Advanced Condensed Matter Physics
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Lecture – 22
Ginzburg Landau Theory, Coherence length and penetration depth

So, in this session we are going to learn the Ginzburg Landau Theory of superconductivity.

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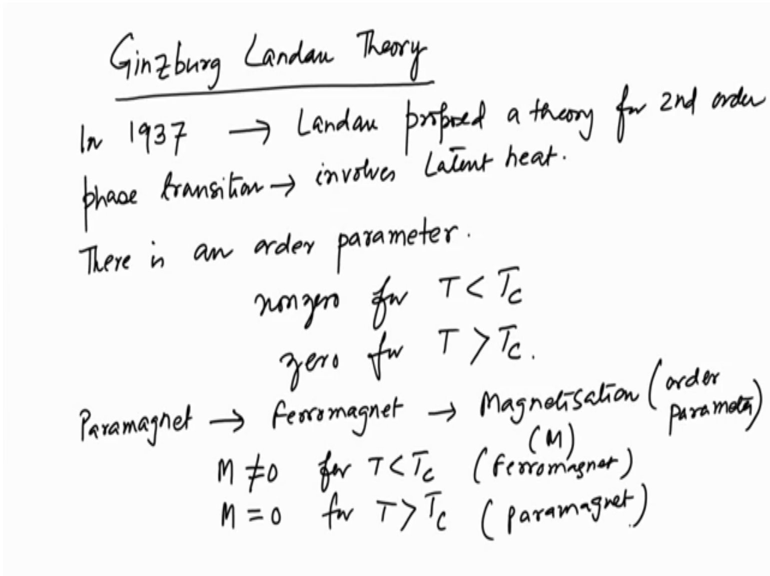

Remember this was evolved before the microscopic theory that is BCS theory was put forward and it is a phenomenological theory of second order phase transition which was extremely successful in describing a few properties of these superconductors such as the penetration depth and the coherence length and thereby a ratio of them could be formed which helps in distinguishing between type 1 and type 2 superconductors.

So, in 1937 landau proposed a theory for second order phase transition now by second order phase transition what we mean is that, that involves latent heat this is one of the modern definitions of phase transition in this theory there is an order parameter that continuously vanishes across the phase transition. So, this order parameter is nonzero for T less than T c and it is equal to 0 for T greater than T c.

So, this is an indicator of the phase transition say for example, in a paramagnet to a ferromagnet transition the magnetization can be the order parameter. So, the magnetization let us call it by m so, magnetization is not 0 for T less than T c which talks about a ferromagnetic state and magnetization being 0 for T greater than T c. So, this is a ferromagnet and this is a paramagnet.

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In Superconductors

Energy gap of the excitation spectrum — order parameter (Δ)

$\Delta \neq 0 \quad T < T_c$: superconducting state

$\Delta = 0 \quad T > T_c$: normal state. T

GL Theory

Free energy $\rightarrow$ expanded in terms of power series of order Parameter.

Free energy $\rightarrow$ scalar

order parameter $\rightarrow$ vector, tensor or a complex quantity.

Similarly, in superconductors we can talk about the energy gap of the excitation spectrum this can be taken as the order parameter call it as delta and delta is not equal to 0 for T less than T c, where T c is the transition temperature for a superconductor which is a characteristic of a material. So, this corresponds to a superconducting state and delta equal to 0 for T greater than T c this is called as the normal state or a non-superconducting state.

Now, since the order parameter continuously evolves across the boundary of the phase transition it should be possible to write down the free energy as a power series in the order parameter. So, this is the whole idea of GL theory a Ginzburg Landau theory I am abbreviating as GL theory. So, it says that free energy should be expanded in terms of order parameter, powers of in terms of a power series of order parameter.

Now, it can be noted that free energy is a scalar; however, the order parameter can be higher dimensional quantity such as a vector or a tensor or a complex quantity. So, this is what we are going to do that is write down the free energy functional the meaning of the word functional is function of a function.

So, we will write this free energy functional which is a function of the order parameter and order parameter could be a function of say the special parameter such as or any other parameter that is relevant to the problem it is usually the space variable which is taken as that that the order parameter being a function of that now we will study the GL theory.

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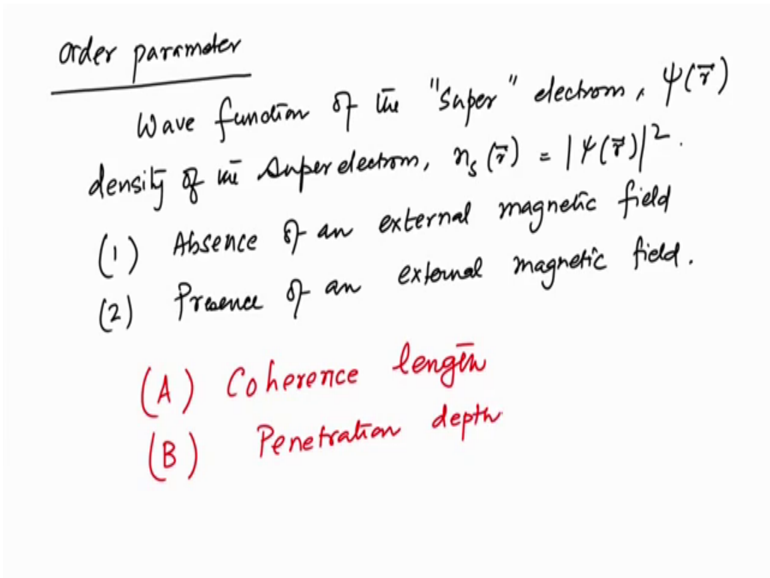

So, the order parameter in GL theory is the wave function of the so, called super electrons call that as psi of r such that the density of the super electrons, what I mean by super electrons is that, they are superconducting electrons which is equal to mod psi r square.

So, the plan is like this that we will do two cases one is absence of an external field magnetic field and then also we will do it in presence of an external magnetic field and finally, we would compute two different quantities one is called as the coherence length and b called as the penetration depth.

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<u>Free energy</u> ($H_{ext} = 0$)

$$F_s = F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$$

F_s : free energy of a superconductor

F_n : Free energy of a normal metal.

α, β : phenomenological expansion Coefficient.

Minimum of F_s is obtained by

$$\frac{dF_s}{d|\psi|} = 0 = 2\alpha|\psi| + 2\beta|\psi|^3 = 2|\psi|(\alpha+\beta)$$

$$|\psi_0|^2 = -\frac{\alpha}{\beta} \Rightarrow F_n - F_s = \alpha\left(\frac{\alpha}{\beta}\right) - \frac{\beta}{2}\left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2}{2\beta}.$$

So, the free energy in H external equal to 0; So, there is no external field and we write down the superconducting free energy which is equal to free energy of the normal state plus alpha psi mod square plus a beta by 2 psi to the power 4 note that we are keeping the expansion only up to the biquadratic that is aquatic terms of psi and the other thing is that there are no odd powers of psi involved because that would be anti-symmetric with respect to the change in the order parameter.

So, F s is free energy of a superconductor F n free energy of a normal metal and alpha and beta are phenomenological coefficients expansion coefficients. So, minimum of F s so, we want to minimize the free energy with respect to the order parameter and find out where the optimum value which minimizes the F s lies minimum of F s is obtained by taking dF s d psi equal to 0 and this is equal to a 2 alpha psi so, and plus 2 beta psi cube.

So, I can take a psi common and this becomes equal to a 2 psi common and this will be like an alpha plus beta. So, psi mod square that does minimize this let us write it with a psi 0 mod square equal to equal to minus alpha over beta thus we can write down F n minus F s it is equal to alpha into alpha by beta and minus beta by 2 alpha by beta whole square this becomes equal to alpha square by 2 beta and it is almost obvious that alpha and beta both could be temperature dependent.

Now, in the first term so, let us write this once more that F n minus F s is alpha square over 2 beta.

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$$F_n - F_s = \frac{\alpha^2}{2\beta}.$$

In the first order, i.e. not too far from T_c

$$\alpha \sim (T - T_c).$$

$$\left.\begin{aligned} \text{i.e.}\quad & \alpha = 0 \quad \text{at } T = T_c \\ & \alpha < 0 \quad \text{at } T < T_c \\ & \alpha > 0 \quad \text{at } T > T_c \end{aligned}\right\} \quad \text{Because } \begin{aligned} & |\psi| = 0 \quad \text{for } T > T_c \\ & |\psi| = \left[\frac{a(T_c - T)}{\beta}\right]^{1/2} \\ & \qquad \text{for } T < T_c. \end{aligned}$$

α changes sign across the phase transition.

β should be a constant (+ive) and indep. of T.

$$\alpha = a(T - T_c). \qquad a : \text{constant}$$

F n minus F s equal to alpha square over 2 beta. So, in the first order that is not too far from T c that is at temperature which is not too far from T c alpha can be T minus T c. The reason is that alpha equal to 0 at T equal to T c alpha is less than 0 at T greater than T c and alpha is greater than 0 at sorry this is T less than T c and this is T greater than T c ok. Now this happens because psi is equal to 0 for T greater than T c. So, psi must be going as a T c minus T whole to the power beta and whole to the power half here for T less than T c.

So, the important thing for us to understand is that alpha changes sign across the phase transition. However beta should not change sign because if both of them change sign then of course, will not have this equation valid which is psi mod square equal to minus alpha by beta and moreover beta also should not have any temperature dependence so, that we will have some pathological.

So, if beta has to be help if beta does not have to change sign across the transition then it has to be like T minus T c whole square because the whole square does not say in sign, but in that case your alpha by beta will be like one over T which would be divergent as T goes to 0 and that should not happen. So, beta should be a constant so, basically it is a positive constant and independent of temperature of course, it means that being constant means that it is independent of temperature.

So, at least for small deviations from T equal to T c this is what should happen. So, alpha has a form which is like a T minus T c beta is a constant and hence will have the psi mod equal to 0 for T greater than T c and psi mod equal going as T T c minus T whole to the power half with as and bs as constant, a is a constant here.

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$$F_n - F_s = \frac{\alpha^2}{2\beta}.$$

In the first order, i.e. not too far from T_c

$$\left.\begin{aligned} &\alpha \sim (T - T_c). \\ \text{i.e.}\quad &\alpha = 0 \quad \text{at } T = T_c \\ &\alpha < 0 \quad \text{at } T < T_c \\ &\alpha > 0 \quad \text{at } T > T_c \end{aligned}\right\} \quad \begin{aligned} &\text{Because} \\ &|\psi| = 0 \quad \text{for } T > T_c \\ &|\psi| = \left[\frac{a(T_c - T)}{\beta}\right]^{1/2} \\ &\qquad \text{for } T < T_c. \end{aligned}$$

α changes sign across the phase transition.

β should be a constant (+ive) and indep. of T.

$\alpha = a(T - T_c)$. $\quad a$: constant.

ψ is homogeneous.

Now, consider a superconductor in an external field magnetic field that is. So, the way it is taken care of is that the momentum is changed by p minus e A by c this you must have seen in your classical mechanics course when we talked about a charged particle in a magnetic field. Now this A is the vector potential which is related to the field as h of r external equal to curl of A. So, it is always derivable from a vector potential by this relation this tells that the del operator should be replaced in presence of a magnetic field by this.

Now, the free energies have to be written down the same free energy that we have written down earlier. So, F s equal to a F n plus alpha psi because of the magnetic field now which is a function of r we have an inhomogeneous order parameters. Now psi starts depending upon r and also because of the magnetic field we have an additional term which is given by i e by h cross c A of r and then a psi of r mod square and plus a magnetic energy because of the external field which is given by H external square by 8 pi. Now free energy F is a functional of psi of r psi star of r del psi of r del psi star of r and A of r.

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Minimizing it with respect to $\psi^*(\vec{r})$

$$\delta F_s = \left\{ -\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}(\vec{r})\right)^2 \psi(\vec{r}) + \alpha\psi(\vec{r}) + \beta|\psi(\vec{r})|^2\psi(\vec{r}) \right\} + \underbrace{\frac{H_{ext}^2}{8\pi}}_{\text{ignore}}$$

for δF_s to be zero, $\{\cdots\} = 0$

1st GL equation

$$-\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}(\vec{r})\right)^2 \psi(\vec{r}) + \alpha\psi(\vec{r}) + \beta|\psi(\vec{r})|^2\psi(\vec{r}) = 0$$

2nd GL equation Minimize F_s wrt $\vec{A}$

$$\vec{\nabla} \times \vec{H}_{ext}(\vec{r}) = \frac{4\pi}{c}\vec{j}(\vec{r})$$

$$j(\vec{r}) = -\frac{ie\hbar}{2m}\left[\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\right] - \frac{e^2}{mc}|\psi|^2\vec{A}(\vec{r})$$

So, minimizing it psi star r we have del F s equal to minus of h square by 2 m del minus i e by h cross c A of r square psi of r plus alpha psi of r plus beta mod psi r square psi of r and we can still write this as this the other term which is h external which we do not need to worry about.

So, let us just write it as h external square over 8 pi, right now ignore this term it is not that it is not important or it is small because we do not need it for our discussion now. So, we ignore that and this when we put this quantity del F s equal to 0 that would tell that these bracket has to vanish this curly bracket has to vanish for delta F s to be 0 this bracket will be equal to 0 and hence we get the first Ginzburg Landau equation as this.

So, note the difference between the earlier case when we do not have magnetic field is that psi was homogeneous we could mention that here as well that psi is and does not depend upon r basically. So, this is called as a first GL equation and in which we have neglected the next the second term there.

Now to get the second GL equation minimize F s with respect to A the vector potential and that gives a very familiar equation which is called as the amperes law which is like this. So, this is the second GL equation where your j r has to be identified as so, that is the second GL equations.

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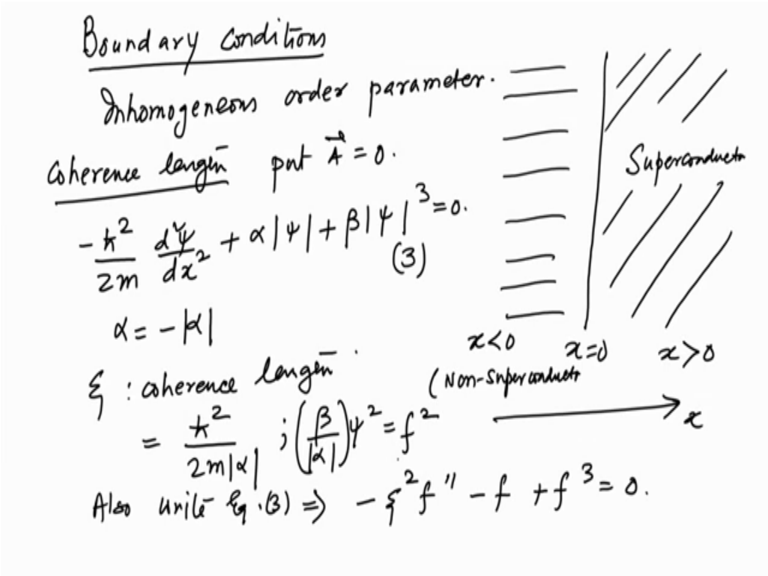

And let us now talk about the boundary conditions and in doing so, we would get or we would sort of obtain the relations or the expressions for the penetration depth and the coherence length. So, let us talk about an inhomogeneous order parameter which we have been talking about since we have introduced the magnetic field.

So, let us consider a half region of so, this is x equal to 0 so, this is where the superconductor exists. So, superconductor exists for x greater than 0 and x less than 0 we do not have a superconductor it could be a normal metal or it could be a magnetic metal for example, so there is a non-superconductor will just simply write it as non-superconductor.

So, these are junction systems which are routinely studied in experimental physics experimental condensed matter physics and one can easily make a junction of a superconductor and a non-superconductor. Now for deriving the expression for coherence length put A equal to 0. So, we do not need the external field there and in that case I get from the first GL equation and the variation is taken to be in one dimension. So, this is the x axis alpha psi plus a beta psi cube this is equal to 0.

So, we have said that earlier that alpha is negative in the superconducting state. So, take alpha to be equal to minus alpha and define a quantity called as xi which is called as the coherence length and this is equal to h cross square over 2 m alpha also this is simple algebra also write equation let us call this as the first one as equation 1 first GL equation, the second GL equation to be equation 2 and this to be equation 3.

So, now write equation 3 in addition define that beta over alpha psi square is equal to f square. So, with these definitions one can write down equation 3 as minus xi square f double prime minus f plus f cube equal to 0. So, I have cast this first GL equation in terms of a scaled variable which is called as a f.

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Now multiply by $f' \left(=\frac{df}{dx}\right)$

$$\frac{d}{dx}\left[\underbrace{-\frac{\xi^2 f'^2}{2} - \frac{1}{2}f^2 + \frac{1}{4}f^4}\right] = 0.$$

$$-\xi^2 \frac{f'^2}{2} - \frac{1}{2}f^2 + \frac{1}{4}f^4 = \text{Const.} \quad [4]$$

Far from the boundary into the Superconducting State,

$$f' = 0 \text{ or } \psi' = 0 \Rightarrow f^2 = 1 \Rightarrow |\psi|^2 = \frac{|\alpha|}{\beta}.$$

Eq (4) becomes,

$$\xi^2 (f')^2 = \frac{1}{2}(1-f^2)^2 \Rightarrow f(x) = \tanh\left(\frac{x}{\sqrt{2}\,\xi}\right)$$

$$\boxed{\psi(x) = \left(\frac{|\alpha|}{\beta}\right)^{1/2} \tanh\left(\frac{x}{\sqrt{2}\,\xi}\right)}$$

Now multiply by a f prime which is equal to df dx and then one should be able to write this as d dx of minus xi square f prime square over 2 minus half f square plus one - fourth f 4 this is equal to 0. So, if d dx of this equal to 0 which means that this is should be equal to a constant. Now far from the boundary that is into the superconducting state. So, this is the boundary and if you are too much into the superconducting state f prime should be equal to 0 or psi prime should be equal to 0.

So, the variation of the order parameter with as a function of x should be equal to 0 and hence this gives that f square should be equal to 1, which tells that psi square should be equal to alpha over beta which is what we have obtained. Then this equation which is equation 4 becomes as xi square f prime square equal to half 1 minus f square, square.

So, this is the equation for psi or f which needs to be solved and the solution is of this form that f of x equal to tan hyperbolic x by root 2 xi. So, that gives that psi equal to alpha by beta whole to the power half tan hyperbolic x by root 2 xi. So, that tells that psi as a function of x has a variation like this. So, that gives the extent of the wave function for the cooper pair or the super electrons and there is a characteristic length that emerges which is equal to xi.

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ξ is the measure of the distance over which the order parameter responds to a perturbation.

ξ is called as the coherence length

Again since. $\alpha = a(T - T_c)$

$$\xi(T) = \left(\frac{\hbar^2}{2maT_c}\right)\left(1 - \frac{T}{T_c}\right)^{-1/2}.$$

$$\xi(T) \text{ diverges as } \left(\frac{1}{1 - T/T_c}\right)^{1/2}$$

Behavior of the coherence length.

So, xi is the measure of the distance over which the order parameter responds to a perturbation. In this case the perturbation is the presence of a boundary that lies between the superconductor and maybe a normal metal again, since alpha equal to a T minus T c the temperature dependence of the coherence length so, this is called as the coherence length.

So, xi of T is equal to h cross square over 2 m a pc and 1 minus T by T c whole to the power minus half. So, xi T diverges as 1 divided by 1 minus T by T c whole to the power whole to the power half. So, as T goes to T c this diverges in this fashion you can see clearly that the divergence is like a square root divergence. So, this is the behavior of the coherence length.

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Penetration depth

The second GL equation (without the first term)

$$j(\vec{r}) = \frac{4\pi}{c}\frac{1}{\lambda_L^2}\vec{A}(\vec{r})$$

$$\lambda_L^2 = \frac{mc^2}{4\pi e^2 |\psi|^2}.$$

The space dependence of ψ will yield the space dependent of λ_L.

$$\lambda_L(T) = \frac{mc^2\beta}{4\pi e^2 a T_c}\left(1 - \frac{T}{T_c}\right)^{-1/2}$$: penetration depth.

Let us look at the other quantity which is the Penetration depth, look at this expression the second GL equation if you drop the first term and only look at the second term that is this term let me just mark it in red. So, if you look at this term then this exactly looks like the London equation this without the first term that is a usual current term.

So, we are purely looking at the current due to the external field and then it looks like that j of r is simply 4 pi by c 1 by lambda l square A of r and immediately the lambda l can be read off as mc square divided by 4 pi e square and mod psi square and this is the vanishing or rather the dependence of the of psi.

So, the space dependence of psi we will yield the space dependence of lambda remember this psi actually falls off as tan hyperbolic x over root 2 xi. So, that will determine that the how lambda l falls off as a distance and to look at the temperature dependence the lambda l which is equal to mc square beta divided by 4 pi e square A T c 1 minus T by T c whole to the power minus half. So, this is lambda l as a function of T this is the penetration depth.

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$\lambda_L(T)$ diverges again $\left(1-\frac{T}{T_c}\right)^{-1/2}$

Define

$$\kappa = \frac{\lambda_L}{\xi} = \frac{mc}{e\hbar}\left(\frac{\beta}{2\pi}\right)^{1/2}$$

$\kappa > \frac{1}{\sqrt{2}}$ type – II Superconductors

$\quad < \frac{1}{\sqrt{2}}$ type – I Superconductors

So, lambda L T diverges again as 1 by T my T c whole to the power minus half exactly like xi of T and one can also define a dimensionless parameter called as lambda by xi which now becomes equal to mc over e h cross beta over 2 pi whole to the power half which has only one unknown parameter which is beta which appears in the Ginzburg Landau theory. Now, we know that kappa greater than a 1 by root 2 is termed as the type 2 superconductors and less than 1 by root 2 the type 1 superconductors.

So, to summarize that without doing explicitly a microscopic theory which we have done for the BCS case here simply writing down the free energy functional as in powers of the order parameter and minimizing it with respect to the order parameter one can actually get the 2 energy scales that are relevant for these superconductors namely the penetration depth and the coherence length and not only that they are how they diverge for T close to T c is also tailed which are of this order as 1 by T minus T c whole to the power minus half.

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<u>Summary</u>

1) Historical introduction
2) Properties – zero resistance
   Meissner effect.
3) origin of the attractive interaction between electrons
   mediated via phonons – Cooper's instability
4) BCS Theory – Superconducting gap
5) Ginzburg Landau theory to obtain
   (i) Coherence length
   (ii) Penetration depth.

So, to conclude the chapter on superconductivity we have given one a historical introduction and then we have said about properties such as Zero resistance, Meissner effect etcetera. And then we have talked about the origin of the attractive interaction between electrons mediated via phonons this is known as Cooper's instability.

And then of course, we have done BCS theory and in which the gap superconducting gap is obtained and the superconducting gap is seen to be non-analytic in powers of the strength of the attractive electron interaction. So, one cannot do a Perturbative theory in order to get this result then we have done a variational theory and also gotten the behavior of the superconducting gap as a function of temperature and then finally, we have done a Ginzburg Landau theory to obtain one coherence length and to penetration depth.

And these are all required for you to learn because this story of super conductivity is fairly old now you understand it is more than it is about 110 years old since it is first discovered and then all also it was the new class of superconductors were discovered in 1986 and a large amount of work had gone in. Since then; however, we have not touched that part because of poor knowledge or still you know evolving knowledge in that particular area, but we have done these studies of the weak coupling superconductor so, called the BCS superconductors in somewhat details.