Advanced Condensed Matter Physics
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Lecture – 25
Bose Einstein Condensation

We shall study Bose Einstein Condensation. This is particularly relevant in the context of 125th birth anniversary of Essen Bose the Indian scientist, whose been associated with this Bose Einstein condensation. We will say a few things about him as we go along. So, this is a special topic and it is distinguished from the earlier topics where we have talked about interacting systems mostly. However, this happens in a this condensation happens in a system of ideal bosons which means that without any interaction.

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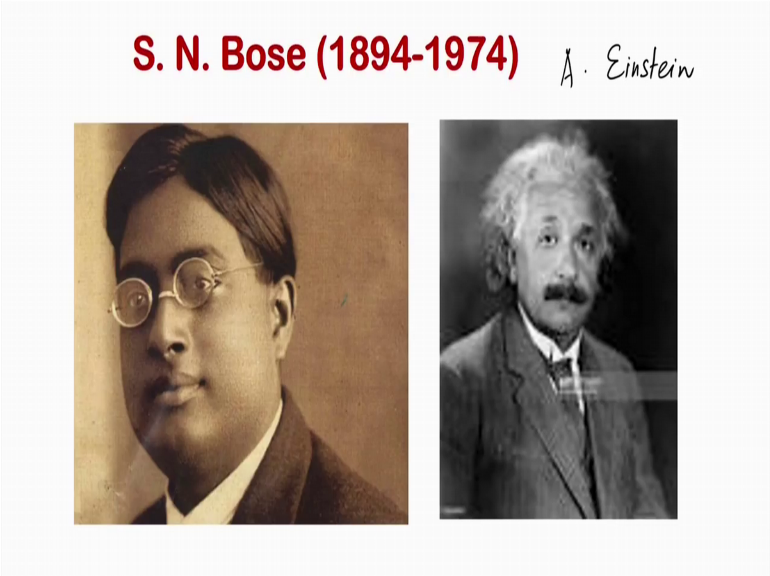

So, this is the picture of S. N. Bose and of course, Einstein here. So, this is Albert Einstein and so we both of them are a responsible for these you know discovery of this phenomena and will let us learn about them.

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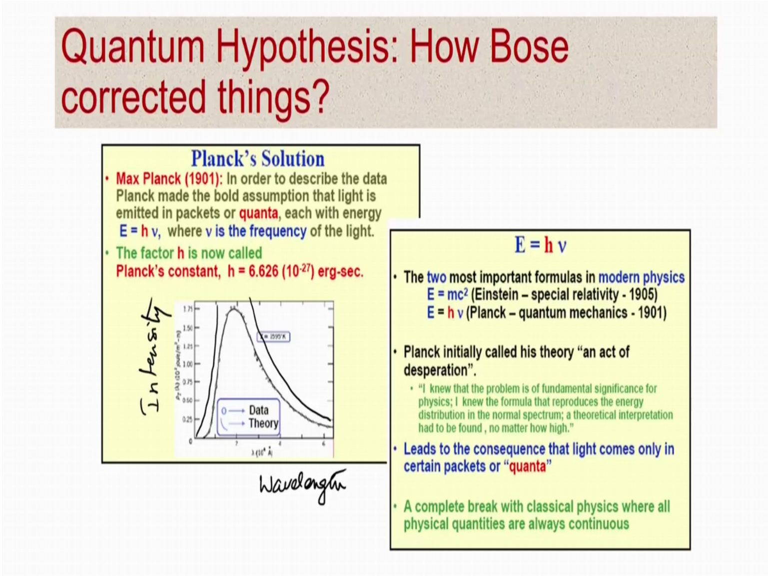

So, what happened or rather how S N Bose got interested in these things? So, what happened is that their discussion on the emission of electromagnetic radiation as a function of temperature. You must have seen that when a piece of iron or a metal is heated it initially becomes red hot and then it becomes white hot and so on. So, the spectrum of the incident radiation changes its frequency or wavelength from one to another and experimentally it is found that the spectrum the incident radiation intensity of the radiation versus this wavelength. So, this is the wavelength if you cannot see it these are on small phones and this is the intensity. So, that looks like this. However, there is a non monotonicity at a given value of the wavelength.

And this was discovered experimentally and the classical existing classical theories predicted that either it is like this or it is like this which are according to the Wien's displacement law or and the Rayleigh's law. However this non monotonicity nobody got theoretically. So, Bose understood that there are a rather in fact, Planck proposed with just conviction and knowing sort of theoretical backing; however, Bose said that there has to be a new statistics for the photons or the incident or the emitted radiation and this laid the Bose Einstein statistics to be proposed which he did in consultation with Einstein and with help from Einstein.

So, it said that the emitted radiation has an energy dependence which goes as h nu or in quantum of h nu and this was a birth of quantum mechanics in some sense. However, the statistics governing this photons are the statistics is given by the Bose Einstein distribution

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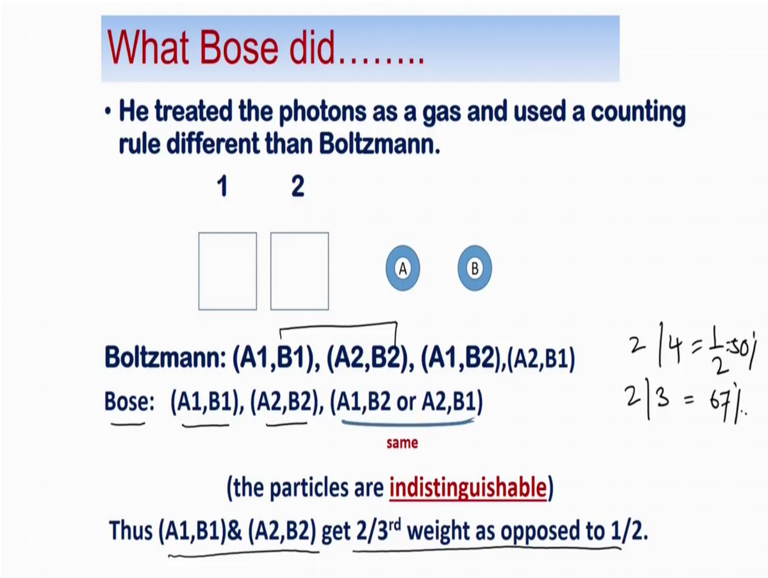

So, let us look at how the condensation phenomena come into the picture. So, say that there are indistinguishable particles. So, bosons are indistinguishable particles. So, bosons photons are bosons phonons are Bosons. So, they are indistinguishable and let us see that how a simple counting procedure can give rise to a condensation like phenomena.

So, let us take 2 boxes and 2 balls marked by as A and B here and let us consider them as classical particle or Maxwell Boltzmann particles, so the particles over Maxwell Boltzmann statistics which means that they are distinguishable. So, if they are Boltzmann particles then we can have A to be in 1. So, the first one refers to the particle nomenclature or the name of the particle and one is the corresponding to the box index. So, A could be in 1, B could be in 1 as well because there is no restriction on the number of particles to be occupying any quantum state or it could be that A could be in 2, B could be in 2 or it could be that A could be in 1 and B could be in 2 or it could be that A could be in 2 and B could be in 1.

So, there are 4 possibilities and if you look at these 4 possibilities there are these 2 possibilities are that they are together. So, they are bunched up in the same box. So, there are 2 out of 4 is the possibility of them being together for a classical particle or a set of classical particles which are represented by the Maxwell Boltzmann statistics which means they are distinguishable.

Now, coming to the Bose particles which are indistinguishable we could have there is no now, no difference between A and B it is just for our reference that we have written as A and B, but they are just both of them to be say A. So, both of them to be in box 1 is one possibility because the bosons do not have that restriction of occupying the same quantum state as the fermions. Fermions obey exclusion principle which we have said a number of times, during the course of this particular advanced condensed matter physics and then both of them could be in the second box remember these nomenclatures are just hypothetical in the sense for our own convenience. They are both A's. So, both of them are in 1, both of them are in 2 or both 1 is in 1 the other is in 2 or the reverse happens.

So, now if you look at the bunching probability then you will see that 2 out of 3 are bunched in the same quantum state. And so this is the crux of Bose Einstein condensation that the statistics says that if they number of particles that is a large number of particles can actually occupy one given quantum state then they will bunch up or rather they will occupy crowd in that quantum state and that quantum state at very low temperatures would be the ground state of the system. So, a macroscopic accumulation of particles in the ground state is what is known as the Bose Einstein condensation.

So, this is what is written that A 1 B 1 and A 2 B 2 get two-third of the weight as opposed to half. So, this is equal to half which is 50 percent and this is two-third is equal to 67 percent. So, there is a larger possibility and this could actually happen in a many particle sector for us to understand Bose Einstein condensation of course, we are going to go to details into that, but this is a very simple idea summarizing that how in-distinguishability of particles can lead to a condensation phenomena.

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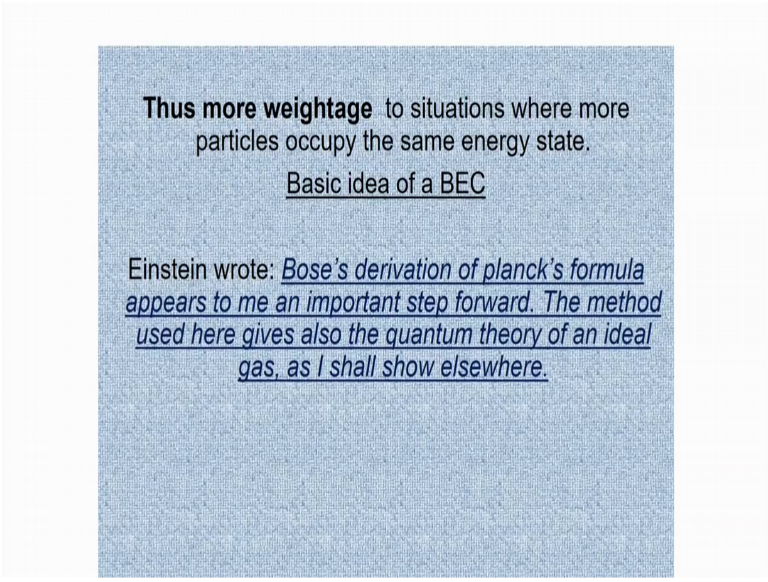

So, the more weightage to situations where particles occupy the same energy state is the central idea or the central focus basic idea of a BEC. And so, Einstein upon receiving a note from S N Bose he understood that there is a lot of merit in Bose's derivation of the Planck's formula, and he says that Bose's derivation of the Planck's formula appears to me an important step forward and the method used here gives also the quantum theory of an ideal gas as I shall show somewhere.

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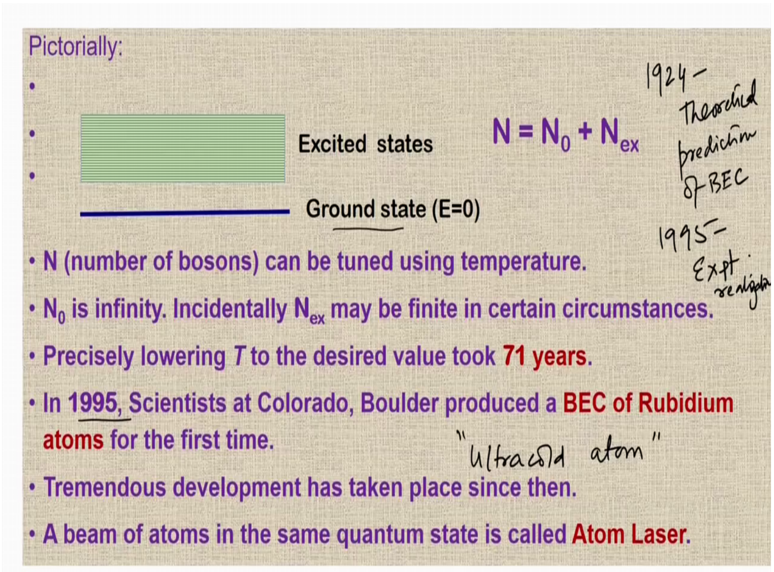

So, pictorially let us see what happens. So, there are a, there is a ground state of a system which corresponds to E equal 0 and these are the spectrum of the quantum of the excited states and we have just shown them almost like a continuous spectrum because in an infinite system or a thermodynamic system they could be infinitesimally close to each other. The reason that we have shown the ground state to be separate from the excited state is something that I am going to discuss later because of the density of states going as a particular fashion as a function of E, which is E to the power half or square root of E this is getting 0 weight which it should not. So, a priori you cannot assign a 0 way to any given quantum state, so we are going to consider this ground state separately as compared to the other excited state.

Now, see say there are N particles N bosons in a given system where N 0 would occupy the ground state just to let you know that the ground state has in principle infinite occupancy. We will show that at lower temperature. And N ex is the number of particles occupying all the excited states put together.

So, now the number of bosons N can actually be tuned using temperature. It is a function of temperature which comes out of the Bose distribution function. Now, N 0 is as I said at very low temperatures is practically infinite. Now, incidentally depending on certain conditions and I will also speak about those conditions N ex may be finite in certain circumstances and in fact, to get any x to be filed rather to get N naught to be infinity you need to go to very low temperature and precisely lowering that temperature lowering t to the desire value to 71 years I will tell you why 71 years because 1924 was the theoretical prediction of BC by boson Einstein.

And it was 1995 as written here experimental realization of BC came in sub system of rubidium atoms and for the first time this actually show a BEC. So, a very large amount of development has taken place since then and the field of what is called as the ultra cold atoms has developed enormously. And, so if the one of the example is that, so a beam of atoms in the same quantum state is called as an atom laser and this is possible one has actually made atom laser and all these are based on the condensation phenomena that was put forward by Bose and Einstein.

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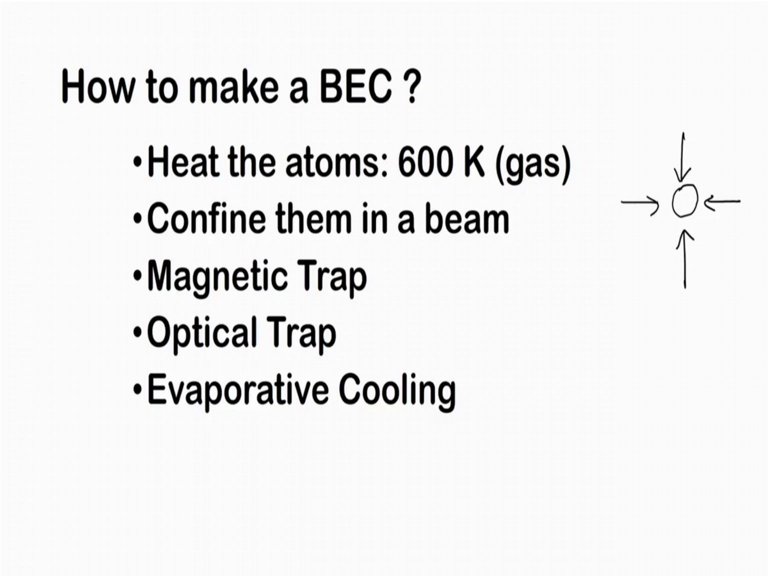

So, let us see how to make BEC. So, you have to heat the atoms say to vedyam atoms at about 600 Kelvin to make a gas of them it is somewhat counterintuitive because we are talking about cooling and then first you have to heat so that we have a very low density of atoms. So, that it forms something like a gas and then confine them in a beam. So, this is a laser beam, so when you, they say a system of atoms. So, if you confine them in such you know and also from top and bottom.

So, then these atom loses its motion and when it loses its motion it cools down and the kinetic energy becomes less, and the temperature corresponding temperature becomes less and this is what is meant by confining them in a beam. Use a magnetic trap will speak a little about them as well you can also talk about an optical trap this is something that these are part of laser cooling we talked a little about laser cooling. And of course, there evaporative cooling which is opening the you know or rather reducing the depth of the trap such that more energetic atoms would escape.

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And then finally there were achievements that 1998 which we are not too keen on ah; however, this is helium 4 was achieved the cooling effects are not very important the cooling effects that we are going to talk about because it happens at 4 Kelvin or say a few Kelvin not 4 Kelvin, but a few Kelvin about 3 3 Kelvin, 3.13 Kelvin. Then in 1995 as we said that Eric Cornell and Curl Wieman the people here they created the BEC in rubidium 87 atoms and then Wolfgang Ketterle in he created BEC in sodium 23 in 1995 just 2 months later after this discovery of rubidium. So, all 3 of them got Nobel Prize in 2001, and these are Kapitsa Allen and Misener these are the people who have liquefied helium in 2000, in 1938.

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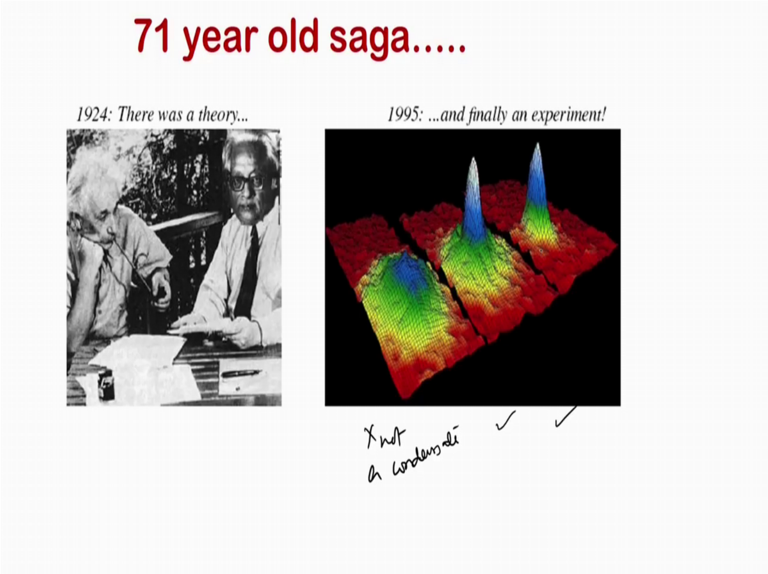

So, this is that old picture that which is connecting the 1925 story when the theory was proposed and 1995 there are experimental evidences. Will try to say a little on this, but just a priori this is still not a condensate here and it starts becoming a condensate and this is a condensate here. So, this is not a condenser and this starts becoming a condensate and so on.

So, what happens is that. So, they have cooled the atoms and then they release the trap in which the atoms were held and when the trap is released the atoms fly away and then a high speed camera device images them and they are the image is converted into Fourier transformed into k space and there is a case phase picture. So, a peak here corresponds to a macroscopic accumulation of particles corresponding to a given k value which is momentum value of the particles and this case the momentum is equal to k equal to 0. So, there is a macroscopic accumulation of particles in a given quantum state which is what BEC is all about.

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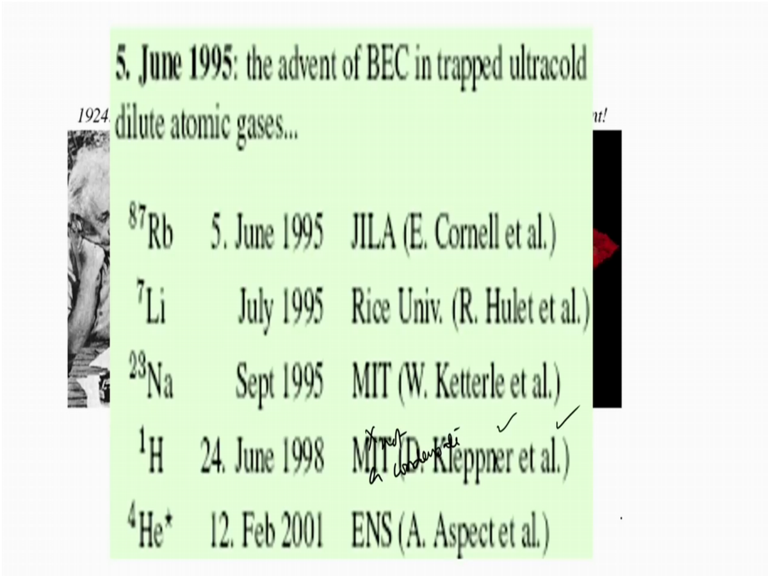

So, just to summarize and the developments that had you know come across in various fields or rather in this trapped ultra cold atoms. So, 87 rubidium was cooled in 1995 and it was in Jila in by Eric Conell. Then it was lithium in July 1995 in Rice University by Randy Hulet, then sodium in September 1995 by Ketterle, hydrogen in MIT and then of course, helium again in 2001 by the group of Allen aspect.

So, now let us start discussing something more you know concrete. So, that you understand the concepts of a BEC. So, let us just talk about the basic features of BEC. So, that things are put in perspective.

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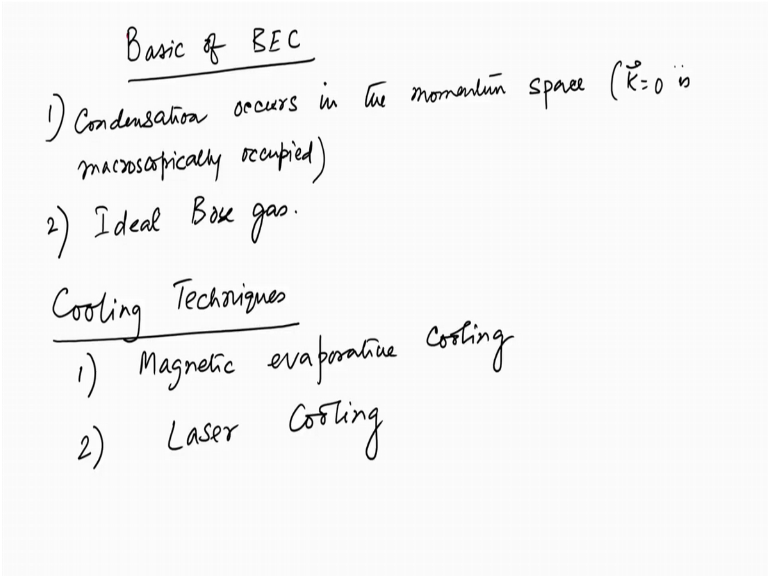

We have said a few things, but will still repeat it. So, the condensation occurs in the momentum space. So, it is a K equal to 0 is macroscopically occupied. So, this is an important key concept rather. So, it is not a real space accumulation of particles. So, it is K equal to 0 is occupied, and it is very important that the particles are non-interacting and we are talking about an ideal Bose gas. In fact, whether the liquefaction of helium is a BEC that is the question that one had to understand, because helium is still an interacting fluid and possibly.

So, there are confusion whether the lambda transition in helium actually falls into the class of a BEC and you will not get into that. Rather let us talk about a few things such as the cooling techniques. And to begin with let us talk about two, one is called as a magnetic evaporative cooling and the second is let us call it as a laser cooling ok.

So, we will briefly describe both of them. So, this is the magnetic evaporative cooling.

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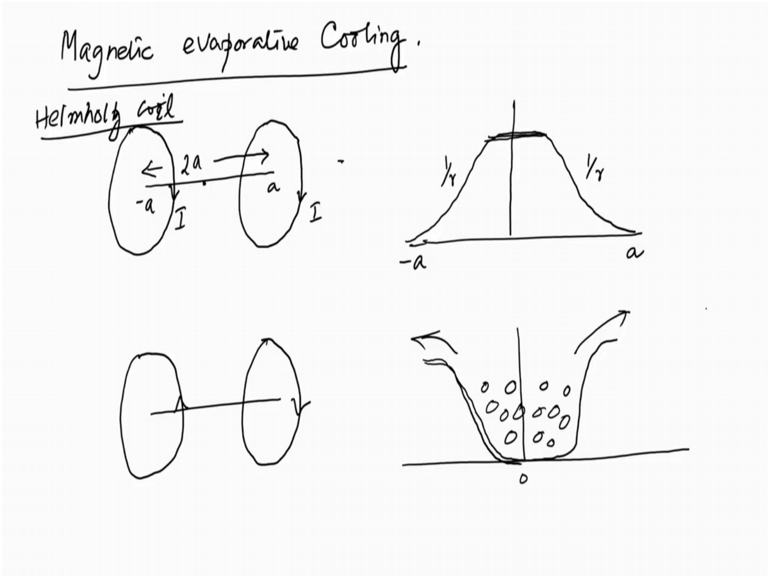

So, this you must be knowing that a system of two coils where the currents are actually moving in the same direction produces. So, this is there are two coils where there is the same current is moving in same direction and the distance between the radii of these two coils is say a or rather this is say 2 a. So, this is 0 to, so this is minus a and this is at a. And then what happens is that then the magnetic field variation because there are current flowing in the coil there will be generation of magnetic field. Now, because there are 2 coils, the magnetic field or the magnetic induction would be a superposition of the effect from both the coils. And this would be like symmetrically it will, so these fall off as one over r whereas, it is fairly constant here and this is a shear minus a to plus a.

So, this is the situation for a Helmholtz coil or Helmholtz double coil as it is said. Now, what person in MIT called David Pritchard did he produced a similar coil with again to such coils; however now, the currents are flowing in different directions in both the coils. So, one with respect to the other and now you will have a magnetic field variation to be having a minima, as opposed to a maxima and which you can do a simple calculation to see that this is of course, symmetric about 0 probably did not draw it as symmetric what it is.

So, this if you can load the atoms here then this acts like a trap for it it is a magnetic trap for it. So, any atom which has more energy will actually escape. So, atoms will escape and leaving only say something like 10 to the power 9 number of atoms in the system which is still you know it is far away from BEC, because BEC requires something around 10 to the power 6 number of atoms. But at least this does one step and it makes a lot of atoms which have which are energetically more I mean they have higher energy they energetically more favorable atoms would escape and leaving behind the slower ones. So, we have a gas of cool atoms and then so this is a one technique, the other one has we say that the laser cooling.

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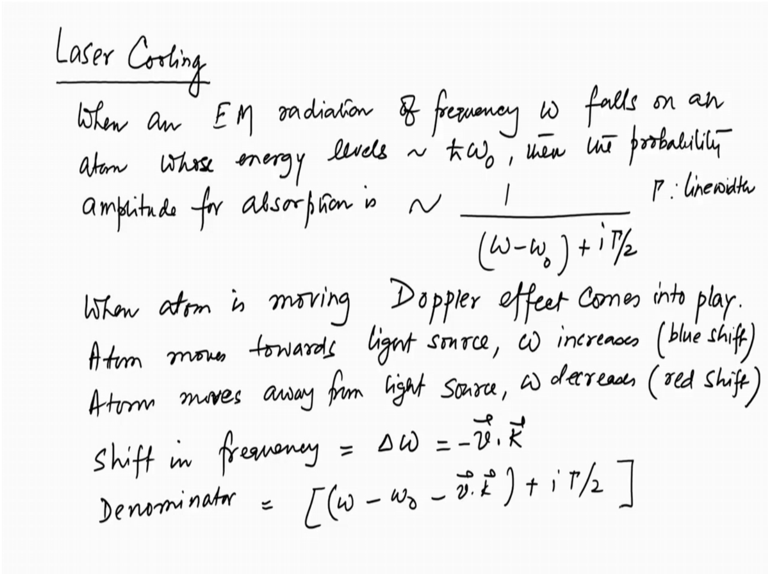

So, this is a very nice technique to slow down atoms and hence a strip their kinetic energies and thereby reducing the temperature. So, what happens is that when an electromagnetic radiation of frequency omega falls on an atom, atom whose energy levels are given by omega naught or h cross omega naught then this I will do without proof, then the probability amplitude for absorption is proportional to 1 divided by omega minus omega not plus some i gamma by 2 where gamma is the line width line width of the radiation this is a natural line width.

So, this happens when the atom is at rest; but when the atom is moving then the Doppler effect comes into play and if we write this. So, what happens when a Doppler effect comes into place, that when the atom moves towards light source, towards light source which means electromagnetic radiation, so the omega increases the effective omega increases. And this is called as blue shift and when the reverse happens that is atoms moves away from light source omega decreases and this is called as red shift.

So, this shift in frequency is say equal to delta omega which is of has a form which is v dot k with or a minus v dot k where v is the velocity of the atom and k is the momentum. So, if you need to slow down atoms then it is beneficial for the atoms to be moving towards the light source or the incident radiation. So, if you can make the atoms move towards the light source then they will slow down and slowing down means the kinetic energy becomes less and when the kinetic energy becomes less the equivalent temperature because the energy is always expressible in terms of temperature with the Boltzmann relation they equal to k t. So, the t decreases.

So, what happens is that the that the denominator here. So, this denominator it takes a form of omega minus omega naught minus v dot k plus i gamma by 2 and so the quantity so delta is equal to minus omega plus omega 0 rather omega naught minus omega 0 is called as the detuning parameter.

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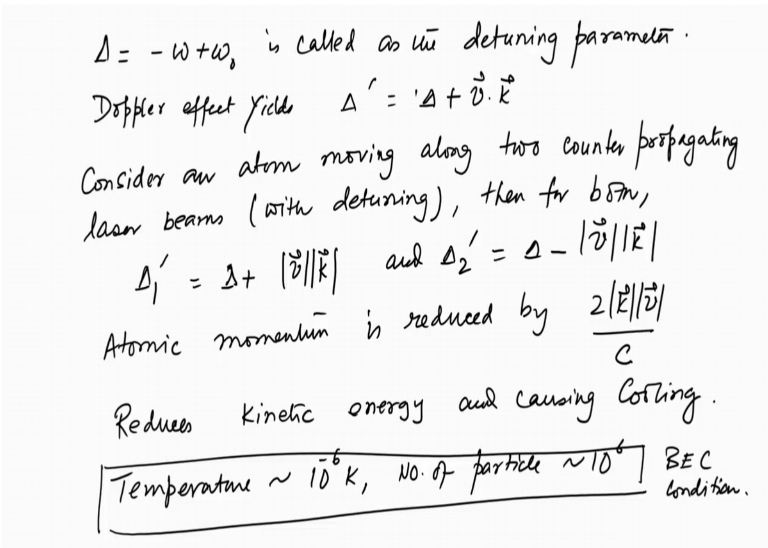

So, the Doppler effect yields delta prime equal to delta plus v k, v our v dot k and so basically now, an atom moving, moving along to counter propagating laser beams this is what I had shown in one of the earlier slides that the atoms are actually held between counter propagating laser beams. Counter propagating means a propagating in opposite directions with detuning. Then for both of them for both delta 1 prime that is this one is equal to delta plus v k. So, this is a 0 angle between them. So, it is a v k cos theta, but cos theta is equal to 0 because they are moving towards each other and the atom is moving along the line. So, there is no angle that it is making and delta 2 prime for the other beam it is equal to delta minus v k.

So, it is clear that the some more energy is actually absorbed from one of the beams as corresponding to the other. So, we are talking about the 2 laser beams. So, one of them is absorbing more energy from the atom as compared to the other, but this further means that the atomic momentum is reduced by twice of this k v divided by C, ok. So, this forces exerted by the beams on the atoms are not balanced. So, there is a resulting in a net force opposite to v and thereby reducing the kinetic energy and so, reduces kinetic energy and causing cooling.

Now, this ends the story on cooling from our side, but you understand that all these engineering of having cold atoms possible in 70 years and they are the best minds that were working in the subject it is still took a very long time, for getting them down to a temperature which is extremely small and BEC really happens in sort of micro Kelvin in the micro Kelvin range. However, well do a calculation and well show that actually for the atoms such as helium the condensation or rather the condensation like temperature is just about a few Kelvin maybe 2 to 3 Kelvin.

However, understand that actually the rubidium atoms required a temperature to be of the order of a micro Kelvin and so let me write that the temperature is a of the order of finally, and there is only a part of the cooling techniques that I have talked about there are the sisyphus cooling, and sympathetic cooling and so on, and so this temperature comes to about 10 to the power minus 6 Kelvin and the number of atoms or number of particles that should be left is about 10 to the power 6.

So, this is the condition for BEC to be formed in lab and. However, as I said that will not, will not discuss the exact magnitude of the temperature, but well do an order of magnitude calculation.

Now, since we are done with the cooling part let us understand that why is that we are talking about bosonic atoms when the atoms are actually new neutral or rather the charge neutral and if we are talking about the electronic the spin then that electronic spin is always half and these alkali atoms rubidium being an alkali atom. So, this that that should correspond to electrons having spin half and that cannot be put in the same class as bosonic atoms. So, let us understand why they are called as bosonic atoms.

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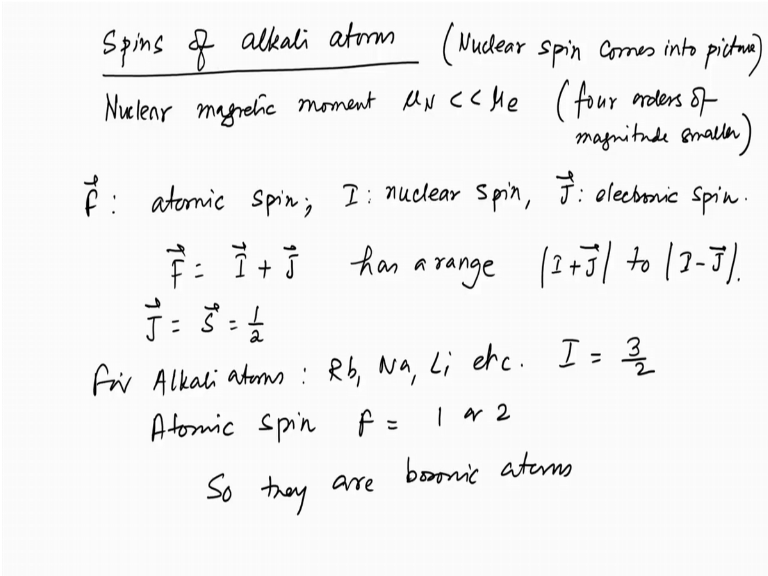

We are just talking about alkali atoms because more it started with alkali atoms as I have shown you rubidium, sodium etcetera potassium lithium again. So, what happens is that even if the electrons have the spin half it is the nuclear spin that comes into picture but because the nuclear magnetic moment, nuclear magnetic moment mu N is far lower than mu e which is the nuclear magnetic moment for electrons by at least of 4 orders of magnitude. And thus to have nuclear spins to play an important role we really need very low temperature.

So, let us say that F is the atomic spin I is the nuclear spin and J is the electronic spin. So, f is equal to I plus J and so it has a range I plus J to I minus J and for J equal to which is a electronics spin equal to half. So, we have F and for rubidium ok, let us for all the alkali atoms such as rubidium, sodium, lithium etcetera we have I which is the nuclear spin is 3 by 2. So, the atomic spin F is equal to 1 or 2, so they are bosonic atoms with integral spins. And as I said for the nuclear spin to play a role the temperature has to be low.

Now, let us start doing some calculations in order to understand the condensation phenomena in an ideal Bose gas. So, let us start with the ideal Bose gas.

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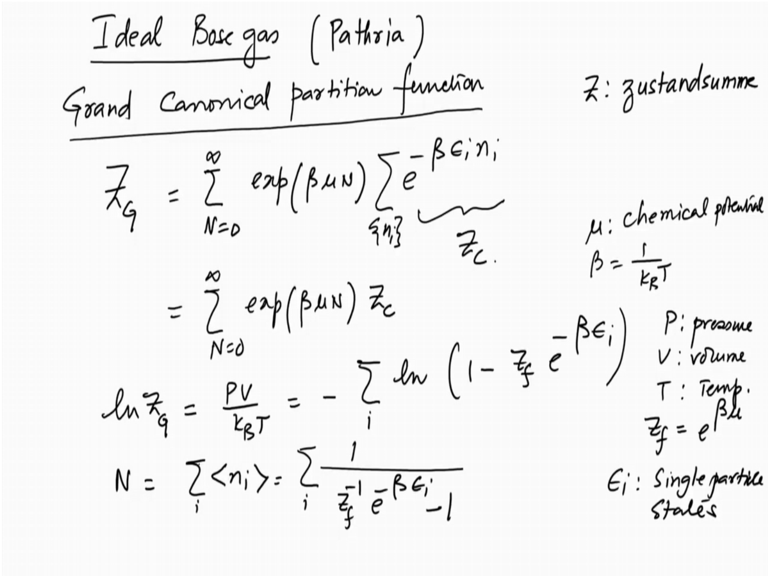

And the discussion is you can follow a statistical mechanics book very good statistical mechanics book by Pathria which gives a very good description of this phenomenon before we proceed let us write down the grand canonical partition function. So, the word grand canonicals grand canonical partition function is that where we also allow in a in addition to exchange of energy among the particles we also allow for exchange of particles between the system and the bath.

So, it is the canonical is about exchanging energies between the system and the bath and this is also in addition to that allowing the number of particles. So, the number of particles is not constant. So, a grand canonical partition function as you know that the partition function is written with Z and this Z comes from a German word probably it is called as; there could be a problem with the spelling, but it means zustandssumme means that it is a sum over states; And this G, subscript G that corresponds to the grand canonical.

So, this is equal to N equal to 0 to infinity as we just said that will allow the particles to be exchanged between the system and the bath, so that is that term and exponential beta mu N and this is equal to an exponential minus beta epsilon i n i and sum over all states let us you know you can write n i and so on.

So, this is the canonical partition function and. So, it is basically N equal to 0 to infinity exponential beta mu N Z C is pretty much the formula for the grand canonical partition function. And you can follow path via to see that the grand canonical partition function the log of that which is related to the free energy a minus K T of that is equal to F which is equal to a PV. So, this is equal to PV over K T and which is equal to a minus sum over i log of 1 minus Z f l right for Z f is and beta epsilon i and so on.

So, let us make this notations all clear. So, P is equal to pressure v 2 be volume t is of course, temperature, Z f is called as the fugacities which is equal to exponential beta mu beta is 1 over K T and mu is the. So, let us write it here mu is chemical potential and beta equal to 1 over K T. And, so this is by and large the expression for this. So, this can actually be the equation of state for an ideal boost gas because we are writing PV by K T which you know for a classical ideal gas PV by K T is equal to some r or it is n r or something like that.

So, it is a constant whereas, in an ideal boost gas at quantum gas which follows an in distinguishability constraint is given by this. And similarly the number of particle which is equal to sum over i n i this is equal to sum over i and a 1 divided by Z f inverse or you can simply write it as exponential minus beta mu and exponential minus beta epsilon i plus this is minus 1 sorry. So, for boson is minus 1 and for fermions it is equal to plus 1 epsilon i's are the single particle states. So, what I mean by signal particle states is that if there are free particles. So, there they will go as the in k space they will go as h cross square k square over 2 m.

Now, before we proceed what we have to do is that we have to convert these or rather these are the working equations for any system whether you are talking about condensation or you are talking about studying other thermodynamic or statistical mechanics mechanical properties you have to start with these equations for both fermions and boson.

Now, we need in order to calculate them we need to, calculate them analytically we need to convert them into integrals. And you know when a sum is converted to an integral you need the density of states you need the number of states in the energy range E and E plus B E and that is an important quantity in condensed matter physics as I might have already told a number of times because this is what creates a difference between a 2 dimensional the properties of a 2 dimensional system with that of a 3 dimension because the density of states go down. In fact, the nano the whole branch of nanoscience and nanotechnology when we talk about quantum wires which means that we are confining electrons or the charge carrier in one dimension while in quantum dots we do it in 0 dimension and in a 2 dimensional electron gas this is in 2 dimension and so on.

So, we need the density of states in order to calculate or rather convert the sums into integrals. Now, this is I will not go into that, but it is a very simple calculation first course of solid state physics would do it that the density of states in short called as DOS.

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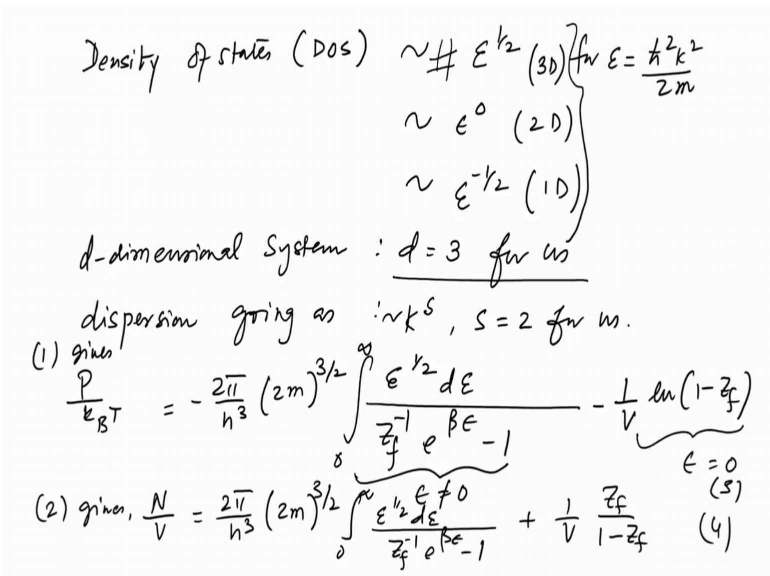

So, that goes as epsilon to the power half for a epsilon equal to h cross square k square over 2 m this is in 3D it goes as constant also let us write it as a epsilon to the power 0 which is also which is a constant in 2D and it goes as epsilon to the power minus half in 1D ok. So, for this dispersion a free particle dispersion.

So, if you see this that the density of say, so we are talking about a 3 dimensional system with k square dispersion. So, we are talking about a d-dimensional system, where d equal to 3. So, d equal to 3 for us and we have a dispersion which is s going as, dispersion going as k to the power s, so s is equal to also, s is equal to 2 for us. In fact, whether a BEC will occur or not will crucially depend upon this value d and s and a you can actually convert it into a single parameter. So, either you call it as by d or d by s. So, depending on certain values of s by d or d by s it is Bose Einstein condensation is possible.

So, we convert using this density of states, we convert the first equation let us call them as first equation and second equation and let us write 1 gives. So, 1 means equation 1 and so this is equal to minus 2 pi over h cube 2 m whole to the power 3 by 2 m is the mass of the particle mass of the bosons here and this comes from the density of states we have not written here a coefficient and that coefficient would involve all these ms and so on m h etcetera etcetera. And as I said its epsilon to the power half d epsilon and I convert the Z f inverse and exponential beta epsilon minus 1.

Now, I should have actually done it for all epsilons, but however, you see if we have epsilon to the power half and for the ground state which is epsilon equal to 0, you would attach 0 weight. This is what I was talking about earlier that you cannot have a 0 weight assigned to a given state then that state is completely unimportant for our computation. So, we will have to take that state out and write it for that state will write this thing as rather it is a minus sign it is a minus 1 by V log of 1 minus Z f. So, this is that contribution for the epsilon equal to 0 and this is for epsilon not equal to 0.

Now, ideally I in this integral we should separate out the epsilon equal to 0, but since this is an integral just having one point less does not make any difference in this integral. So, we can still put it as 0 to infinity. So, so the integral becomes 0 to infinity and this integrand and then this is the epsilon equal to 0 component of that equation number 1.

So, now let us call this as equation number 3, and 2 gives in the same manner N by V is equal to 2 pi over h cube 2 m by 3 by 2, and a 0 to infinity just in the same way and d epsilon and you have a Z f inverse exponential beta epsilon minus 1 and the plus 1 over V, Z f by 1 minus Z f let us call this as equation 4.

Now, if you look at these 2 equations we have clearly as separated them into two terms, one corresponding to epsilon equal to 0 the contribution to the pressure corresponding to epsilon equal to 0 and the contribution to the density of particles or the number of particles to be f for epsilon equal to 0 and for epsilon not equal to 0. This is in one of my slides that I have shown the ground state energy to be separate as compared to the excited state energies.

So, now, let us see the second terms of both of both 3 and 4.

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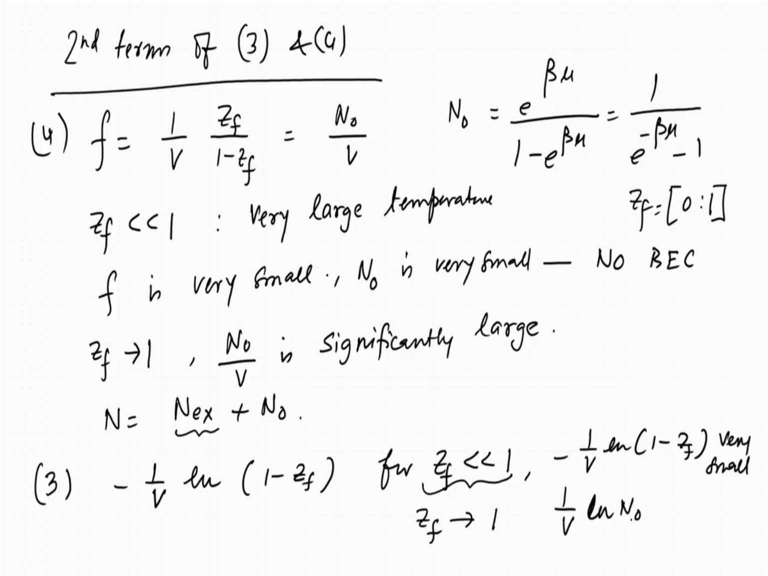

So, see what the second terms are. Let us write that second term as this its equal to Z f 1 minus Z f and that is equal to N naught by V. You understand N naught, N naught is the occupancy or the number of particles that the ground state can hold and this is equal to exponential beta mu 1 minus exponential beta mu which is nothing, but equal to exponential minus beta mu minus 1. So, this is the N 0 by V that is the second term in equation 4. So, this is 4.

Now, when your Z f is much smaller than 1, now Z f is actually between 0 and 1 ok so, it can take a maximum value 1 and can take a minimum value 0. So, if Z f is close to 0 that is a smaller much smaller than 1 then we have, this corresponds to very large temperature. And if it talks about very large temperature then you have the number of particles is very low because this f is equal to f is a very small and so N 0 is very small and of course, that corresponds to a no Bose Einstein condensation because you have a classical physics that is taking over. So, thermal effects should you know drive all the particles away.

Now, you try to understand that as you reduce the temperature N 0 increases. So, at very low temperature N 0 is very high and so as Z f goes to 0. So, that is, so mu is actually a chemical potential which is negative. So, if Z f goes to 0 then N 0 by V is significantly large. In fact, it is so large that your N equal to N ex plus N 0 somehow if you can show that your N ex is equal to a finite number then since N 0 is infinitely large. So, if your N is infinite or rather very large then all the particles will go to the ground state because it is has infinite occupancy whereas, a very smaller number of particles would actually go to the excited states because it has limited occupancy.

Now, if you look at 3 then you will see that this is equal to the 0 epsilon equal to 0 contribution is equal to this and this tells that for a Z f is much smaller then this is equal to negligible. So, this minus 0 over V log of this thing is very small that of course, we know that the pressure due to the all the particles that would go to the ground state is very small because there is no almost no particles at large temperatures this is the limit for large temperature.

However if you go to the other limit that is Z f going to 1, I am sorry the Z f should go to one here not 0. So, Z f is 1. So, if Z f goes to 1, so this is for Z f going to 1. So, Z f going to 1; however, this thing would take a form which looks like 1 over V log of N naught. Now, N naught could be large, but log of that would be you know still small. So, the second term in equation 3 can still be neglected even if at low temperature that is Z f going to 1, of the fugacities going to 1; however, that cannot be neglected in equation 4. This is the main central message of this discussion; that even the pressure contribution from all the particles in the ground state of the system could be infinitesimally small; however, the number of the number density is significant.

So, now well write this. So, basically our 3 becomes its P over K T its equal to minus 2 pi over h cube or 2 m K T whole to the power 3 by 2 and 0 to infinity x to the power half. So, its log of 1 minus Z f exponential minus x, dx, where x equal to exponential minus sorry x equal to beta E, that is the thing.

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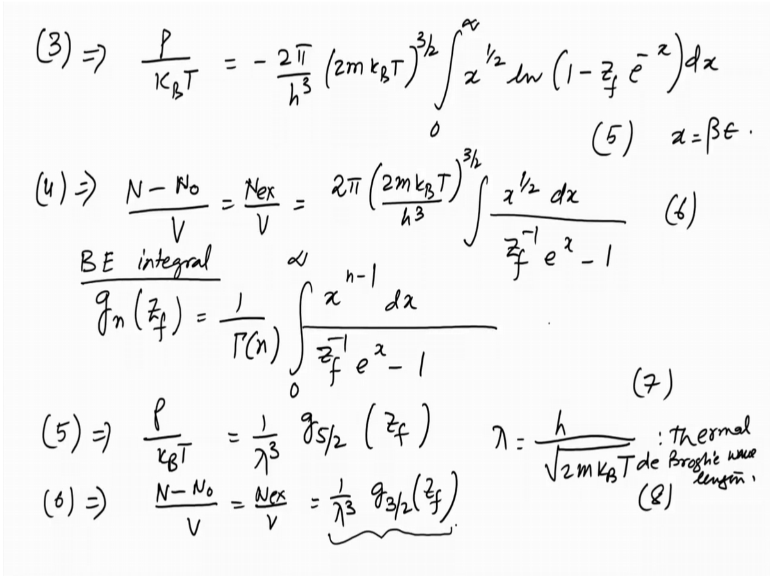

And so beta has come out and the 4 a gives let us call this as maybe 5 and 4 gives N minus N 0 by V, which is N ex by v E x, N ex is the occupancy or the number of particles in the excited states and this is equal to 2 pi 2 m K T by h cube and this is equal to, so this is 3 by 2 and this is equal to x to the power half d x Z f inverse exponential x minus 1.

Now, these are called as the oh science stein integral and a g n of z. So, this these are Bose Einstein integral lifts in short called as BE integral and this BE integral takes a form its equal to 1 minus gamma N x to the power N minus 1 you can write it as Z f and this is equal to dx divided by 0 to infinity and its equal to Z f inverse exponential x minus 1. So, these integrals look very similar to that excepting for its N equal to say 3 by 2 and so on. So, and this is equal to it can be shown this I leave it to you it is a matter of you know doing a partial integral integration, sorry integration by parts of this integral and a express it in the form of this and finally, what one gets is, so let us call this a 6.

So, 5 becomes equal to P over K T its equal to 1 over lambda cube and g 5 by 2, Z f. We are less interested in this formula though it is important nevertheless; however, we are interested in this formula the one that we are going to write later. So, let us call this as 7 and this is equal to N ex by V which is equal to 1 by lambda cube g 3 by 2 Z f. That is the second equation that is equation 6. So, 6 yield this.

So, our the number of particles which is what we wanted in the excited states is given by some quantity which is this where lambda is equal to root over h over h divided by root over 2 pi m 2 m K T sorry not 2 pi m, 2 m K T it is called as a thermal de Broglie wavelength. So, this is equation a thermal de Broglie wavelength.

Now, we are almost done, we have obtained an equation for the number of particles in the excited states. If this quantity is finite is not infinitely large which would depend upon certain criteria then we are done we would again leave this thing and show that your g n, Z f is equal to sum over l equal to 1 to infinity, it is a Z f to the power l, l to the power n which is equal to Z f plus Z f square by 2 to the power n plus Z f cubed by 3 to the power n and so on.

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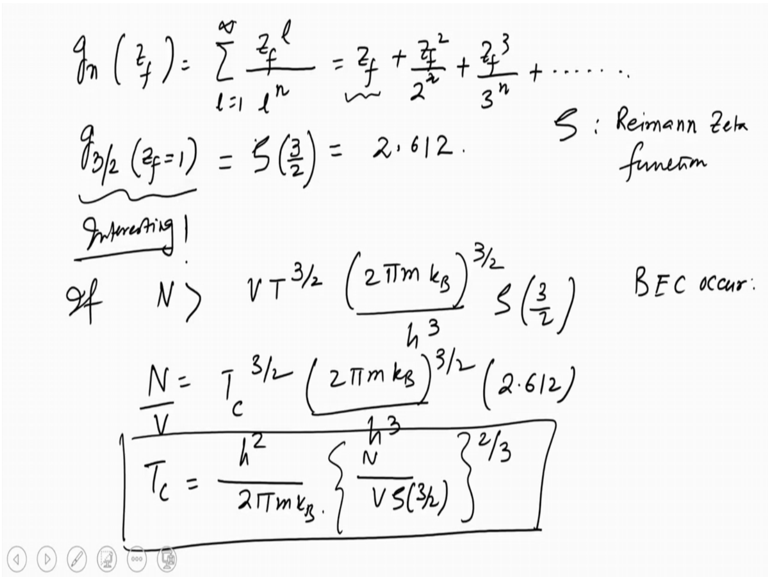

Now, for small z that is that if much smaller than one the classical limit you can be satisfied with Z f, ok. So, if you put that equal to a Z f then of course, that becomes will depend on the excited state occupancy is you know is goes as exponential beta mu.

However, at low temperature that is when Z f goes to in the limit goes to 1, you have to take all of these terms into consideration and cannot stop at a finite terms, but fortunately this is equal to for g 3 by 2 said f g 3 by 2 and Z f equal to 1 which is of interest to us this is interesting to us and this its equal to a Riemann zeta function which is 3 by 2 which has a value 2.612.

So, this is a finite value this is what we were hinting at time and again that the excited state occupancies which are coming out from this equation number 8 has an the excited state occupancy has a finite value. So, if there are more number of particles they will all go to the ground state. If there are a macroscopically large number of particles they would all go to the ground state and this is what the condensation. So, N greater than you know a V T to the power 3 by 2 and 2 pi m k divided by 3 by 2 h cube etcetera it becomes equal to a 3 by 2 and the psi is this is not the way to write it is like this, it is called as the Riemann zeta function.

So, the Bose Einstein integral is related to the Riemann zeta function when you take this entire sum. The entire sum has a closed form which is called as Riemann zeta function and for an argument equal to 3 by 2 it has a value equal to 2.612. So, if N is greater than, this N is the total number of particles then BEC occurs and for the BEC to occur the critical condition is that your N by V has to be equal to some T C 3 by 2 it will happen at that value. So, this is equal to a k 3 by 2 and by h cube and this value which is equal to 2.612.

So, if you put everything there then what we get is that a T C has an expression which is equal to h square by 2 pi m k and its equal to N by V and its equal to 3 by 2 and its equal to a 2 by 3. So, this is the expression for T C which means the temperature if its lowered below this then all the particles will go to the ground state and the excited state because the excited states have a a finite occupancy they will avoid the excited states and go to the ground state because the temperature is also very slow.

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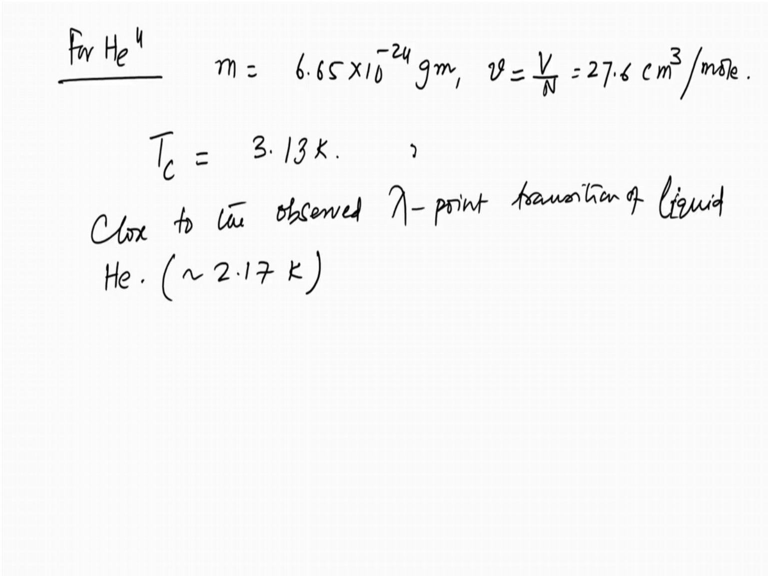

Now, this we will just do it for helium 4 which as we said that is not a a prototype case for Bose Einstein condensation, but still if you take this values 10 to the power minus 24 gram. And so the density or the inverse density which is called as a specific volume V by N equal to 27.6 as centimeter cube by per mole then if you put all these things T C comes out to be equal to 3.13 Kelvin and 13 Kelvin and this is close to the observed lambda point transition of helium liquid helium, liquid helium which is equal to 2.17 Kelvin.

So, there was a initially misconception that the liquification of helium is actually or rather they heal the lambda point transition in helium is actually a BEC transition. We will not elaborate on that, but what we have got is a condition for the Bose Einstein condensation also we have explained various steps that are associated with the cooling process and finally, when the cooling happens the whole atom the 10 to the power 6 atoms number of atoms.

They form a structure which has very low temperature which is like 10 to the power minus 6 to minus 7 Kelvin which is like a less than a nano, I mean about micro Kelvin temperature which is the probably the coldest temperature in the universe and they are imaged as I said that releasing the traps and letting them fly apart. So, when they fly they are imaged and they are finally, Fourier transformed into k space to see there is a macroscopic accumulation of particles in the k space.

So, this is a k space phenomena. It is a real momentum space accumulation of particles B the BEC is a a a example of that and so this the credit goes to Bose and Einstein who have proposed this in 1924, nearly a 100 years from now, 100 years earlier. However, as I said the realization had take place a very large number of years.