Advanced Quantum Mechanics with Applications
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Lecture – 11
Spin Angular Momentum, Addition of Angular Momentum, Clebsch gordan coefficients

So, today we well talk about Spin Angular Momentum. We have talked about the angular momentum in general, but however, for our purpose the discussion on spin angular momentum is required. And we will also talk about the total angular momentum, their algebra, the commutation relations and various things that are related to this and we have already learnt that the generators of rotations are the angular momentum.

So, a system having a rotational symmetry that is it is invariant under rotational transformation in space has its angular momentum conserved and the corresponding quantum numbers are good quantum numbers for the problem. So, let us look at the spin angular momentum. So, all these all the elementary particles that we know they possess an internal degree of freedom which behaves as angular momentum and termed as the spin or the spin angular momentum S.

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Spin Angular Momentum

All elementary particles possess an internal degree of freedom which behaves as angular momentum and termed as 'spin' or spin angular momentum, $\vec{S}$.

Commutation relations : $\vec{S}\times\vec{S} = i\hbar\vec{S} \Rightarrow \begin{cases} [S_x, S_y] = i\hbar S_z \\ [S_y, S_z] = i\hbar S_x \\ [S_z, S_x] = i\hbar S_y \end{cases}$

In a compact form,
$[S_i, S_j] = i\hbar\, \epsilon_{ijk} S_k$ $\qquad \epsilon_{ijk}$ = Levi civita symbol

$\epsilon_{ijk} = 1$: on cyclic reshuffle of indices
$= -1$: on non-cyclic reshuffle of indices
$= 0$: on having more than one index identical

The computation relations are written as S cross S equal to i h cross S. This is the commutation relation in shorthand notation. This if you, break it up into components it will look like the commutation of S x S y equal to i h cross S z S y S z equal to i h cross S x and S z S x equal to i h cross S y and it's a cyclic permutation of that. So, it in a very compact fashion it can be written as the commutator of S i S j, where i and j are components of the spin angular momentum is equal to i h cross and epsilon i j k S k..

Epsilon i j k is called as a Levi Civita tensor or it is also called as a Levi Civita symbol which has a value equal to 1 on cyclic reshuffle of indices and it has a value minus 1 on non-cyclic reshuffle of indices and it is equal to 0 on having more than one index being identical. So, if you have i i k that will be equal to 0 which means equal to that, if you want to take the commutation between S x S x that will be 0 for obvious reasons. And, if you break the cyclic symmetry that is if you right it as instead of i j k you write it as j i k then it will pick up a minus sign.

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Similar to the orbital angular momentum $\vec{L}$, one can choose $\vec{S}^2$ $(= S_x^2 + S_y^2 + S_z^2)$ and S_z as the two preferred operators.

Since $[\vec{S}^2, S_z] = 0$: They have common eigenfunctions: $|\chi_{s m_s}\rangle$

$$\vec{S}^2 |\chi_{s m_s}\rangle = s(s+1)\hbar^2 |\chi_{s m_s}\rangle \; ; \; S_z |\chi_{s m_s}\rangle = m_s \hbar |\chi_{s m_s}\rangle$$

These relations are all similar to orbital angular momentum, $\vec{L}$ with the difference that, l, m_l, the quantum numbers for $\vec{L}^2$ and L_z are all integers. Whereas for S, they can be integers (e.g. 0, 1, 2 . . .) for Bosons and half integers (e.g. $\frac{1}{2}, \frac{3}{2}$ etc) for Fermions.

As earlier, m_s will have values

$$-s, -s+1 \; \ldots\ldots \; 0 \; \ldots\ldots \; s-1, s \quad (2s+1 \text{ distinct values of } m_s)$$

So, similar to this orbital angular momentum which we have learnt we can also choose this S square which is equal to the sum of the squares of its individual components. And, S z just like l square and l z they are the two preferred operators and we can write down the angular momentum spin angular momentum in a representation in which its diagonal in S z and S square.

And of course, S square commutes with any of the components of the angular momentum S square and S z that is equal to 0. So, they have common eigenfunctions and let us write the common eigenfunctions as chi of S and m s these are the two quantum numbers that describe the problem of spin or spin angular momentum. S is so, they are defined in the following fashion. So, the eigenvalue of S square acting on chi s m s is equal to S into S plus 1 h cross square and returns back the eigenfunction.

And, similarly S z acting on the eigenfunction will give us m s h cross and a ket chi S m s. These relations are very similar to the orbital angular momentum that we have seen, excepting that there we have written it as l and ml. However, one thing that you need to notice that there is there no dependence on the space coordinates for this chi. While, the angular momentum the eigenfunctions of the angular momentum they are the y length functions of the spherical harmonics which depend upon the angular variables theta and phi. These are simply numbers or so, they are simply quantum numbers which are which denote the eigenfunctions of S square and S z.

So, there are of course, other differences as well because l and m l both were integers is like 0, 1, 2 etcetera. However, for S these are integer for Bosons such as 0, 1, 2 etcetera and half integers such as half three half five half etcetera for fermions. So, where l and m l are necessarily integers, these could be integers or maybe half integers for namely for a Bosons and fermions respectively. And of course, m s as earlier will have values which are minus S to minus S plus 1 through 0 to S minus 1 and S. So, there are 2 S plus 1 distinct values of m s exactly like that we have for ml the orbital angular momentum.

So, in many aspects they are similar, the algebra is very similar. However, we still need to go through some of the algebra in order to make our self familiar and comfortable with this analysis of the spin angular momentum.

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The orthonormality condition of the eigenfunction is written as,

$$\langle \chi_{s'm_s'} | \chi_{s m_s} \rangle = \delta_{ss'} \, \delta_{m_s m_s'}$$

The basis of $\vec{S}^2$ and components of $\vec{S}$ (for us S_z is important) has dimension $(2s\times1) \times (2s+1)$.

For spin $S=1$

Use relations (Proofs in Tutorial) for raising and lowering operators (Borrowed from relations for $\vec{L}$)

$$(S_+)_{s'm_s', sm_s} = \langle s'm_s' | S_+ | s m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)} \, \delta_{ss'} \, \delta_{m_s', m_s+1}$$

$$(S_-)_{s'm_s', sm_s} = \langle s'm_s' | S_- | s m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)} \, \delta_{ss'} \, \delta_{m_s', m_s-1}$$

and $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$

The orthonormality condition of the eigenfunctions is written as this S prime m s prime and chi s m s is equal to chi s prime and m s m s prime, which means that both the indices have to be same in order to give 1. And, if any of the indices either S or m s if they are different than that gives 0 because, of the orthonormality relation and the basis of S square and the component of S that is S x S y S z. However, as we have said that its only S z that is important, all these operators have a dimension 2 S cross 1 into 2 S cross 1 because, of the m s value taking values m s taking values from minus S to plus S through 0.

So, there are so, this the size of the vector space for each one of the components of S or the S square. And, let us write down the for spin S equal to 1 and maybe we shall look at this in a tutorial for proofs. And so, these are borrowed from relationships for l and S plus which is the spin raising operator, it has the matrix elements of that between S prime m s prime and s m s states are given by this h cross root over S into S plus 1 minus m s into m s plus 1 delta ss prime and delta m s prime m s plus 1.

So, this is an off-diagonal opera[tors]- or rather these are the off-diagonal matrix elements and is very easy to understand that they cannot be diagonal because, if they are diagonal then s m s would have been an eigenfunction of S plus or S minus which it is not. Because, we are writing S plus is as we have seen earlier that for l plus and l minus they are written in terms of the linear combinations of S x and S y in this particular fashion. And, neither S x nor S y are have eigenfunctions s m s are not ei[gen]- I mean s m s none of the s m s are eigen functions of this S x and S y, hence they are not eigenfunctions of S plus and S minus.

And, hence we will have off-diagonal terms in the matrix elements as it is given by this, similarly for the S prime sorry S minus which is a spin lowering operator that has a change in this term where it is m s plus 1 whereas, it is m s minus 1 and so on. So, this also connects the m s prime to m s minus 1 the later one alright. So, let us look at some of the spin operators in the matrix form; so, that all these things that we have just seen the matrix element written in terms of indices general indices can be verified.

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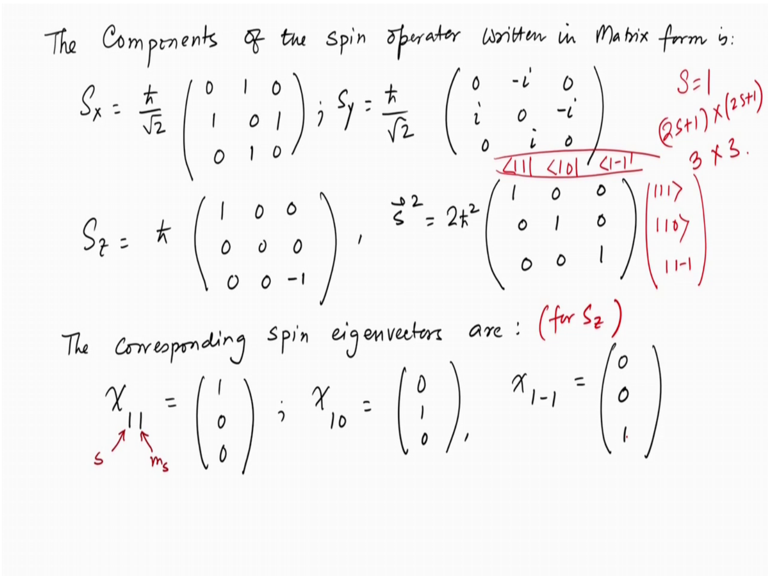

So, we are writing it for a S equal to 1 and because S equal to 1 we have a 2 S plus 1 into 2 S plus 1 that is the size of the matrix. So, S equal to 1 means is 2 S plus 1 is 3. So, we have a 3 cross 3 matrix as you can see here. So, S x is equal to h cross by root 2 and 0 1 0 1 0 1 and 0 1 0. And similarly, S y has got a similar form with it is of course, imaginary because of this relationship that we have seen. It is 0 minus y minus i 0 and i 0 minus i and 0 i 0 whereas, S z as we know it has to be diagonal. So, it is equal to 1 0 minus 1 as the diagonal entries, all the off-diagonal entries are 0 and similarly the S square is also diagonal in the s m s basis. So, and this is equal to 1 0 0 1 1 1 0 and a 1 0 0 1 and if you want to know I will show it for one particular.

So, these are the s m s values that you need to write down. So, S is equal to of course, 1 and m s is equal to 1 so, this is S m s. So, this is like writing like a conjugate and it is a S this and then this is equal to 1 1 minus 1 and I will write it the ket of that its 1 1 and 1 0 and 1 minus 1. And, if you take the matrix elements according to this relations that we have seen here then of course, you will get this all these things. And, if we so, the corresponding spin eigenvectors for say S z is given by its a 1 0 0. So, these are for S z and this is equal to 1 0 0 0 1 0 and it is a 0 0 minus 1. And so, this is actually same for everything or rather it is for S z, you can check for others what are the spin eigenvectors. First calculate the eigenvalues and then put them into the equation to calculate the eigenvectors alright.

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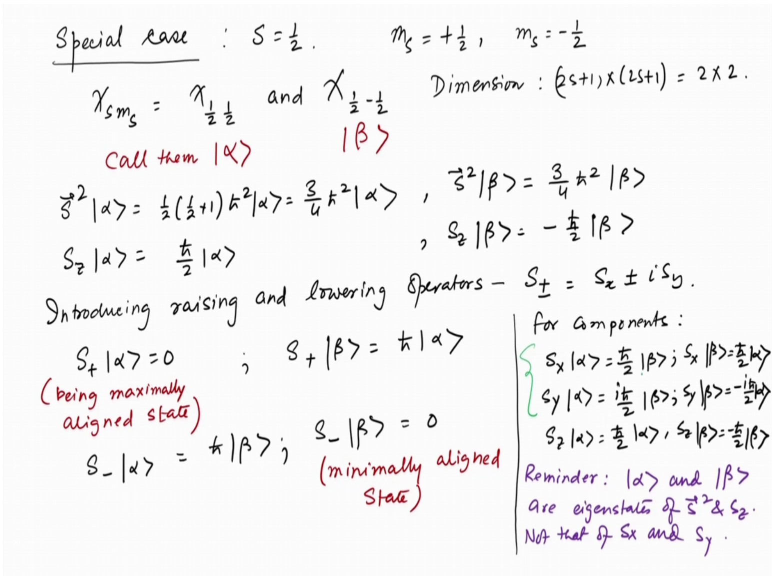

So, let us go to the special case of S equal to half which is of interest because we usually talk about fermions, who which have the spin angular momentum having a value half. So, the eigenfunctions are chi half and chi half minus half because, these are for S equal to half we can have the m s value is equal to plus half and minus half. So, of course, here again the dimension of a vector space as we have seen earlier x equal to are 2 S plus 1 cross 2 S plus 1 and for S equal to half it is equal to 2 cross 2 and these are the two eigenfunctions calling it as alpha and beta.

So, S square acting on alpha which is half is half into half plus 1 h cross square which is 3 by 4 h cross square alpha. S square acting on beta is 3 by 4 h cross square beta and S z acting on alpha will give h cross by 2 alpha; S z acting on beta will give minus h cross by 2 beta. Again, remember that h alpha and beta are only eigenfunctions of S square and S z they are not eigenfunctions of S x S y or S plus S minus. Now of course, this is the maximally aligned state because if S is equal to half m s the maximum value of m s is equal to half. So, alpha is called as a maximally align state.

If you operate S plus on the maximally allowed maximally aligned state pardon me that will give a 0 because, there is no way that you can raise the m s value any farther. And, S plus is known to raise the m s values. Similarly, the S plus acting on beta will of course, raise the m s value from minus half to plus half. So, it will give an alpha and of course, we will give the coefficient which is according to this relation that we have shown here. And similarly, for a S minus acting on alpha will give a h cross times beta and S minus acting on beta will give a 0 because, this is a minimally aligned state because this is equal to half and minus half.

So, m s value the minimum value of m s is equal to minus half. So, you cannot have anything lower than that. So, if S minus which is a spin lowering operator acting on a state which has a m s equal to minus half then of course, it will be it will give 0. And similarly, you can work out this is something very simple for you to workout is S x acting on alpha will give a h cross by 2 beta. S x acting on beta will give a h cross by 2 alpha and S y acting on alpha will give i h cross by 2 beta and S y acting on beta will give a minus i h cross by 2 alpha. S z acting on alpha will give h cross by 2 alpha, S z acting on beta will give minus h cross by 2 beta.

So, remember that once again repeating the same statement is that alpha and beta are eigenstates of S square and S z, but not of S x and S y. And, that is why these two relations adequately make it clear that that they are not eigenfunctions because, S x acting on alpha does not return alpha, but it returns beta. So, let us now define a set of matrices which are quite important in the study of quantum mechanics and sure condensed matter physics. And, every other subject that we come across these are called as a Pauli spin matrices which are simply related to the S matrix that we have just studied by a factor 2 by h cross.

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Pauli spin matrices

Define $\vec{\sigma} = \frac{2}{\hbar}\vec{S}$ (or $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$)

$[\sigma_x, \sigma_y] = 2i\sigma_z$; $[\sigma_y, \sigma_z] = 2i\sigma_x$; $[\sigma_z, \sigma_x] = 2i\sigma_y$.

Also, $\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 1$

Further they anticommute,

$$[\sigma_i, \sigma_j]_+ = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \qquad (ij) \in (x, y, z).$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\sigma_x, \sigma_y, \sigma_z$ and $\mathbb{1}$ can represent any 2×2 matrix as a linear combination

$\boxed{[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z + \delta\mathbb{1}$

$\alpha, \beta, \gamma, \delta$ are coefficients.

(1) σ_i are traceless and determinant $= -1$

(2) In this representation σ_z is diagonal.

$\boxed{\text{All have eigenvalues } +1, -1}$

So, if I multiply the spin operators that we are just learned first spin equal to half by 2 by h cross or write S as h cross by 2 and sigma. So, these sigmas are called as a Pauli matrices. In fact, there is let me write down that sigma x which is the x component of the Pauli matrix, sigma y sigma z and a 2 by 2 identity matrix can represent any 2 by 2 matrix as a linear combination and this can be proved. Let me tell this statement a little more clearly so, that you comfortable with this. So, let us take any 2 by 2 matrix such as a b c d, where a b c d are arbitrary numbers and this is equal to some alpha sigma x plus a beta sigma y plus a gamma sigma z plus a delta i by suitable choices of alpha, beta, gamma and delta.

So alpha, beta, gamma and delta are simply coefficients. So, what about the commutation relations of sigma x and sigma y, sigma x and sigma y have the commutation relations which is 2 i sigma x sigma y commutator equal to 2 i sigma z. And, similarly y z is 2 i sigma x sigma z sigma x is 2 i sigma y and which is basically again that you one can write it using the Levi Civita symbol that is sigma i sigma j equal to 2 i epsilon i j k sigma k. So, and they also have an anti commutation relation which are often very important, that is sigma i sigma j anti commutator which is written as the commutator is written with a minus sign in between the two terms whereas, this is written with a plus sign it is a sigma i sigma j plus sigma j sigma i equal to 2 delta ij, where ij is belong to the xyz components which are shown here.

And, we can write down the sigma x as a 0 1 1 0 sigma y as 0 minus i i 0, sigma z is 1 0 0 minus 1. See again we are interested in the in basis or a representation in which sigma z is diagonal and it is shown as diagonal. A very interesting similarity between these 3 matrices they are all traceless, that is the sum of the diagonal elements are 0. So, it is 0 plus 0 and 0 plus 0 and it is 1 plus minus 1. So, they are traceless that is why they called traceless and the determinant we calculate, that is equal to 1 which is sorry which is equal to minus 1 and of course, as I said that in this representation all the sigma z is diagonal.

And all sigma all of sigma x, sigma y and sigma z have eigenvalues a plus 1 and minus 1 ok. So, little bit of algebra if you do it reveals that the eigenvectors are for sigma x are written as a 1 1 and 1 minus 1, corresponding to this eigenvalue that is here lambda equal to plus 1 and lambda equal to minus 1 respectively. Sigma y equal to 1 i and 1 minus i corresponding to lambda equal to 1 and lambda equal to minus 1 and sigma z has 2 eigenvectors corresponding to lambda equal to 1 and lambda equal to minus 1 as 1 0 0 1 respectively. It may be given as an exercise maybe, in the tutorial that for any two arbitrary vectors A and B it can be proved that a sigma dot a that sigma is the sigma xx cap sigma yy cap.

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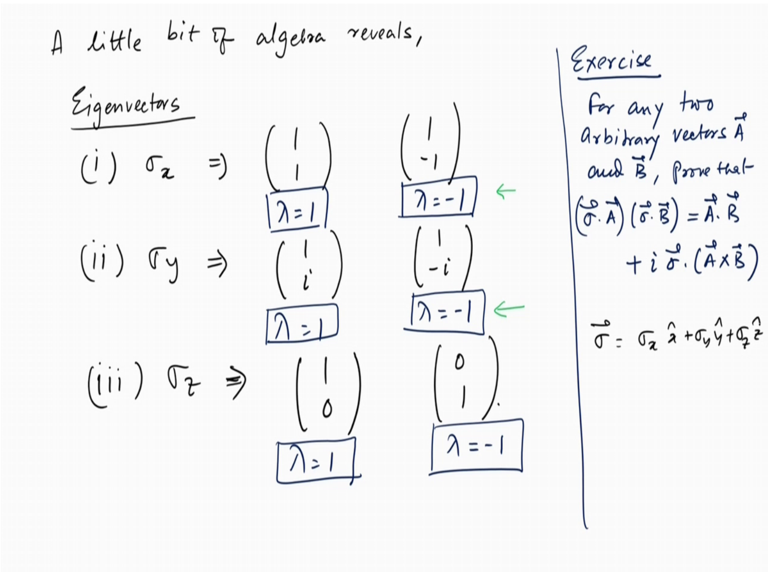

So, this is let me write this here. So, sigma equal to sigma x x cap plus sigma y y cap plus sigma z z cap and of course, A has this components A xx cap A yy cap and A zz cap, it can be written as sigma dot A multiplied by sigma dot B which is equal to A dot B and i sigma dot A cross B. Now, move onto the total angular momentum and now this total angular momentum is important in a context that I will not immediately discuss, but whenever there is spin orbit coupling of the form, L dot S neither L nor S are good quantum numbers for the reason that L dot x can be written as L x S x plus L y S y and L z S z.

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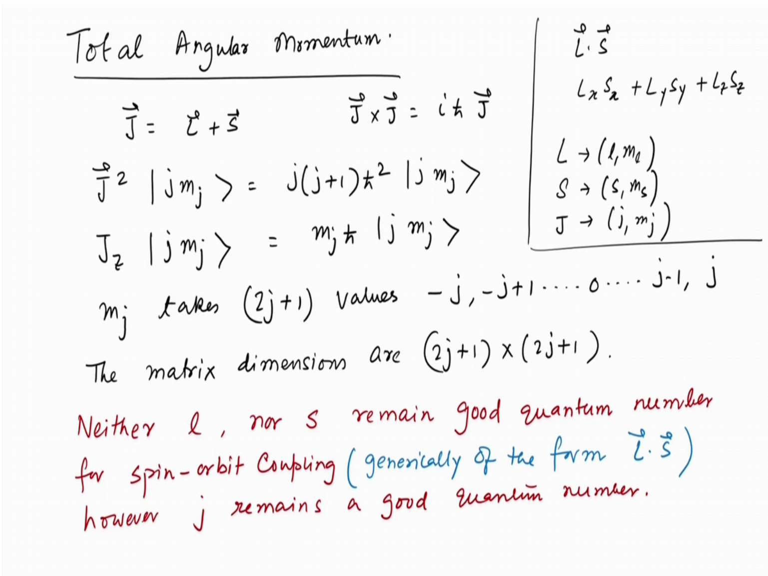

Thus, since the each of these individual components do not commute neither L nor S will remain as good quantum numbers, but luckily the J which is the vector sum of L plus S that remains as a good quantum number. And, again the commutation relations are written as J cross J equal to i h cross j, a J square j m j. Now, I am changing my symbol from s m s to j m j which is equal to j into j plus 1 h cross square j m j. Same relations accepting that for L we have written it as l m l, for S we have written it as s m s these are quantum numbers and for J we are writing it as j m j.

So, J square acting on j m j is will give you j into j plus 1 h cross square and returns back the state j m j and the J J z acting on j m j will give a m j h cross j m j. So, m j takes values 2 j plus 1 2 j plus 1 values minus j 2 j and again the matrix dimensions has earlier is that it is 2 j plus 1 into 2 j plus 1. So, as I said earlier the neither l nor S remain a good quantum number in the spin presence of a spin orbit coupling. So, j is a good quantum number and the eigenfunctions of such a Hamiltonian can be represented in terms of j m j.

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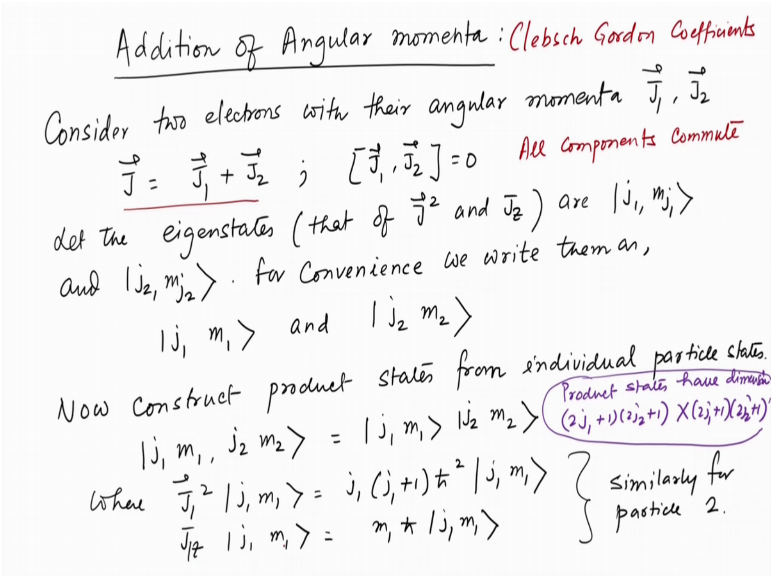

Now, we are going to talk about addition of angular momentum and very importantly something called as Clebsch Gordan coefficients. Now, these are quite important in a various branches of physics mainly quantum mechanics and atomic and molecular physics etcetera in even a nuclear physics. So, I am having a good understanding of Clebsch Gordan coefficients is a quite helpful. Let us see what they are, but before that let us do a little bit of algebra to familiarize ourselves with j vector.

Now, we talking about two electrons with their total angular momentum as J 1 and J 2. So, the total angular momentum is equal to J 1 plus J 2 so, which is here. So, this is the total angular momentum all components of J 1 and commute with all other components of J 2 ok. So, they have all components commuting and let the eigenstates for each one of them be taken as j 1 m j 1 and j 2 m j 2. So, just we wish to discard this j index or suffix with m. So, we simply write it as j 1 m 1 and j 2 m 2 just shorthand notation for that, they mean the same thing.

So, now we can construct product states from individual particles states. So, it is a j 1 m 1 and j 2 m 2 which are products of j 1 m 1 and j 2 m 2 just what we said, but we will write them as just like within a single ket. The products says of course, of dimensions 2 j plus 1 into 2 j 2 2 j 1 plus 1 multiplied by 2 j 2 plus 1 and then 2 j 1 plus 1 multiplied by

2 j 2 plus 1 that is the size of the vector space. And obviously, the J 1 square will act on only j 1 m 1 not on j 2 m 2, will give us j 1 j 1 plus 1 j 1 into j 1 plus 1 h cross square j 1 m 1. And similarly, J 1 z will give j 1 m 1 and j acting on j 1 m 1 will give m 1 h cross j 1 m 1. And similarly, you can just simply change the index or the suffix from rather subscript from 1 to 2 and can write it for the particle 2, same relationships will hold alright.

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Let us now work out the operations of the total angular momentum operator, $\vec{J}^2$ and J_z on the product state.

$$J_z|j_1 m_1, j_2 m_2\rangle = (J_{1z} + J_{2z})|j_1 m_1, j_2 m_2\rangle$$
$$= (m_1 + m_2)\hbar|j_1 m_1, j_2 m_2\rangle$$

$\boxed{\text{Thus simultaneous eigenfunctions of } \vec{J}_1^{\,2},\ \vec{J}_2^{\,2},\ J_{1z} \text{ and } J_{2z} \text{ are also eigenfunctions of } J_z.}$

What about $\vec{J}^2$?

Now, let us work out the operations of the total angular momentum operator J square and J z, which are the for the composite system the square of the total angular momentum and the of course, there will not be any vector on J z it simply J z ok; no vector on that on the product sets. So, J z acting on j 1 m 1 j 2 m 2 will give me a J z can be written as a J 1 z plus J 2 z, they will act on each one of those. And so, it is m 1 plus m 2 h cross j 1 m 1 j 2 m 2. Thus, simultaneous eigenfunctions of J 1 square J 2 square J 1 z and J 2 z are also eigenfunctions of J z.

So, at least one thing we have been able to settle is that a for the total angular z component of the total angular momentum, that is J z has same eigenfunctions as each one of those J 1 square J 2 square J 1 z and J 2 z. Now, what about J square? What are the eigenfunctions? Do they have the same eigenfunctions at J 1 square J 2 square J 1 z and J 2 z? J square is written as J 1 plus J 2 whole square and it is written as J 1 square plus J 2 square and J 1 J 2 plus a J 2 J 1. Now, since every component of J 1 commutes with every other component of J 2 we can combine this to write as 2 J 1 J 2. So, so the J square contains a twice of J 1 dot J 2.

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$$\vec{J}^2 = \left(\vec{J}_1 + \vec{J}_2\right)^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + \underbrace{\vec{J}_1\cdot\vec{J}_2 + \vec{J}_2\cdot\vec{J}_1}_{=\,2\vec{J}_1\cdot\vec{J}_2}$$

(because every component of $\vec{J}_1$ commutes with every component of $\vec{J}_2$)

Now $\vec{J}_1\cdot\vec{J}_2 = J_{1x}J_{2x} + J_{1y}J_{2y} + J_{1z}J_{2z}$

Since $\left[J_{1x/y}, J_{1z}\right] \neq 0$

So, $\vec{J}^2$ does not commute with J_{1z} or J_{2z}

Thus simultaneous eigenfunctions of $\vec{J}^2$ and J_z are eigenfunction of $\vec{J}_1^{\,2}$ and $\vec{J}_2^{\,2}$, but not of J_{1z} or J_{2z}.

Thus there exists two distinct descriptions of the system:

(i) In terms of eigenfunctions of $\boxed{\vec{J}_1^{\,2}, \vec{J}_2^{\,2}, \vec{J}^2 \text{ and } J_z}$ (i)

(ii) In terms of eigenfunctions of $\boxed{\vec{J}_1^{\,2}, \vec{J}_2^{\,2}, J_{1z}, J_{2z}}$ (ii)

Now, J 1 dot J 2 is J 1 x J 2 x plus J 1 y J 2 y plus a J 1 z J 2 z. Now, since the J 1 x and y they do not commute with J 1 z so, J square as it contains J 1 dot J 2 does not commute with J 1 z or J 2 z ok. These are very important result because, we just found that J z commutes with J 1 square J 2 square J 1 z and J 2 z. However, J 2 I mean the whole J square does not commute with these. So, there is a simultaneous eigenfunctions of J square and J z are eigenfunctions of J 1 square and J 2 square, but not J 1 z and J 2 z; even though J z has same eigenfunctions as J 1 z and J 2 z, but J square does not have.

So, thus there exist two distinct descriptions of the system, in terms of the eigenfunctions of J 1 square J 2 square J square and J z and J 1 square J 2 square J 1 z and J 2 z ok. Now, this is important because if there are two such identical descriptions of a system then there has to be a unitary transformation or a relation that connects these two bases. So, the eigenvalues corresponding to 1, 1 is this let us this called this as 1 and this as 2 is written there of course, on the left hand side.

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The eigenvalues corresponding to (i) are

$$\boxed{j_1(j_1+1)\hbar^2,\quad j_2(j_2+1)\hbar^2,\quad j(j+1)\hbar^2,\quad m\hbar}$$

The eigenvalues corresponding to (ii) are

$$\boxed{j_1(j_1+1)\hbar^2,\quad j_2(j_2+1)\hbar^2,\quad m_1\hbar,\quad m_2\hbar}$$

Similarly the eigenfunctions corresponding to (i) $\boxed{|j_1, j_2, j, m\rangle}$

the eigenfunctions corresponding to (ii) $\boxed{|j_1, j_2, m_1, m_2\rangle}$

Since both define the complete vector space, there has to be an unitary transformation connecting the two bases.

The coefficient of the unitary transformation define Clebsch Gordon (CG) coefficients.

And this is equal to j 1 into j 1 plus 1 h cross square and j 2 into j 2 plus 1 h cross square and j into j plus 1 h cross square and m h cross. As the eigenvalue corresponding to 2 which are these J 1 square J 2 square J 1 z and J 2 z are j 1 into j 1 plus 1 h cross square and j 2 into j 2 plus 1 h cross square and m 1 h cross and m 2 h cross. So, the eigenfunctions corresponding to 1 are j 1 j 2 j m and the eigenfunctions corresponding to 2 are j 1 j 2 m 1 and m 2.

Since both define the complete vector space, there has to be a unitary transformation connecting the two bases. The coefficient of the unitary transformation are known as the Clebsch Gordan coefficients which we intend to compute for a given case. So, once again so, both define a complete vector space for a problem of j square and j z. So, now, there has to be a united transformation connecting the unitary transformation that connects the two bases, the coefficients corresponding to that are called as the Clebsch Gordan coefficients.

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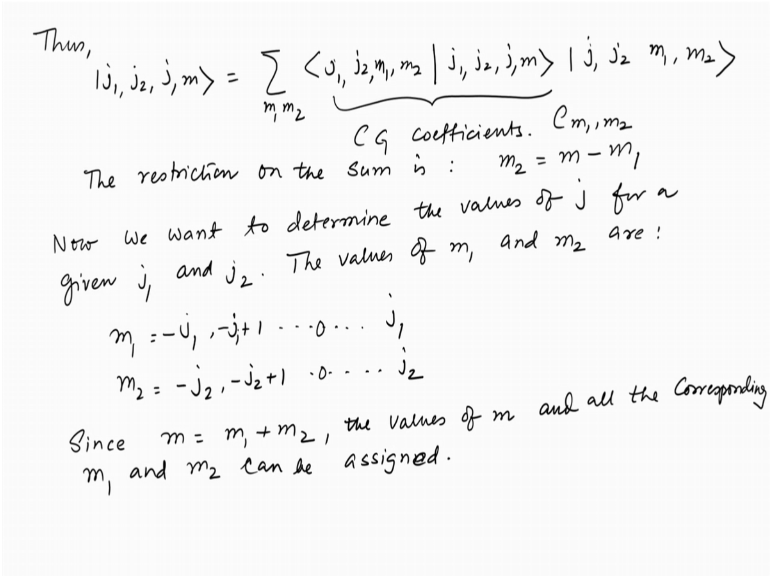

So, we are writing these states as j 1 j 2 j m and the j 1 j 2 m 1 m 2 and now the coefficients are the overlap of this j 1 j 2 m 1 m 2 and j 1 j 2 j m. The restriction on the sum is that that m 2 has to be, there is a sum over m 1 and m 2, the restriction on m 2 should be that is the total m minus m 1. So, we want to determine the values of j for a given j 1 and j 2 ok. The values of m 1 and m 2 of course, range from minus j 1 to plus j 1 through of course, 0 which we have not written here; there is a 0 here and there is a 0 here. Similarly, for m 2 it is minus j 2 2 plus j 2, since m equal to m 1 plus m 2 the values of m and the all the corresponding value m 1 values for m 1 and m 2 can now be assigned.

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Let us show it in a tabular form,

m	m_1	m_2	degeneracy of m
j_1+j_2	j_1	j_2	1
j_1+j_2-1	j_1-1 j_1	j_2 j_2-1	2
j_1+j_2-2	j_1-2 j_1-1 j_1	j_2 j_2-1 j_2-2	3
$\vdots$	$\vdots$		$\vdots$
$-(j_1+j_2)$	$-j_1$	$-j_2$	1

Let us look at the table, for m to be having the maximum value which is j 1 plus j 2 my m 1 can be j 1 and m 2 can be j 2 and that is the only way to achieve the maximum value of m. However, when we take the next to the maximum value that is j 1 plus j 2 minus 1, there are two possibilities that is either m 1 can be j 1 minus 1 and m 2 could be j 2 and also j m 1 can be j 1 and m 2 can be j 2 minus 1. So, these are the two possibilities that we are showing here and similarly for the third-one that is the next to that is j 1 plus j 2 minus 2. So, there is a j 1 minus 2 and j 1 minus 1 and j 1 and j 2 and j 2 minus 1 and a j 2 minus 2 it is a threefold degenerate.

So, we are writing down the degeneracy on the rightmost column. The first-one having a degeneracy 1, the second-one having a degeneracy 2's, third-one having a degeneracy 1 because, all combinations are possible and the lowest value of a m is minus of j 1 minus j 2. So, each of m 1 and m 2 can take values minus j 1 and m 2 can take value minus j 2 and again the degeneracy is equal to 1. So, these are the values that the total angular momentum can take for the composite system, that is m values and viz a viz be the m 1 and m 2 values which are the individual quantum numbers for the total angular momentum, they can take alright. So, now still that question which we have posed earlier remains that is how to figure out the allowed values of j for a given value of j 1 and j 2.

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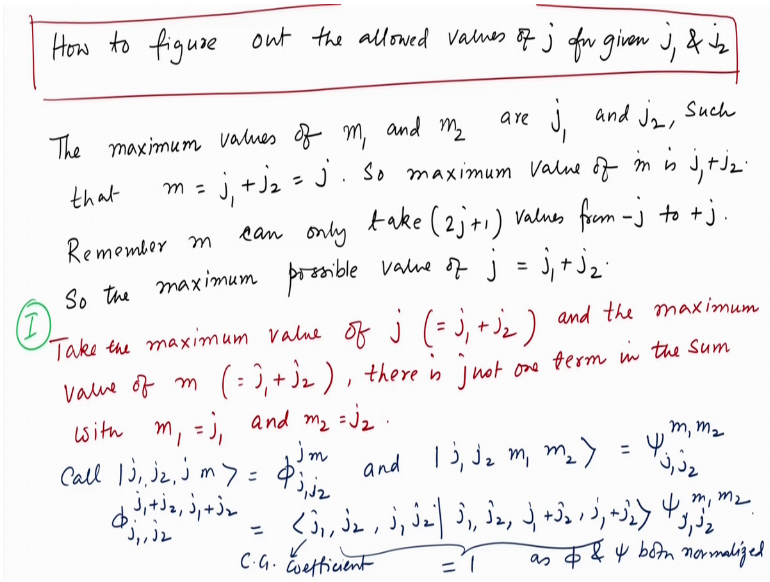

So, the maximum values of m 1 and m 2 are j 1 and j 2, such that m equal to j 1 plus j 2 equal to j. So, the maximum value of m is j 1 plus j 2. So, remember m can take only values which are from a 2 j plus 1 values from minus j 2 plus j. So, the maximum possible values of j is j 1 plus j 2. So, case 1: that is the top of the table that we have seen here. So, take the maximum value of j equal to j 1 plus j 2 and the maximum value of m which is equal to j 1 plus j 2, again there is just one term in the sum with m 1 equal to j 1 and m 2 equal to j 2. Go back to the sum that we have shown here for the Clebsch Gordan coefficients we need to sum over m 1 and m 2.

So, we need to know the precise values of m 1 and m 2 such that we can perform the sum. So, m 1 equal to j 1 and m 2 equal to j 2. And, now we introduce another shorthand notation that is j 1 j 2 j m as phi j 1 j 2 and in the superscript j m. And, similarly for the other bases we write psi of j 1 j 2 m 1 m 2. Thus, phi which is the one of the bases which is j 1 j 2 and for j equal to j 1 plus j 2 and m equal to j 1 plus j 2, again that is the which has the degeneracy equal to 1. So, there is nothing to you know sum up and this is written the Clebsch Gordan coefficients written as j 1 j 2 j 1 j 2 j 1 j 2 j 1 plus j 2 and j 1 plus j 2. And, then the psi the other bases and since both phi and psi are normalized this Clebsch Gordan coefficients must be equal to 1, it has to be normalized. So, this Clebsch Gordan coefficients equal to 1.

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(II) Let us now take same $j = j_1 + j_2$, but the next lower value of m. that is $m = j_1 + j_2 - 1$. This has 2 possibilities $(j_1 - 1, j_2)$ and $(j_1, j_2 - 1)$ for (m_1, m_2).

Thus $\phi^{j_1+j_2,\, j_1+j_2-1}_{j_1, j_2}$ should be linear combination of two linearly independent eigenfunctions, $\psi^{j_1, j_2-1}_{j_1, j_2}$ and $\psi^{j_1-1, j_2}_{j_1, j_2}$.

Moreover two such linear combinations appearing, one for $j = j_1 + j_2$ and the other for $m = j_1 + j_2 - 1$.

(III) Proceeding further to $m = j_1 + j_2 - 2$, we shall have 3 linearly independent sets; $(m_1, m_2) = (j_1 - 2, j_2); (j_1 - 1, j_2 - 1), (j_1, j_2 - 2)$ and so on....

Let us now take the same j which is j 1 plus j 2, but take the next lower value of m which is j 1 plus j 2 minus 1. This is the term in the second term in the table that we have just shown. So, in this particular case this has two possibilities namely, see here it is j 1 minus 1 and j 2 and also a j 1 and j 2 minus 1 for m 1 m 2. Thus, phi j 1 j 2 and j j 1 so, j 1 plus j 2 minus 1. So, this is equal to this should be equal to j 1 plus j 2, should be linear combination of two linearly independent eigenfunctions namely psi j 1 j 2 j 1 j 2 minus 1 and psi j 1 j 2 j 1 minus 1 and j 2.

Moreover two such linear combinations are appearing, one for j equal to j 1 plus j 2 and the other for m equal to j plus j 2 j 1 plus j 2 minus 1. Proceeding further to the next one, m equal to j 1 plus j 2 minus 2. So, we shall have 3 linearly independent sets for 3 for values of m 1 and m 2 which are j 1 minus 2 j 2 j 1 minus 1 j 2 minus 1 and j 1 and j 2 minus 2 and so on. So, this will go on.

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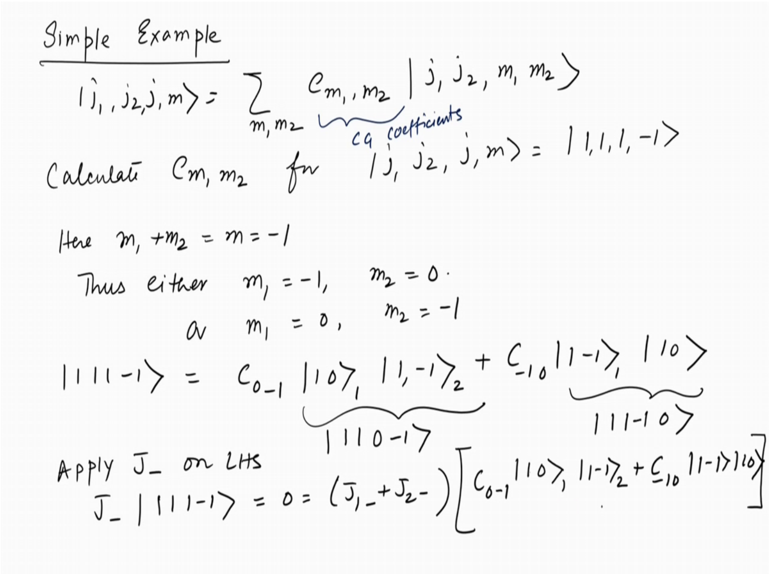

Let us take a simple example to calculate the Clebsch Gordan coefficients. Now, this is a quite a task to do it for complicated problem and which we want to avoid here, but we will still the essence still remains the same and we will show it for a particular case. So, once again to remind you that we will write down the kets as this where C m 1 m 2 are the CG coefficients. Let us have so, we have to calculate C m 1 m 2 for j 1 j 2 j m as 1 1 1 and minus 1, that is j 1 equal to 1, j 2 equal to 1 j equal to 1 and m equal to minus 1.

So, since total m is equal to minus 1 m 1 plus m 2 must be equal to minus 1. So, either m 1 equal to minus 1 m 2 equal to 0 or m 1 equal to 0 and m 2 equal to minus 1. So, I can write down this as C 0 minus 1 and 1 0 and this is for the first particle and this is 1 minus 1 is for the second particle. This is a shorthand notation for 1 1 0 minus 1 and similarly it is a C minus 1 0 1 minus 1 and the 1 0 ket which is a 1 1 minus 1 0.

Now, in order to calculate this a trick is required that is, if we since this is a minimally aligned state that is m has the value which is minus 1, if we apply a j minus 1 because your j is equal to 0. So, m is the minimal value of m is minus 1, if we apply j minus on this state will give me 0. So, j minus simply nothing, but j 1 minus plus j 2 minus which will act on the state and will give me 0.

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Remember

$$J_{\substack{1\\2}\pm} |j_{\substack{1\\2}}, m_{\substack{1\\2}}\rangle = \hbar\sqrt{j_{\substack{1\\2}}(j_{\substack{1\\2}}+1) - m_{\substack{1\\2}}(m_{\substack{1\\2}}\pm 1)}\; |j_{\substack{1\\2}}, m_{\substack{1\\2}} \pm 1\rangle.$$

$$\text{Thus, } J_- |1 1 1 -1\rangle = 0 = \sqrt{2}\,(\underbrace{C_{0-1} + C_{-10}}_{=0})\, \underbrace{|1 -1\rangle_1 |1, -1\rangle_2}_{|1 1 -1, -1\rangle}$$

$$\text{Thus } C_{0-1} = -C_{-10}$$

$$C_{0-1} = \frac{1}{\sqrt{2}} = -C_{-10}.$$

So, that will use those relations that we have learnt earlier that is a j plus minus acting on j 1 m 1 is equal to j 1 j 1 plus 1 and minus m 1 and m 1 plus 1 and or minus 1 depending on whether you are applying the raising operator or the lowering operator; here we are interested in the lowering operator. So, that gives a state which is 1 1 minus 1 minus 1 and because this is equal to 0 these kets are not equal to 0 and neither root 2 is.

So, we have this a bracket equal to 0. So, if this is equal to 0 then we have C 0 minus 1 equal to minus of C minus 1 0. And so, each one of them if you normalize it becomes equal to 1 over root 2 this is of course, one is a 1 over root 2, the other one is minus 1 over root 2. So, this is the way we have calculated the Clebsch Gordan coefficients for this state given by here alright.

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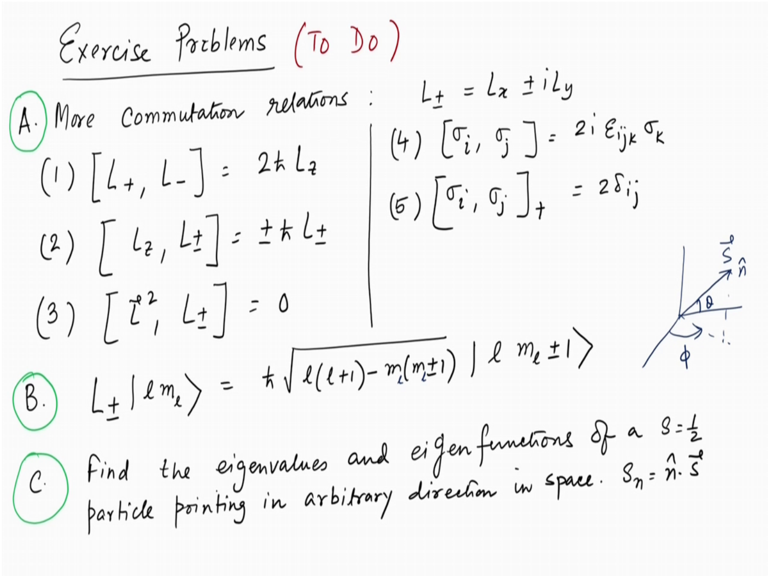

Let us go through some of the problems which you may see in the tutorial and these are exercise problems, nevertheless is important for me to just browse through them which are the relations that you need. The some commutation relations which you have seen in some form, I am writing them down again. It is a commutation between L plus L minus which gives a 2 h cross L z and L z and L plus minus gives plus minus h cross and L plus minus.

And, similarly L square and L plus minus commutator always give 0 because, L plus minus is nothing, but a linear combination of L x and L y and because, L square commutes with all components of L that is why this is equal to 0. And, similarly for the spin angular momentum sigma i sigma j equal to 2 i epsilon ijk sigma k and sigma i sigma j the anti commutator is equal to 2 delta i j. So, these are the problems that you should do and practice in order to have a familiarity with the angular momentum algebra.

So, L plus minus acting on l ml equal to a we have a we may not write this ml, but since we have written let me just add that ml here. L plus minus acting on l ml is h cross root over l into l plus 1 minus ml into ml plus 1 and l ml plus minus 1. Of course, as I said that these are not the eigenfunction so, one gets the different state after one operates L plus minus on l ml and let say the find the eigenvalues and eigenfunctions of a spin half particle pointing arbitrary direction in space. So, we have now so, far talked about z being the preferred direction in which case S z has always been diagonal.

Suppose the spin vector is pointing in some particular direction in space which is the n cap direction then what are the you know the so, these are the so, this phi and this is theta or so, this is a theta and so on. And so, this can be worked out and you can find out the forms for the Pauli matrices which you have learnt.

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D. Write down the matrices Corresponding to L_x, L_y and L_z for $l = 2$.

Ans. $$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}; \quad L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & -i\sqrt{6} & 0 & 0 \\ 0 & i\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{6} & 0 & -2i \\ 0 & 0 & 0 & 2i & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\text{Size} = (2l+1)\times(2l+1) = 5\times 5$$

Look at this problem that, write down the matrices corresponding to L x L y and L z for l equal to 2. So, l equal to 2 the size is equal to 2 l plus 1 cross 2 l plus 1 and this is equal to a 5 cross 5. So, we will have a 5 by 5 matrix which are written as L x is equal to h cross by 2 and these all these elements that are written there L y equal to h cross by 2 and all these elements. L z of course, is diagonal when you have l equal 2 the ml values are 2 1 0 minus 1 and minus 2. So, these are appearing at the diagonal entries and all the off-diagonal entries are 0 and of course, this is the eigenstate of L z. The basis is an Eigen basis of L z.

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E. Using a two dimensional rotation operator, $U_R(\theta)$, show the following group property,

$$U_R(\theta_1)\,U_R(\theta_2) = U_R(\theta_1+\theta_2)$$

Ans $U_R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$; $U_R(\theta_1)U_R(\theta_2) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \\ -\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = U_R(\theta_1+\theta_2)$$

Let us do a problem which is the two-dimensional rotation operator, we have looked at rotation operator in three-dimension which you know that, if you rotate a system or an object about the z axis then what rotation matrix is. We are simply talking about a rotation in the xy plane which has a form which is written here, for this U R of theta so, show the group property. So, basically U R forms a group and there are elements of the group and they have certain properties. So, this is one group property which is U R theta 1 and U R of theta 2 which is U R theta 1 plus theta 2.

And so, this called as a associative property. So, U R of theta is written as cos theta sin theta minus sin theta cos theta. So, U R of theta 1 into multiplied by U R of theta 2 is the product of these two matrices and this is equal to, if you do the simplification it comes out as this and this is equal to U R of theta 1 plus theta 2. So, that property can easily be proved.

And so, we stop here with I mean having told you most of the relevant things that are needed for the angular momentum, including it is a commutation relations, the algebra that is needed and various things such as the being the generator of rotation. And the reason that the rotation operators do not commute because the components of the angular momentum do not commute and in general when you write it as matrices one should understand easily that their own commute because, matrices do not matrix multiplication is non-commutative in general.