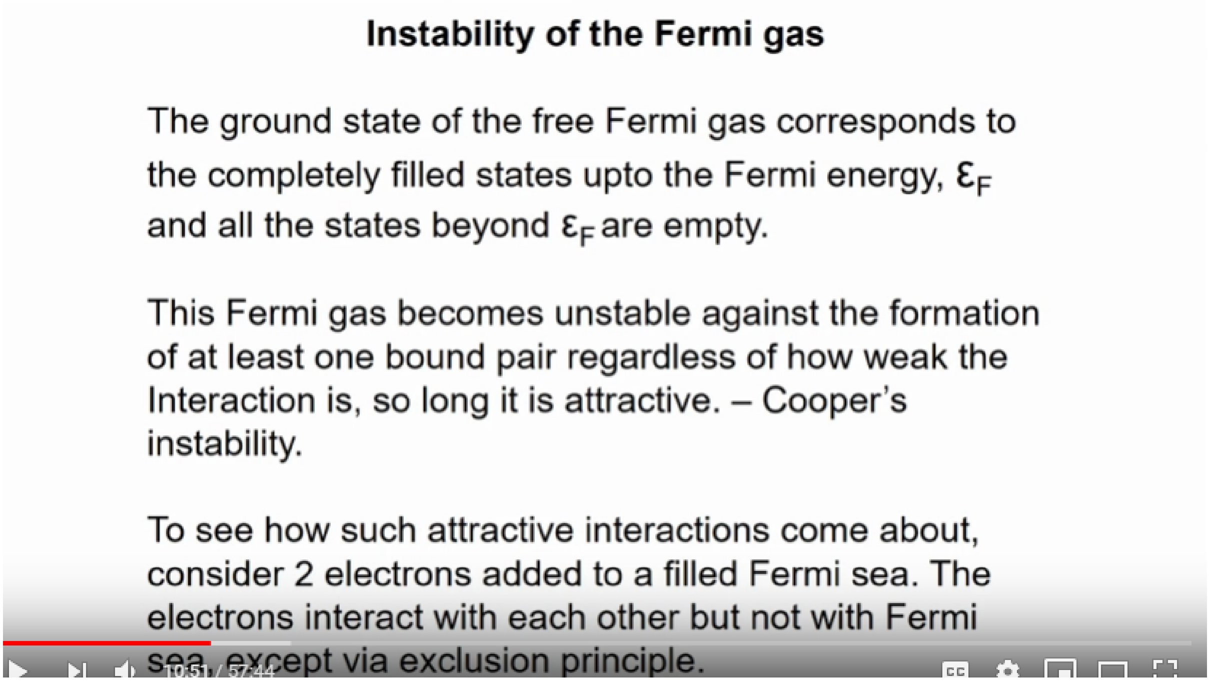

Instability of the Fermi gas

The ground state of the free Fermi gas corresponds to the completely filled states upto the Fermi energy, ε_F and all the states beyond ε_F are empty.

This Fermi gas becomes unstable against the formation of at least one bound pair regardless of how weak the Interaction is, so long it is attractive. – Cooper's instability.

To see how such attractive interactions come about, consider 2 electrons added to a filled Fermi sea. The electrons interact with each other but not with Fermi sea, except via exclusion principle.

So, what exactly is happening how, are the electrons interacting with themselves by, an attractive interaction by, an attractive potential such that they form bound pairs and what do, these bound pairs have in common with the superconducting state so, it can be understood as the instability of the Fermi gas so what happens is that the ground state of the free Fermi gas as we said that this corresponds to the filled Fermi sphere, corresponds to all completely filled states up to the Fermi level and all states beyond, the Fermi energy they are all empty this is the definition of the Fermi distribution function but, this Fermi gas is going to get unstable against the formation of at least one bound pair irrespective of regardless of how, weak the interaction is so long it's attractive and this was known as Cooper's instability so, to rephrase this we have a field Fermi sea, or a Fermi sphere if by some means, what means we are going to discuss just in a while if by some means that two electrons form bound pair in the vicinity of that Fermi sphere, lying within the energy scale which is given by the phonon frequency or, phonon energy then this Fermi gas, the free Fermi gas is going to be unstable and this is called as the “Cooper's instability” to see how such attractive interactions come about now, consider two electrons added to a filled Fermi sea, the electrons interact with each other but, not with the Fermi sea except via the exclusion principle one sort of hand waving way of seeing it is that suppose you have a horse, moving through a dusty field and when it runs through the dusty field it gathers a lot of dust around it, and anybody say another horse, would not see that horse but, would see a cloud of dust that is moving forward and pretty much something similar to that is happening that an electron passes through the lattice and because, of these the ions being present these pics of in a positive charge that it looks for another, electron it looks like a positive charge and they form bound pairs, now it could also be argued that why doesn't this other electron also is perceived, as a cloud of positive charge which when they will repel and we'll see that this happens at different time scales and that's why there is not enough time that both of them, would be like an ionic cloud moving together and they would repel rather it happens at a shorter life time scale, that the, the electron, one electron sees the other ionic cloud and gets attracted to it and forms a bound pair but, this is just a hand waving way of seeing looking at it we will see this more carefully so, as has been told that the first hunch, or the inkling, of the BCS theory what was put forward by

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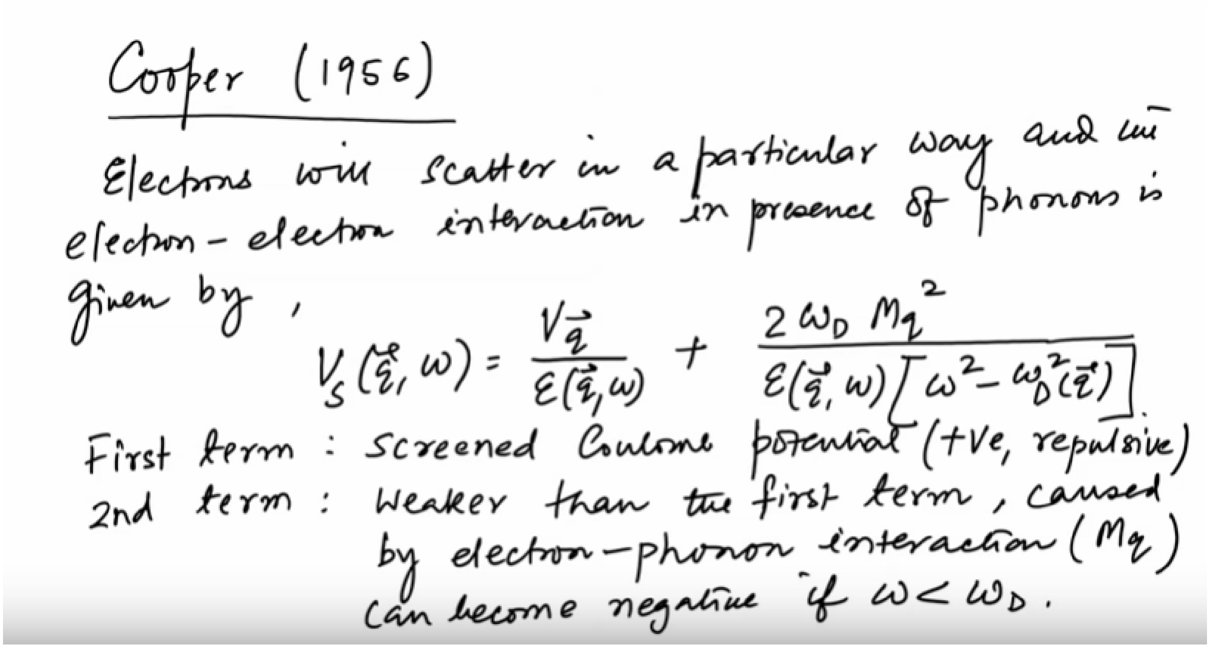

Cooper (1956)

Electrons will scatter in a particular way and the electron-electron interaction in presence of phonons is given by,

$$V_s(\vec{q},\omega) = \frac{V_{\vec{q}}}{\varepsilon(\vec{q},\omega)} + \frac{2\omega_D M_q^2}{\varepsilon(\vec{q},\omega)\left[\omega^2-\omega_D^2(\vec{q})\right]}$$

First term : Screened Coulomb potential (+ve, repulsive)

2nd term : Weaker than the first term, caused by electron-phonon interaction (M_q) Can become negative if $\omega < \omega_D$.

Cooper in 1956, just a year ahead of the discovery of BCS theory .so, he pointed out, that a normal metal, will become unstable, to the formation of pairs, if the I mean, no matter how small the attractive interaction between the pair is and so he proposed that ,such pairing would be preceded or would be facilitated by the presence of the lattice and hence phonons. So, the electrons would scattered in a particular way and the electron-electron interaction, in presence of phonons is given by, I will explain what all these quantities are? So, the first term is the screened Coulomb potential and it's always positive and hence repulsive. So, in no way, that this term can give rise to, an attractive interaction, the first term. Now, the second term is generally, weaker than the first term and caused by electron phonon interaction, the strength of which is given by, MQ. Now, this term can become negative, if Omega square is less than, Omega D square. So, in general, this the frequencies lie around the Debye frequency, as has been told earlier. So, in a small range, that e in the vicinity of Omega D, if Omega becomes, less than Omega D, the energy denominator becomes small and negative. So, the whole interaction term or this whole second term becomes, negative and large and this can cause super conductivity. So, it will happen inner

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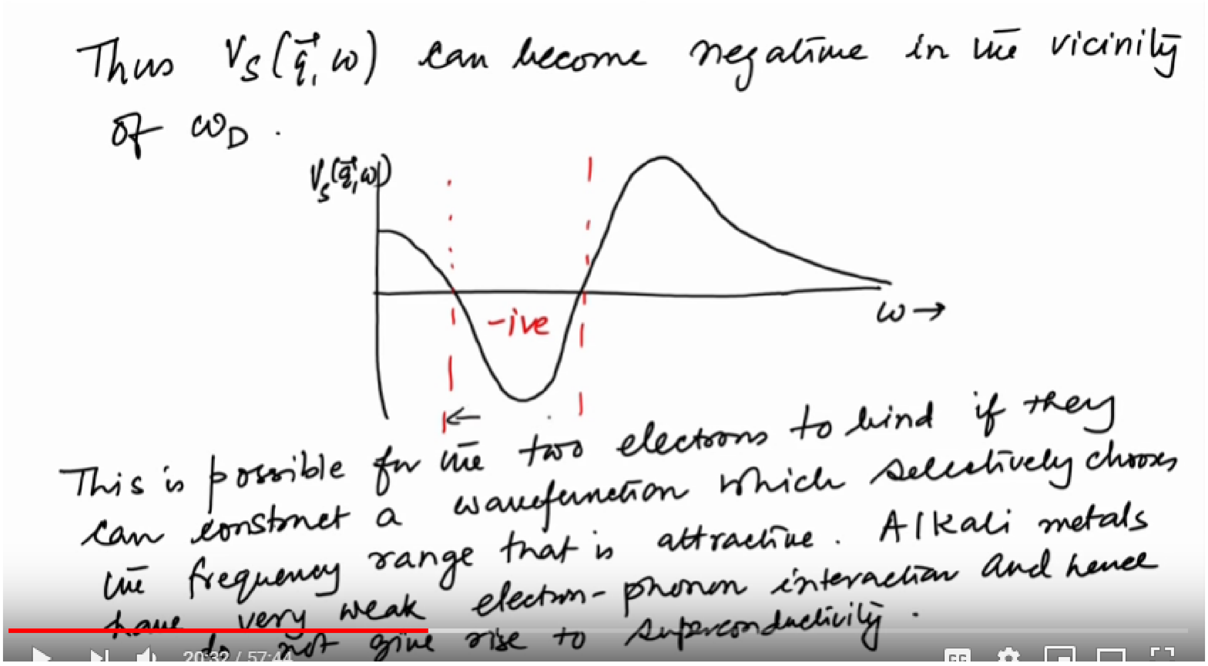

Thus $V_s(\vec{q},\omega)$ can become negative in the vicinity of ω_D.

This is possible for the two electrons to bind if they can construct a wavefunction which selectively chooses the frequency range that is attractive. Alkali metals have very weak electron-phonon interaction and hence do not give rise to superconductivity.

Thus, I mean, vs. q Omega. Now, when the second term actually, takes over the first term. The second term being negative, then the whole interaction, can become negative, in the vicinity of the, of Omega D .Okay? so, it is possible that the two electrons, to bind, if they construct a relative wave function ,which selectively uses a frequency range, that is given by so, if you plot this, then it looks like that, it is so, this is a vs Q Omega and this is Omega and it is in this range so it, it is in this range, that it's negative and one can have, an attractive interaction .so ,once again to repeat it is that ,this is possible for the two electrons to bind , if they can construct ,a wave function, which selectively chooses, the frequency ,the frequency range, that is attractive, attractive, attractive. And in fact alkali metals, like sodium, potassium, etcetera; alkali metals have very weak, electron phonon interaction and hence, do not give rise to superconductivity .Okay? So now, so this interaction has to be in this regime that is shown here

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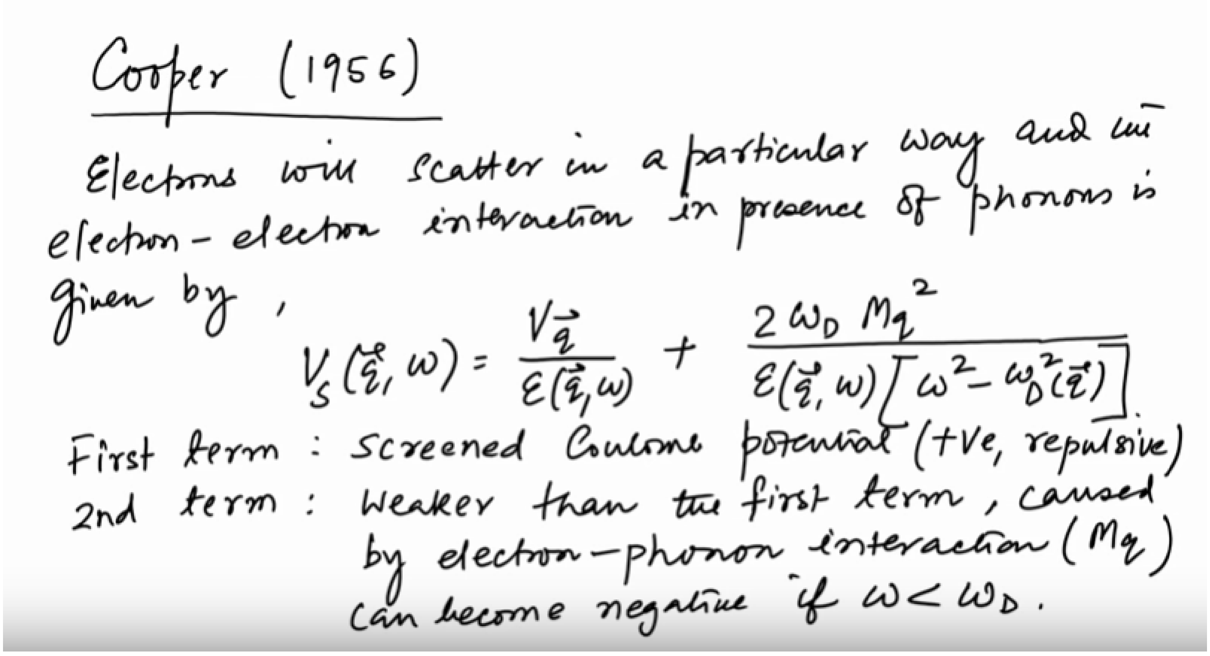

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First term : screened Coulomb potential (+ve, repulsive)

2nd term : weaker than the first term, caused by electron-phonon interaction (M_q) can become negative if $\omega < \omega_D$.

And it is usually is as a competition of two terms, one term being necessarily positive and repulsive, which is the screen Coulomb term, and the other term which arises because of the electron phonon interaction, the strength of that being mq .but, because of a denominator, which is a competition between, the energy of the pair versus, the energy of the ,the Debye energy .so and then since Omega is close to Omega D, Omega is measured from the field for me C , Omega is close to Omega D, if it can happen that Omega is less than Omega D, then this term the denominator becomes small and negative and then we can have an attractive interaction and that attractive interaction ,will actually give rise to super conductivity .but, we haven't still derived or at least explain that, how this interaction term comes from ? And that can be understood in two different ways, one is that rigorously taking an electron photon matrix element or I draw the diagrams and calculate this term, the second term particularly. There is another way, of doing that which is what we are going to do

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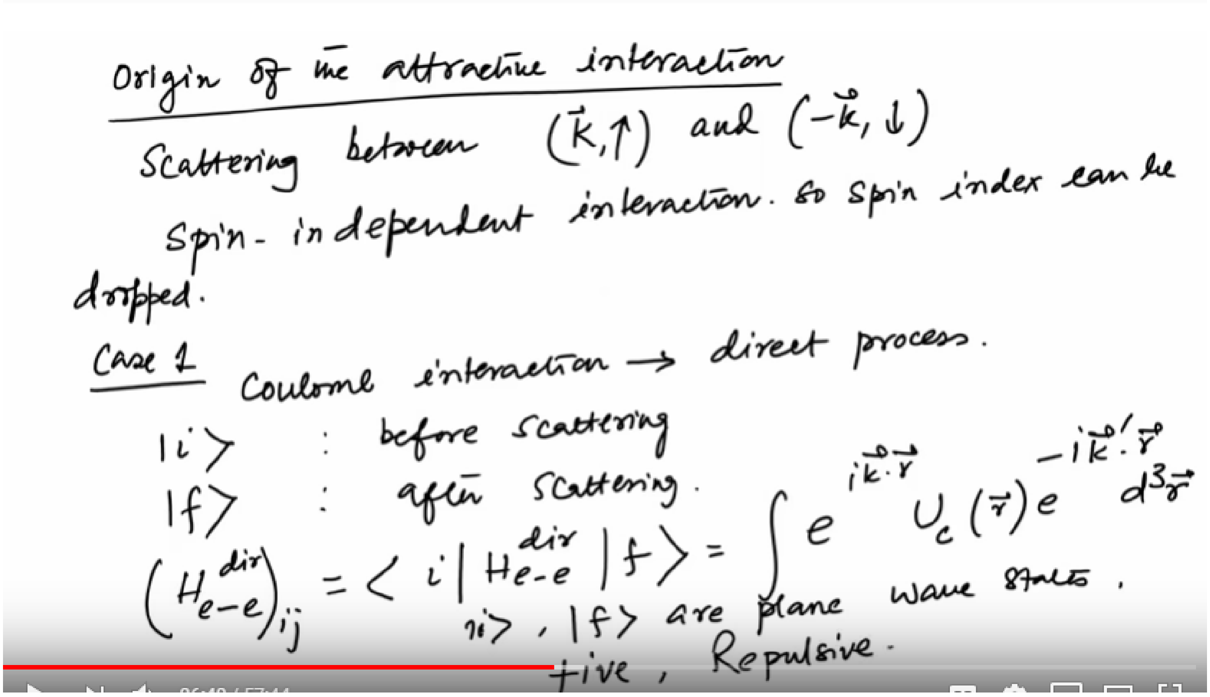

Origin of the attractive interaction

Scattering between $(\vec{k},\uparrow)$ and $(-\vec{k},\downarrow)$

Spin-independent interaction. So spin index can be dropped.

Case 1 Coulomb interaction → direct process.

$|i\rangle$: before scattering

$|f\rangle$: after scattering.

$$(H^{dir}_{e-e})_{ij} = \langle i | H^{dir}_{e-e} | f \rangle = \int e^{i\vec{k}\cdot\vec{r}}\, U_c(\vec{r})\, e^{-i\vec{k}'\cdot\vec{r}}\, d^3\vec{r}$$

$|i\rangle$, $|f\rangle$ are plane wave states.

+ive, Repulsive.

So, let's do the origin of .Okay? So, we have a scattering between case k, up and a minus K, down electron .so, one could ask the question, that why is it? that one has one electron has a momentum K, the other electron has a momentum minus K ,it can be shown that, with the Momentum's being chosen in this particular fashion, a two particle state yields the minimum energy, which is the most stable configuration and for a bound pair to happen ,because they're in the same state, we cannot expect a both up ,up pair to occur or an up down, down pair to occur. So, the pairing has to be between up and down, so these are the momentum and spin of the participating electrons, which are participating in the bound pair. So, let's see that, how these pairing takes place? Now, just to say that it's a spin independent interaction and so, spin index can be dropped .Okay? So, let's say, the case one, the case one is that, that there is a direct electron-electron interaction, which is facilitated by the Coulomb forces. So, let us , consider an initial state, so this case one is Coulomb interaction and it's a direct process, what do I mean by direct process is? That, there is an initial state I, before scattering and there is a final state F, after scattering. So, the matrix element of such a electron-electron interaction, in such a direct process ,will just write e, e and dir, let's just give a bit of space here and this is written as, is I He-e , dir and F and this can be written as exponential I, K dot R, which is a plane wave initial state and you see, that's what we write as Coulomb and then exponential minus I K prime dot R and D cube R ,so, this is the matrix element and this matrix elements, K is the incoming state momentum, so the electron comes with a momentum K and leaves with a momentum – I, mean leaves with a momentum K Prime, this minus sign is coming because you are taking a kept here, whereas this is a bra here, so this is exponential I ,K daughter. So, we have assumed that both I and F are plane wave states, I and F are plane wave states. So, this term cannot give any negative contribution and hence it cannot account for the two electrons to form a bound pair, so this is positive and hence it's repulsive. So, we are not yet there, for getting the attractive interaction.

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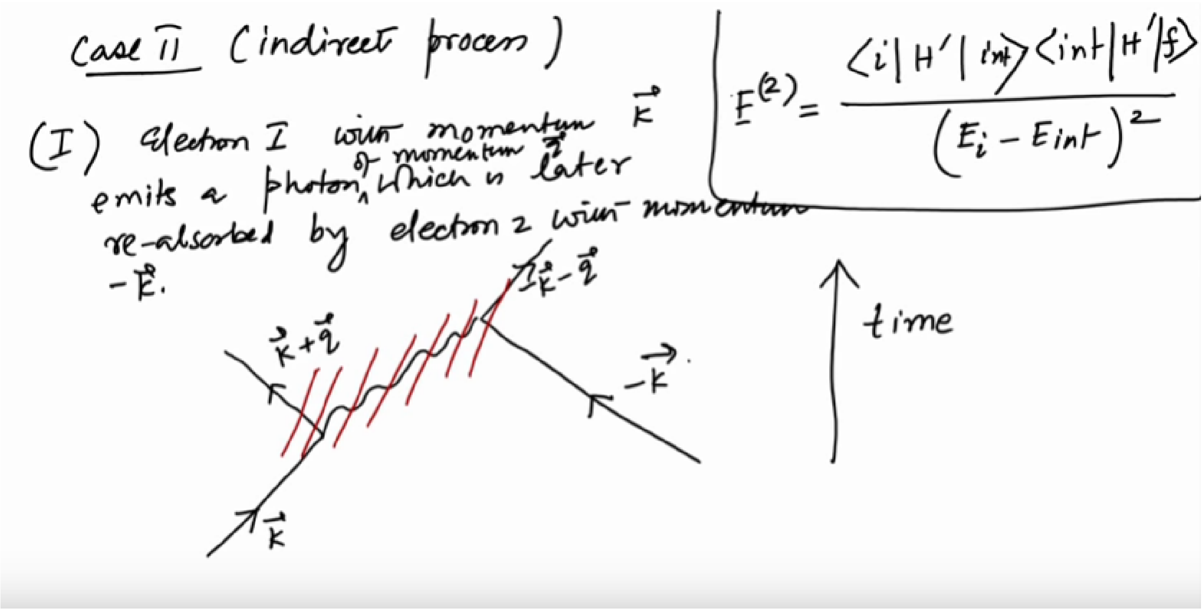

Case II (indirect process)

(I) electron I with momentum $\vec{k}$ emits a photon of momentum $\vec{q}$, which is later re-absorbed by electron 2 with momentum $-\vec{k}$.

$$E^{(2)} = \frac{\langle i|H'|int\rangle\langle int|H'|f\rangle}{(E_i - E_{int})^2}$$

Now, let us look at, a case 2, which is an indirect process ,it takes place waier exchange of phonons and this there is an intermediate process involved and it is that's, why it's a second order process, remember your second order perturbation Theory ,where the way it is written is that there is an intermediate ,there is an initial state and then there is a perturbation and then there is a final state, I'm sorry, there's an intermediate state here and then this, another matrix element and the final state and then here, one gets the energy , so I minus the intermediate energy square. So, this is the correction, let's call it as e2, in the second order perturbation theory. this is just a result that we are going to use, so there are two ways that this indirect thing can take place, one is number one, in which the electron one, let's just label them, even though they are indistinguishable this is for our convenience, so with momentum key, emits a photon, which is later reabsorbed by electron 2 with momentum – K. let's just say, this process once more. So, there is an electron, called electron one, which has a momentum K, it emits a photon, which later, at a later time, the electron 2 absorbs this, which had a momentum minus K and this process would be looked upon as this. So, this is the key, which is electron number one and this is an electron minus K, this is electron 2, this one M its ,k m its, photon of momentum Q .so, this becomes K plus Q and this becomes minus K, minus Q and because two so ,this is time axis. So, this is how time increases and we are particularly, you know talking about, that so this is the, this is how the time grows, so later on, when after it emits it gets reabsorbed. Now, the same thing can be thought of for the reverse case,

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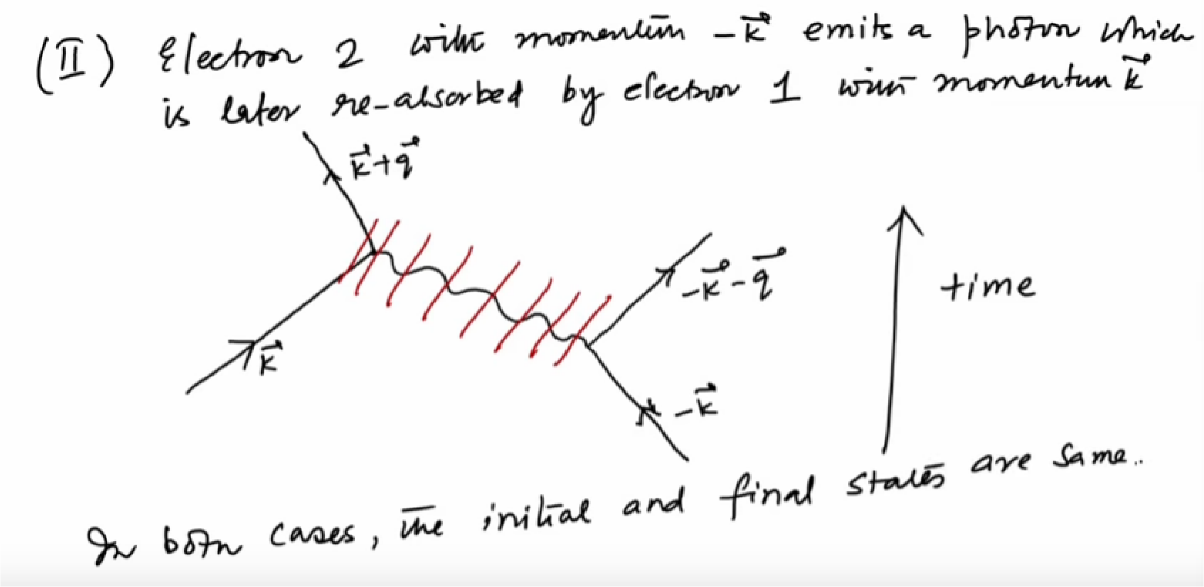

(II) Electron 2 with momentum $-\vec{k}$ emits a photon which is later re-absorbed by electron 1 with momentum $\vec{k}$

In both cases, the initial and final states are same.

that is 2 ,is that electron 2 , with momentum K, - k emits a photon, which is, is later reabsorbed by electron ,electron one with momentum K .so, this is a similar thing, it's just that the alignment of the figure will be a little different, so they'll be K here and it goes like this and now because the process happens in the reverse order, so we'll draw it this way, so this is electron one, with K, again it's K plus Q here and it is, minus K and this is minus K, minus Q and we are talking about ,so this is the again the time, so these are the two possibilities of the indirect process, that one emits the photon and the other absorbs it and there's a time delay ,between that and in principle both can occur and both will have to be considered. Now, you see that in both cases, the initial and final states are same.

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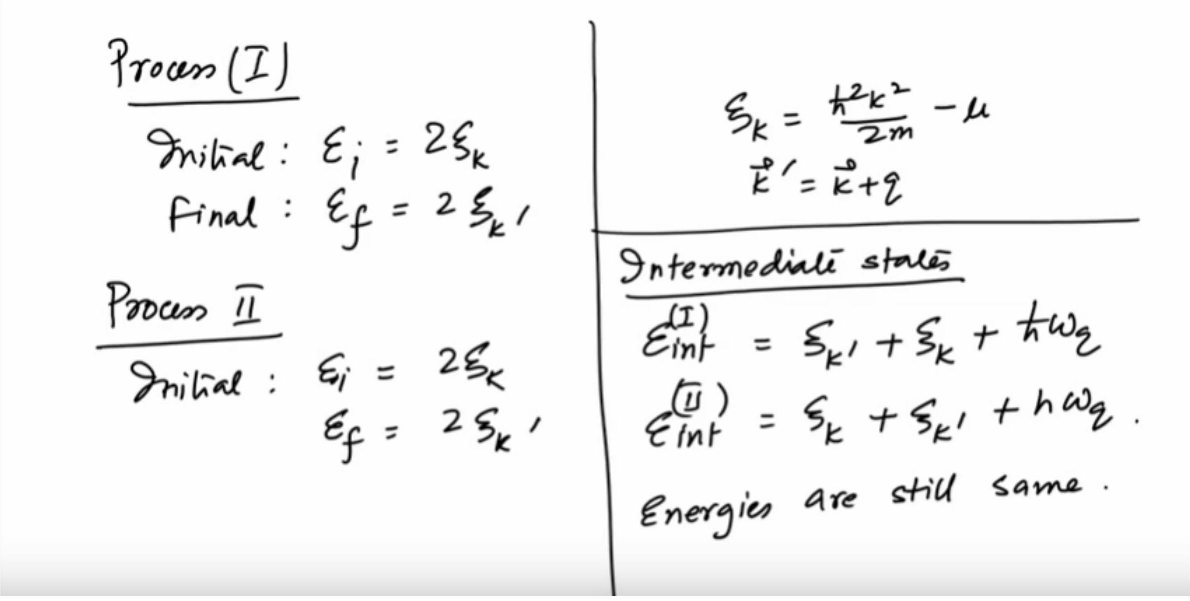

Process (I)

Initial : $\mathcal{E}_i = 2\xi_k$

Final : $\mathcal{E}_f = 2\xi_{k'}$

Process II

Initial : $\mathcal{E}_i = 2\xi_k$

$\mathcal{E}_f = 2\xi_{k'}$

$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu$

$\vec{k}' = \vec{k}+\vec{q}$

Intermediate states

$\mathcal{E}^{(I)}_{int} = \xi_{k'} + \xi_k + \hbar\omega_q$

$\mathcal{E}^{(II)}_{int} = \xi_k + \xi_{k'} + h\omega_q$.

Energies are still same.

So, let us do this, that process one, process one, had initial is epsilon I, which is two Zi K, which is epsilon K minus mum, so Zi K is equal to H cross square ,K square over 2m minus mu and final is epsilon F which is two Zi K Prime and K prime equal to K plus Q. So, that's the, that's for the process one and similarly for the process 2 , its initial is Zi equal to two Zi K and F is equal to 2 Zi K prime, well I mean, this K is our vector. But, we are ignoring it at the moment and so even if the initial and the final states are saying, I mean, the energies are same. But, the inter amine, the initial and the final states are same. But, the intermediate states are different and let's say, the intermediate step, so for process one, the intermediate is equal to Zi k prime plus, Zi k plus, h cross Omega Q and for the second one, it's equal to Zi K plus, Zi k prime plus, h cross Omega Q .so, energies are still same of the intermediate state. Now, calculate the matrix elements, including these intermediate states.

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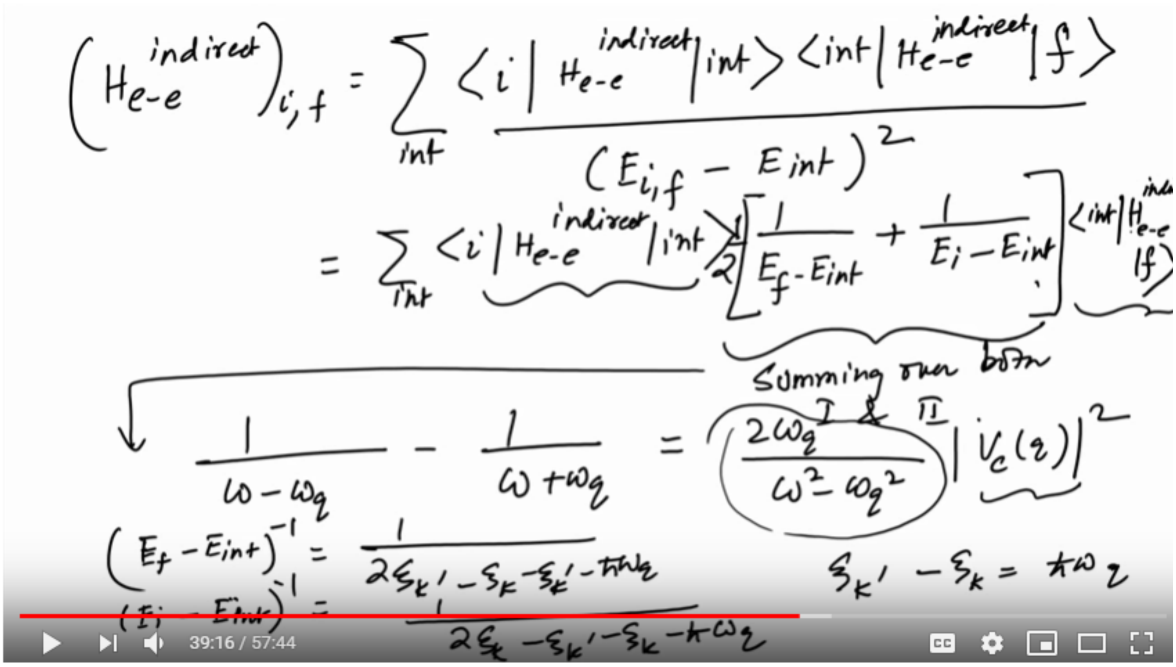

$$\left(H_{e-e}^{\text{indirect}}\right)_{i,f} = \sum_{int} \frac{\langle i | H_{e-e}^{\text{indirect}} | int\rangle \langle int | H_{e-e}^{\text{indirect}} | f\rangle}{(E_{i,f} - E_{int})^2}$$

$$= \sum_{int} \underbrace{\langle i | H_{e-e}^{\text{indirect}} | int\rangle}\ \underbrace{\frac{1}{2}\left[\frac{1}{E_f - E_{int}} + \frac{1}{E_i - E_{int}}\right]}_{\text{Summing over both I \& II}} \underbrace{\langle int | H_{e-e}^{\text{indirect}} | f\rangle}$$

$$\frac{1}{\omega - \omega_q} - \frac{1}{\omega + \omega_q} = \frac{2\omega_q}{\omega^2 - \omega_q^2} | \underbrace{V_c(q)} |^2$$

$$(E_f - E_{int})^{-1} = \frac{1}{2\xi_{k'} - \xi_k - \xi_{k'} - \hbar\omega_q} \qquad \xi_{k'} - \xi_k = \hbar\omega_q$$

$$(E_i - E_{int})^{-1} = \frac{1}{2\xi_k - \xi_{k'} - \xi_k - \hbar\omega_q}$$

so, H e-e, in direct, using the same notation as earlier between ,I and F ,you have to sum over the intermediate States and its I , H e-e in direct, I'll just write it, intermediate and intermediate H e-e in direct, indirect means ,they exchange your phone on .so, they don't directly interact and F and then there is a square of the matrix element. Now, it's either I or F both of them being the same. and the square of that, so this is gives rise to an intermediate i, H e-e indirect , intermediate and 1/2 of, 1/2 and 1 divided by E f minus e nth plus, 1 by E I plus minus Eint and then we have, intermediate H e-e indirect and F and so on. So, these are simply numbers. Because, these are matrix elements. So, this is summing over, over both 1 and 2 .so, this is written here, it's equal to 1 divided by Omega minus Omega Q, 1 divided by Omega plus Omega Q. So, this becomes equal to 2 Omega Q, divided by Omega square minus Omega Q Square and a VC, Q square, which is, so in, in essence, this exactly looks like the second term, that we have gotten, with an energy denominator which is here, given by this and if you want how this comes, then you can look at this thing, that e F minus e int, inverse it's equal to ½ zi K prime, minus Zi K minus Zi K prime minus H cross Omega Q and E I minus e int, inverse it's equal to 1/2 Zi K, minus Zi K prime, minus Zi K, minus H prime Omega Q and also Zi K prime minus k, Zi has been used as H cross Omega Q .so, this matrix elements are positive definite, this one, however because, of this energy denominator and in that narrow that we have talked about, this could be attractive and hints would give rise to the attractive interaction .

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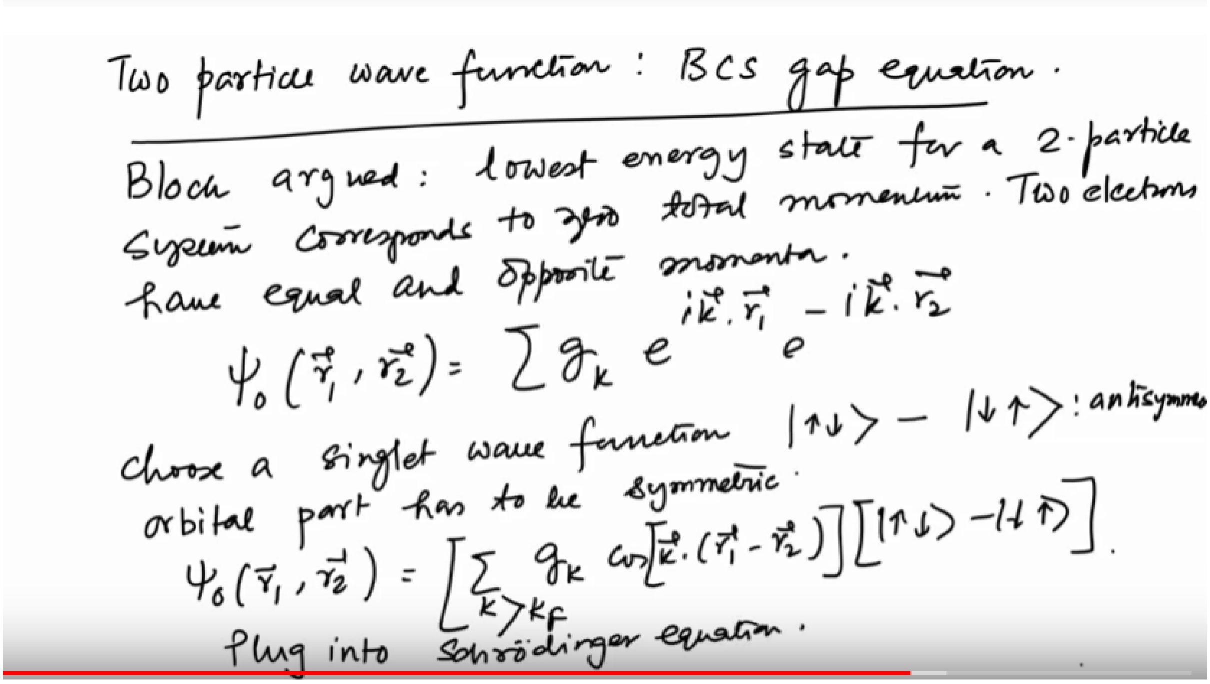

Two particle wave function : BCS gap equation.

Bloch argued : lowest energy state for a 2-particle system corresponds to zero total momentum. Two electrons have equal and opposite momenta.

$$\psi_0(\vec{r}_1, \vec{r}_2) = \sum g_k e^{i\vec{k}\cdot\vec{r}_1} e^{-i\vec{k}\cdot\vec{r}_2}$$

choose a singlet wave function $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$: antisymme

orbital part has to be symmetric.

$$\psi_0(\vec{r}_1, \vec{r}_2) = \Big[\sum_{k>k_F} g_k \cos[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)]\Big]\Big[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big].$$

Plug into Schrödinger equation.

Now, let us look at, the two particle wave function: BCA's gap equation .Okay? so, block actually argued: lowest energy State for a two particle system, corresponds to zero total momentum. So, the two electrons are equal and opposite momentum, two electrons which are taking part in binding, so we can write down, the orbital wave function, it's only the orbital part that we are writing it down ,two have, R 1 and R 2 equal to a gk exponential I K dot R 1, exponential minus I K dot R 2, where GK is some weight or the amplitude of the wave function. Now, when we write the full wave function, the orbital part would be symmetric or anti-symmetric depending on the spin part beings anti-symmetric or symmetric because the full wave function, consisting of the orbital part and the spin part will have to be anti-symmetric. So, if we choose symmetric for this, then the spin part would be anti symmetric and if we choose anti-symmetric part for this, then the spin part would be symmetric. But, remember that we have done this earlier, for two spins, we can have four states, one of them is called as a singlet   state, which is an anti symmetry and the other three are called triplet states, which are symmetric ,which means that if you interchange the particles then the wave function does not change. So, because the spin part for a singlet wave function, is anti-symmetric. So, the orbital part will choose will be chosen as symmetric. Now, this is also shown, that it is generally true, for a two particle system, that the singlet one has a lower energy .so, we take a singular choose a singlet wave function and BCS theory corresponds to, singlet pairing. So, we have the singlet wave function as, up down - are down up: and this is anti-symmetric and because this is anti-symmetric the orbital part has to be symmetric. So, we take this, orbital part as and also we'll have to take, that K less than, I mean, K greater than K. Because, we are writing down the two particle wave function ,which is residing just outside the Fermi surface .so, it's GK and the cosine K dot R 1 minus R 2 and then up down - r down. Now, this can be plugged, into the call this as equation 1.

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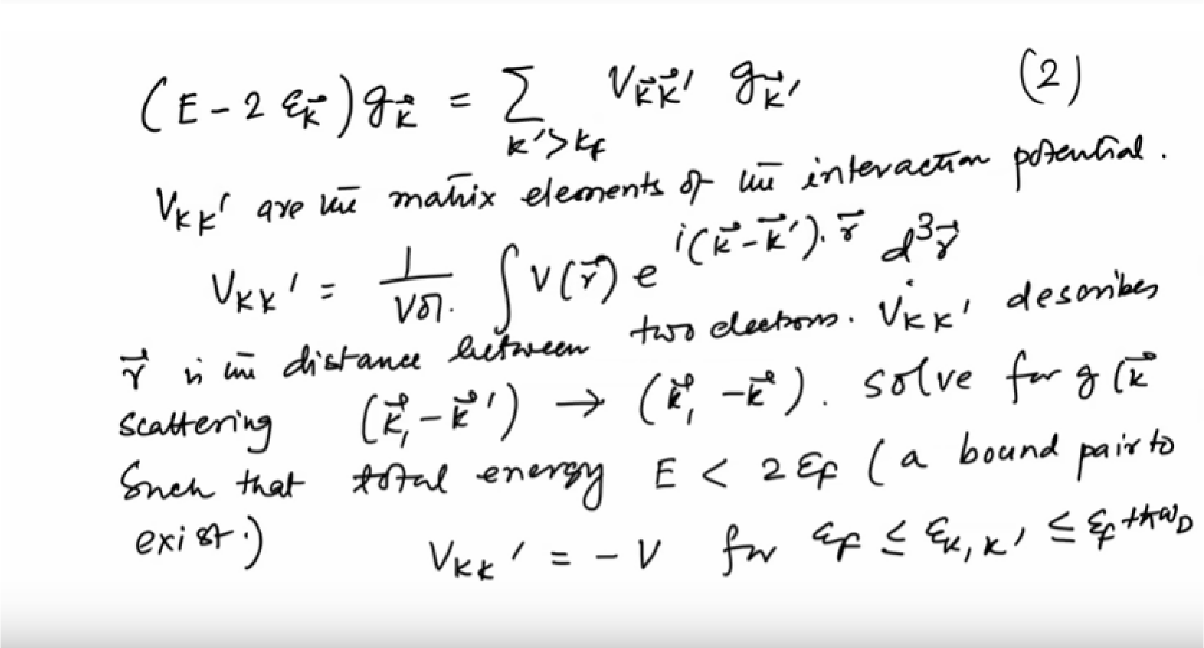

$$(E - 2\varepsilon_{\vec{k}})\, g_{\vec{k}} = \sum_{k' > k_f} V_{\vec{k}\vec{k}'}\, g_{\vec{k}'} \qquad (2)$$

$V_{kk'}$ are the matrix elements of the interaction potential.

$$V_{kk'} = \frac{1}{V\Omega} \int V(\vec{r})\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, d^3\vec{r}$$

$\vec{r}$ is the distance between two electrons. $V_{kk'}$ describes scattering $(\vec{k}, -\vec{k}') \to (\vec{k}, -\vec{k})$. solve for $g(\vec{k}$ such that total energy $E < 2\varepsilon_F$ (a bound pair to exist.)

$$V_{kk'} = -V \quad \text{for } \varepsilon_F \le \varepsilon_{k,k'} \le \varepsilon_F + \hbar\omega_D$$

And this is plugged into the Schrodinger equation, so this is e minus 2 EK and a GK equal to K prime greater than K F, so you used a vkk Prime and a G K Prime. So, all these are k's are of course vectors. so ,VK k prime, are the matrix elements of the interaction potential and vkk prime is something that we have seen, just a while back, but just to keep our discussion simple, we'll take this as Coulomb term or rather, you can consider this VR to be arising from that kind of a interaction ,which is mediated by phonons. So, R is the R is the distance between two electrons, with momentum K and distance between two electrons and the VK k prime of course 8, so VK k prime describes scattering of K prime, minus K prime, 2 K and minus K .so, we are to solve for this is equation number to solve for G K ,such that, such that total energy, E is less than two EF is basically that's the condition for a bound pair to exist . it is very hard to do this calculation for a general potential. So, what you can do is that, a V KK prime ,can be taken as a minus V, for epsilon F to be smaller than equal to epsilon KK prime and smaller than equal to epsilon F plus H cross Omega D. So, this is as we have said earlier, that it will be overall negative, because mediated by phonons, of course that has to be satisfied, by the participating electrons, they have to choose the energy range, where it becomes negative, but suppose it becomes negative and that the K and K prime for both the electrons would have to lie between epsilon F to, epsilon F plus h cross Omega T, it's equal to zero, otherwise.

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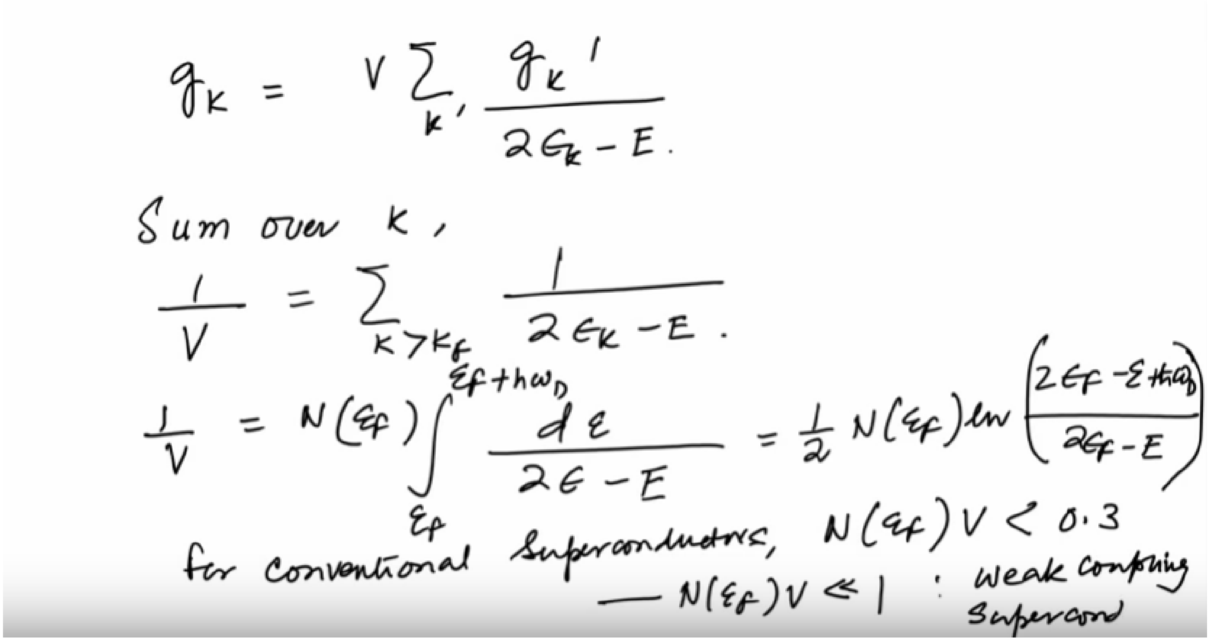

$$g_K = V \sum_{K'} \frac{g_{K'}}{2\epsilon_K - E}.$$

Sum over K,

$$\frac{1}{V} = \sum_{K > K_F} \frac{1}{2\epsilon_K - E}.$$

$$\frac{1}{V} = N(\epsilon_F)\int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} \frac{d\epsilon}{2\epsilon - E} = \frac{1}{2} N(\epsilon_F) \ln\left(\frac{2\epsilon_F - E + \hbar\omega_D}{2\epsilon_F - E}\right)$$

for conventional Superconductors, $N(\epsilon_F)V < 0.3$

— $N(\epsilon_F)V \ll 1$: Weak coupling Supercond

Now, with this, we have GK, if you put it there, it becomes equal to a K Prime and G K Prime and two epsilon K minus e. Now, sum over K, so it becomes one over V, this V is the strength of the interaction and we just reiterate that, this V could be infinitesimally small, a super conductivity will still occur. So, this is equal to sum over K and greater than KF,1 by 2 epsilon K minus e. Now. We can replace the summation by integration and we had to just invoke the density of states. But as we have told several times .that the super conductivity is a phenomenon that happens very near to the Fermi surface. So, while we convert the sum into an integral, we don't need to take, the energy dependence of the density of states, rather we can take the density of states at the Fermi level. So, this is equal to n epsilon F and the integration has to be done, not over all energies. but ,what is the relevant scale is this and then it is 2 epsilon minus e, this epsilon is a running variable, this capital epsilon is the two particle energy, its n EF once we do the integration ,it becomes log of 2 epsilon F minus epsilon plus h cross Omega D and we have 2 epsilon f minus e .so, that's the equation that one gets and now, for conventional superconductors, n epsilon F into V ,is usually of the order of 0.3 or less than 0.3. So, this is what is meant by the weak coupling approximation which demands, that n epsilon F, V has to be less than 1 or, or maybe much lesser than 1. So, this is weak coupling superconductors.

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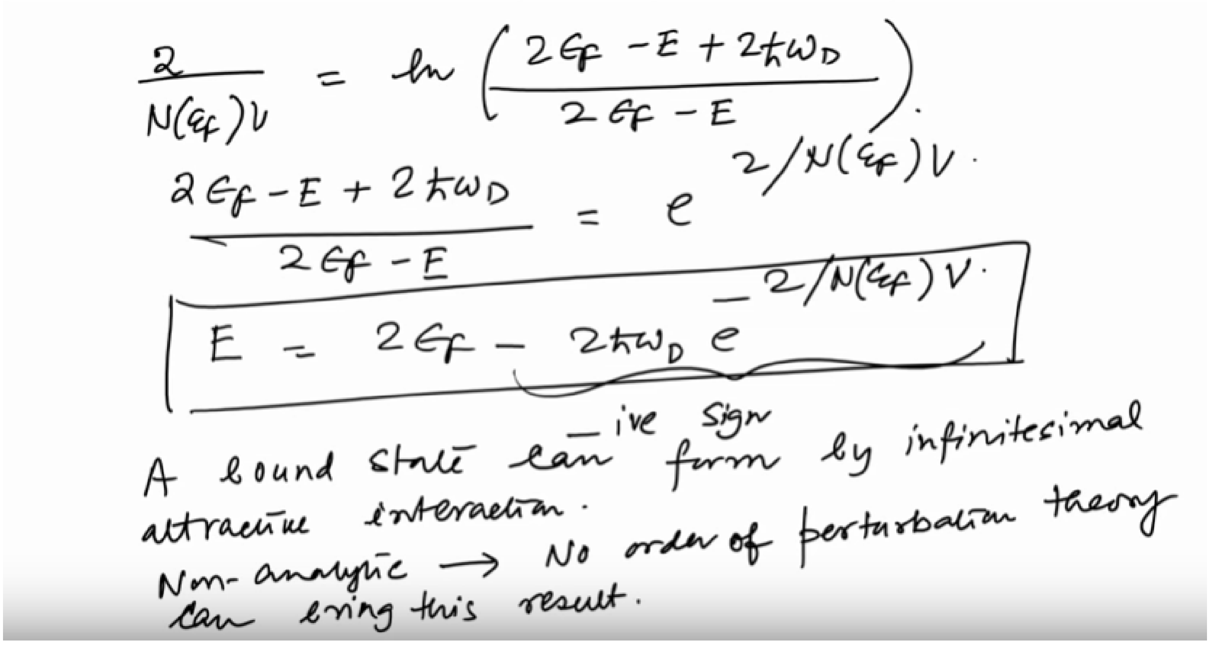

$$\frac{2}{N(\epsilon_F)V} = \ln\left(\frac{2\epsilon_F - E + 2\hbar\omega_D}{2\epsilon_F - E}\right).$$

$$\frac{2\epsilon_F - E + 2\hbar\omega_D}{2\epsilon_F - E} = e^{2/N(\epsilon_F)V}.$$

$$\boxed{E = 2\epsilon_F \underbrace{- 2\hbar\omega_D\, e^{-2/N(\epsilon_F)V}}_{-\text{ive sign}}}$$

A bound state can form by infinitesimal attractive interaction.

Non-analytic → No order of perturbation theory can bring this result.

So then, we have 2 divided by n e F into V ,equal to log of 2 epsilon F - e + 2 H cross Omega D ,divided by 2 epsilon F minus E and now, if I want to free this log, we have a to epsilon F - e + 2 H cross Omega D ,divided by 2 epsilon F minus E and this has to be exponential 2 divided by n EF into V and if you solve for e ,it becomes equal to so I can, I can multiply this here and just doing one more step, I would land up with this 2 H cross Omega D exponential minus 2 by n EF into V. So, this is the energy of, total energy of the two particle system and so, this is with respect to , if we take it with respect to the Fermi surface of the two electrons, this is comes with a negative sign and this negative sign ,talks about that, they have a bound ,they are bound , they are in bound state .so, a bound state can form off infinitesimally, this is not committed to the magnitude of V, any magnitude of V would do. So, by an infinitesimal attractive interaction, also you see, that the binding energy is non analytic, it comes as an exponential and hence no orders of perturbation theory, can bring this, this result. our perturbation theory in orders or in terms of V, the strength of the attractive interaction between the electrons, just one more thing that we need to do here, is that so, for the wave function,

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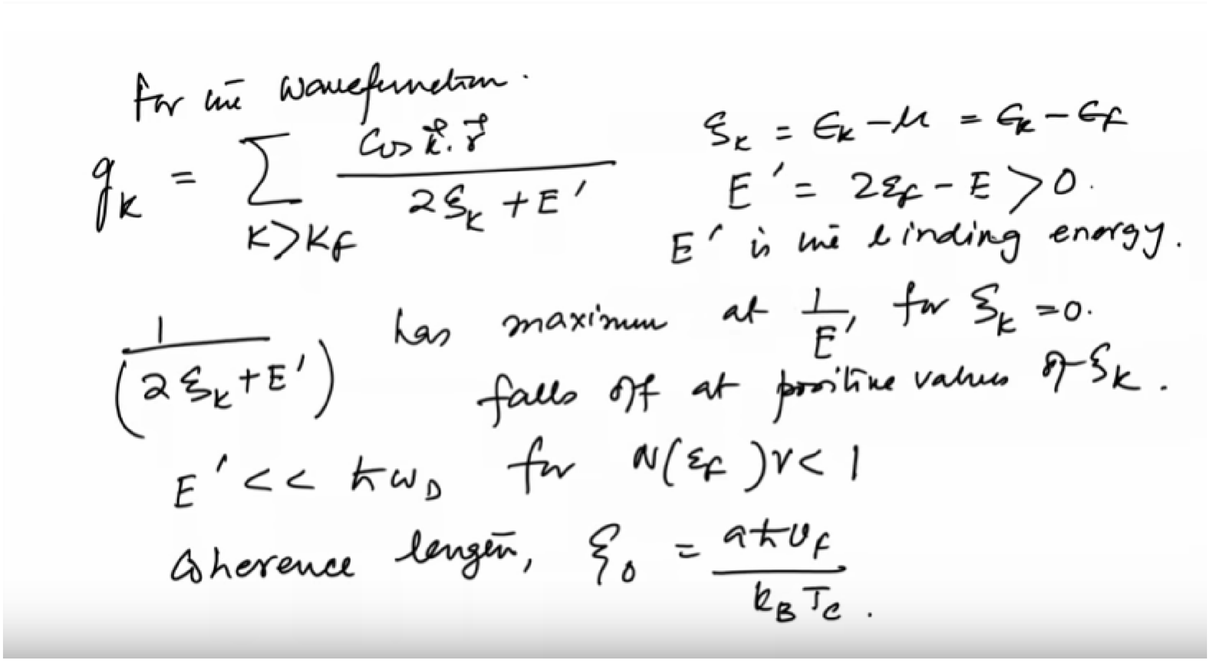

for the wavefunction.

$$g_k = \sum_{k>k_f} \frac{\cos \vec{k}\cdot\vec{r}}{2\xi_k + E'}$$

$\xi_k = \epsilon_k - \mu = \epsilon_k - \epsilon_f$

$E' = 2\epsilon_f - E > 0$.

E' is the binding energy.

$\frac{1}{(2\xi_k + E')}$ has maximum at $\frac{1}{E'}$, for $\xi_k = 0$.

falls off at positive values of ξ_k.

$E' << \hbar\omega_D$ for $N(\epsilon_f)V < 1$

Coherence length, $\xi_0 = \frac{a\hbar v_f}{k_B T_c}$.

we have, the amplitude of the wave function which is GK, this is equal to a cosine K dot R, divided by 2 Zi k plus e prime, where as Zi k equal to epsilon K minus mu and which is equal to epsilon K minus epsilon F at T equal to 0 and E prime equal to 2 epsilon F minus E ,which is of course, greater than 0. So, e Prime can now be called as, binding energy .and this K has to be greater than KF, just a few comments, this weighting factor ,which is given by 1 divided by Zi K Plus, has a maximum at, at one over e prime for Zi k equal to zero. which are valid or which are applicable for electrons at the Fermi surface and it falls off at positive values of zi K. so, if you note that ,the e prime The Binding energy is much smaller than ,the H cross Omega D ,for n epsilon F to be into v to be less than one. so this makes sure that the detailed behaviour of V KK prime is not important, also the small energy range, allows the estimation of the range of the Cooper pairs ,which we know is called as the coherence length and that coherence length, is given by ,this is what has been told earlier and this is of the order of 1 in BCS theory it's sort of the order of 0.8 and that gives a good estimation of the coherence length of the Cooper pair, wave function and that matches with the experimental values which are of the order of maybe 3 to 5000 angstrom. So, this is a theory ,which gave, the calculation had given rise to a gap ,the main features of the gap is that, it is valid for any strength of the electron-electron interaction, so long is attractive, it is a non perturbative, the result is non perturbative and hence you cannot do a perturbation theory of any order in order to get this result and also it allows the estimation of the, the extent of the Cooper pair wave function, which is called as the coherence length.