Advanced Condensed Matter Physics

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Lecture – 20
BCS Theory

Let us look at BCS Theory the Bardeen cooper Schrieffer theory; the first microscopic theory of superconductivity, which was extremely successful in describing the behavior and the properties of the conventional superconductors, which are also known as weak coupling superconductors.

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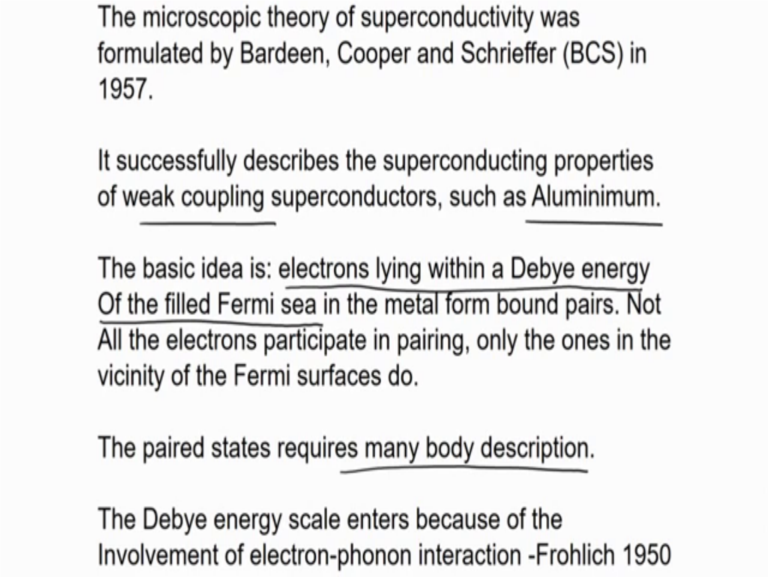

The microscopic theory of superconductivity was formulated by Bardeen, Cooper and Schrieffer (BCS) in 1957.

It successfully describes the superconducting properties of weak coupling superconductors, such as Aluminimum.

The basic idea is: electrons lying within a Debye energy Of the filled Fermi sea in the metal form bound pairs. Not All the electrons participate in pairing, only the ones in the vicinity of the Fermi surfaces do.

The paired states requires many body description.

The Debye energy scale enters because of the Involvement of electron-phonon interaction -Frohlich 1950

So, I said that it was the first microscopic theory and it was formulated by Bardeen cooper and Schrieffer and that is where the name BCS came in 1900 and 57. So, which was approximately 50 years after the discovery of superconductivity experimentally, which as we have seen before that it was done by Kimberling owns in 19 O 8.

This theory successfully describes the superconducting properties of weak coupling superconductors will try to make it clear that what weak coupling the word weak coupling means such as Aluminium and other materials mostly they are metals and not all metals from or give rise to super conductivity, but there are certain metals which give this also will discuss which are the potential candidates for superconductivity.

So, the basic idea is that the electrons lying within Debye energy of a filled Fermi sea ok. So, that 2 electrons are needed and they are lying within the Debye frequency or Debye energy of a filled Fermi sea in a metal they can form bound pairs.

So, this is something that has to be contrasted from the usual behavior of electrons which would repel each other; however, if you can create a situation in which they are in a very close proximity of a filled Fermi sea, the electrons do not interact with the Fermi sea other than by exclusion principle. And then these 2 electrons actually can have an attractive interaction between them and can form bound pairs ah. And it is important to say that not all electrons participate in pairing and only the electrons which are in the vicinity of the Fermi surfaces do not in the say the near the center of the Fermi surface they do not take part in pairing, but it is only which lie in the vicinity of the Fermi surface or close to the Fermi energy they do.

This paired state requires a many body description and this is what is the central focus of our discussion at least for this one and a discussion that is going to be held after this, now this hint that the Debye energy scale is important for this problem it actually signals that there is an involvement of electron phonon interaction and this was proposed by Frohlich as early as 1900 and 50. So, before this theory came up.

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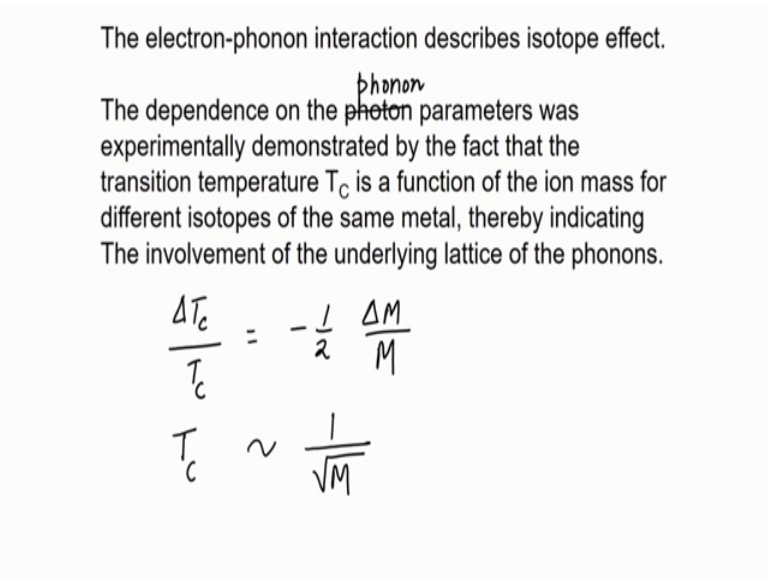

The electron-phonon interaction describes isotope effect.

The dependence on the ~~photon~~ phonon parameters was experimentally demonstrated by the fact that the transition temperature T_C is a function of the ion mass for different isotopes of the same metal, thereby indicating The involvement of the underlying lattice of the phonons.

$$\frac{\Delta T_c}{T_c} = -\frac{1}{2}\frac{\Delta M}{M}$$

$$T_c \sim \frac{1}{\sqrt{M}}$$

Now, this electron phonon interaction describes isotope effect this give me a minute to say what that is. So, the dependence on the sorry this should be phonon, on the phonon parameters were experimentally demonstrated by the fact that the transition temperature we have seen this concept of transition temperature. A temperature that divides superconducting state with that of a non-superconducting one or rather a metallic state the temperature at which this transition occurs is called as a superconducting transition temperature.

So, this transition temperature T C becomes a function of the ionic mass of our different isotopes of the same metal. Now that tells that if that happens then the lattice, because the ions are involved and hence the lattice is involved and then the phonons are involved as well. And this is an experimental demonstration of an electron phonon interaction or the rather the phonons are playing an important role in the whole process.

So, if you want to write this goes as a delta T C over T C this is equal to a minus half delta M over m where M is an ionic mass and delta M is the difference in ionic masses of 2 isotopes, or very simply this is written as a T C to be a function of you know 1 by square root of m where M being the ionic mass.

So, these are experimental demonstration that the phonons are involved and which is also central to the discussion that we are going to have.

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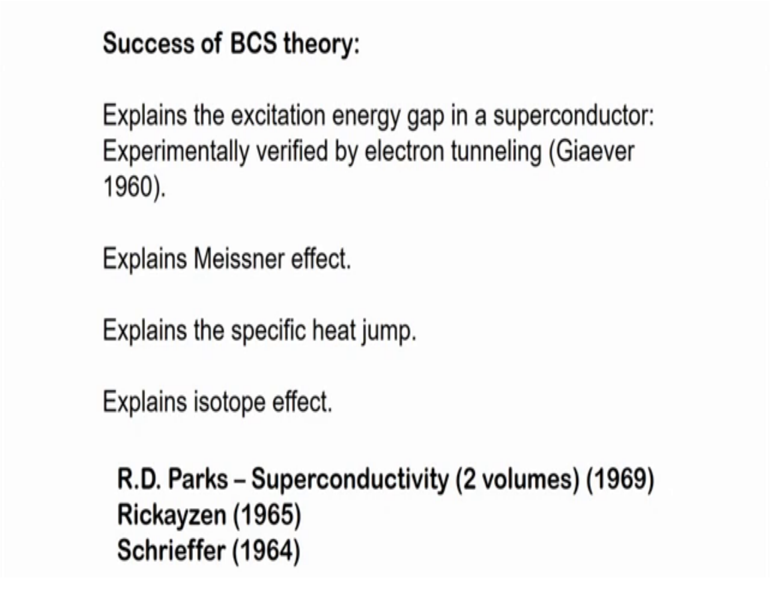

Success of BCS theory:

Explains the excitation energy gap in a superconductor: Experimentally verified by electron tunneling (Giaever 1960).

Explains Meissner effect.

Explains the specific heat jump.

Explains isotope effect.

R.D. Parks – Superconductivity (2 volumes) (1969)
Rickayzen (1965)
Schrieffer (1964)

So, coming to the success of the BCS theory a priori of course, we have not talked about BCS theory yet, but it is important to know that why should we be doing it? In a course like this there are many successes. In fact, it is it gives the excitation energy gap the energy gap. So, called in a superconductor there is certain amount of energy that is required to be supplied to the superconductor in order to break the or rather the break the cooper pairs and promote them to the excited state.

So, this is done experimentally done by the electron tunneling which was done by giver in 1900 and 60 ah. So, when an electron is sent across a metal in metal superconductor junction the superconducting energy gap to delta, it should not have been able to pass through because there is a additional barrier that is there which the electron of which the metallic side does not have.

But, however, there is a process called as a Andreev reflection, which we are not going to discuss here and because of that Andreev reflection there is a pair if not the electron there is a pair that can propagate into the superconducting region and this shows up in a peak, which is called as a un reef peak and this happens just below the energy of the excitation energy of the of the gap of a superconductor. And hence it is it gives these electron tunneling measurements they yield the magnitude of the gap and there are other experiments, which actually probe the symmetry of the gap the symmetry of the gap in the momentum space.

So, to say it explains Meissner effect we have seen extensively what Meissner effect is it explains the specific heat jump, it explains isotope effect as we have seen and the important books in this context are parks R D parks he had 2 volumes of super conductivity, which were published in 1969. And then there is a Rickayzen which there is a book by Rickayzen which was in 1965 and then this book by Schrieffer in 1964 one of the proposers of BCS theory.

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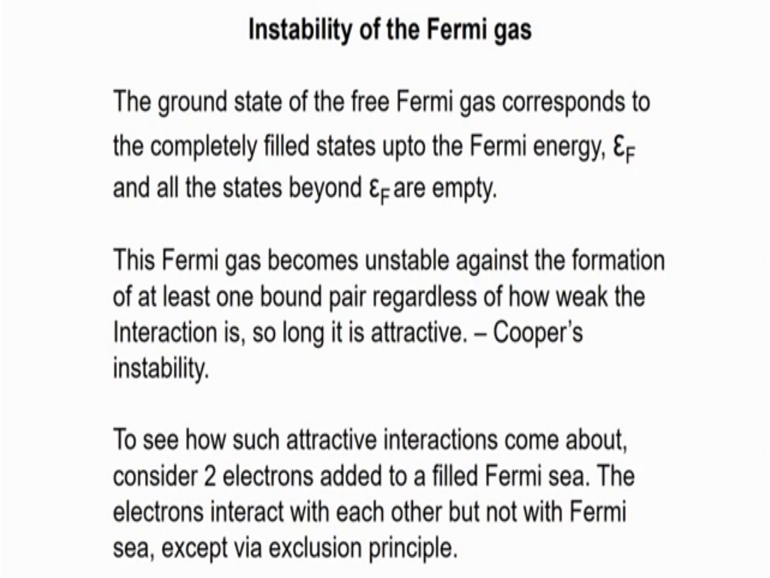

Instability of the Fermi gas

The ground state of the free Fermi gas corresponds to the completely filled states upto the Fermi energy, ε_F and all the states beyond ε_F are empty.

This Fermi gas becomes unstable against the formation of at least one bound pair regardless of how weak the Interaction is, so long it is attractive. – Cooper's instability.

To see how such attractive interactions come about, consider 2 electrons added to a filled Fermi sea. The electrons interact with each other but not with Fermi sea, except via exclusion principle.

So, what exactly is happening how are the electrons interacting with themselves why an attractive interaction, why an attractive potential such that they form bound pairs? And what do these bound pairs have in common with the superconducting state? So, it can be understood as the instability of the Fermi gas.

So, what happens is that the ground state of the free Fermi gas as we said that this corresponds to the filled Fermi sphere corresponds to all completely filled states up to the Fermi level and all states beyond the Fermi energy they are all empty this is the definition of the Fermi distribution function, but this Fermi gas is going to get unstable against the formation of at least one bound pair irrespective of regardless of how weak the interaction is so, long it is attractive and this is was known as coopers instability.

So, to rephrase this we have a field Fermi sea or a Fermi sphere if by some means, what means we are going to discuss just in a while if by some means, that 2 electrons form bound pair in the vicinity of that Fermi sphere lying within the energy scale which is given by the phonon frequency or phonon energy.

Then this Fermi gas the free Fermi gas is going to be unstable and this is called as the cooper's instability. To see how such attractive interactions come about now considered 2 electrons added to a filled Fermi sea, the electrons interact with each other, but not with the Fermi sea except via the exclusion principle ah, one sort of hand waving way of seeing it is that suppose you have a horse moving through a dusty field.

And when it runs through the dusty field it gathers a lot of dust around it and anybody say another horse would not see that horse, but would see a cloud of dust that is moving forward. And pretty much something similar to that is happening that an electron passes through the lattice and because of these the ions being present this picks up in a positive charge that it looks or another electron it looks like a positive charge and they form bound pairs.

Now, it could also be argued that why does not the this other electron also is perceived as a cloud of positive charge, which when they will repel and we will see that this happens at different time scales and that is why there is not enough time that both of them would be like an ionic cloud moving together. And they would repel rather it happens at a shorter life time scale that the electron one electron sees the other ionic cloud and gets attracted to it and forms a bound pair, but this is just a hand waving way of seeing looking at it we will see this more carefully.

So, as has been told that the first hunch or the inkling of the BCS theory what was put forward by cooper, in 1900 and 56 just a year ahead of the discovery of BCS theory.

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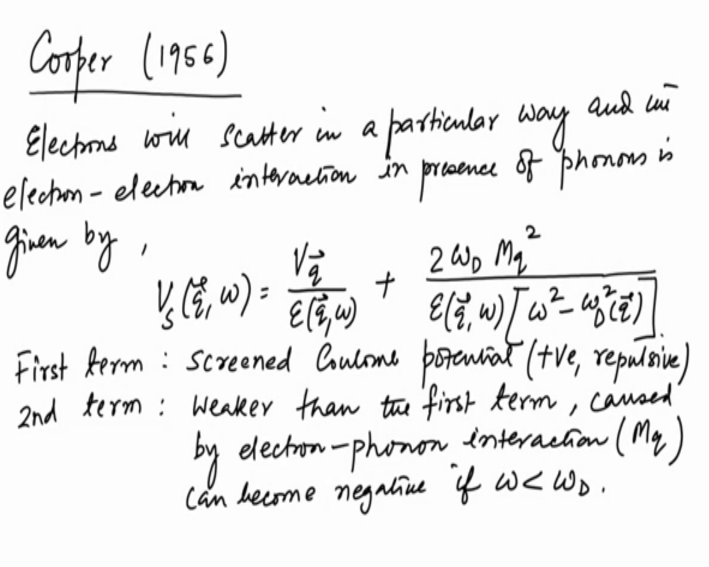

Cooper (1956)

Electrons will scatter in a particular way and the electron - electron interaction in presence of phonons is given by,

$$V_s(\vec{q},\omega) = \frac{V_{\vec{q}}}{\varepsilon(\vec{q},\omega)} + \frac{2\omega_D M_q^2}{\varepsilon(\vec{q},\omega)\left[\omega^2-\omega_D^2(\vec{q})\right]}$$

First term : Screened Coulomb potential (+ve, repulsive)
2nd term : Weaker than the first term, caused by electron-phonon interaction (M_q) can become negative if $\omega < \omega_D$.

So, he pointed out that a normal metal will become unstable to the formation of pairs if the I mean no matter how small the attractive interaction between the pair is and so, he proposed that such pairing would be preceded or would be facilitated by the presence of the lattice and hence phonons.

So, the electrons would scattered in a particular way and the electron interaction in presence of phonons is given by I will explain what all these quantities are. So, the first term is the screened coulomb potential and it is always positive and hence repulsive.

So, in no way that this term can give rise to an attractive interaction the first term. Now the second term is generally weaker than the first term and caused by electron phonon interaction the strength of which is given by M q, now this term can become negative if omega square is less than omega D square. So, in general this, the frequencies lie around the Debye frequency as has been told earlier.

So, in a small range that e in the vicinity of omega D if omega becomes less than omega D the energy denominator becomes small and negative. So, the whole interaction term or this whole second term becomes negative and large and this can cause superconductivity.

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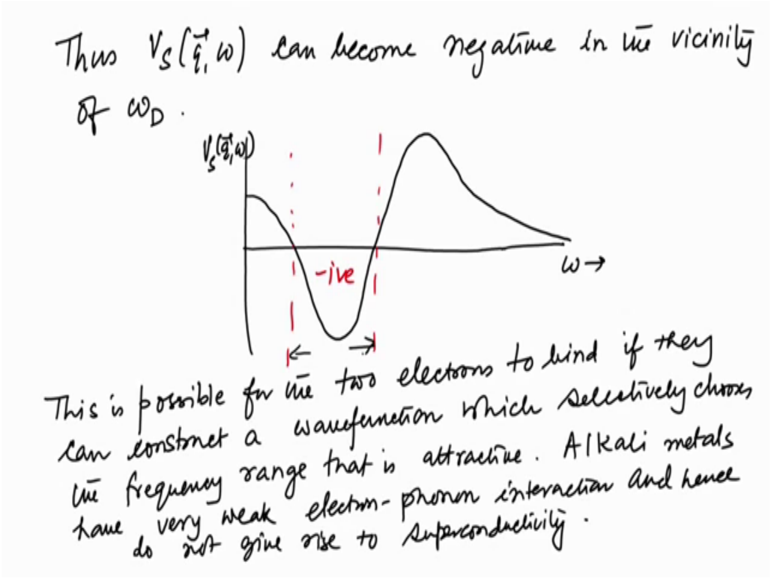

Thus $V_S(\vec{q},\omega)$ can become negative in the vicinity of ω_D.

This is possible for the two electrons to bind if they can construct a wavefunction which selectively chooses the frequency range that is attractive. Alkali metals have very weak electron-phonon interaction and hence do not give rise to superconductivity.

So, it will happen in a thus I mean V S q omega now when the second term actually takes over the first term, the second term being negative then the whole interaction can become negative in the vicinity of the of omega D. So, it is possible that the 2 electrons to bind if they construct a relative wave function which selectively uses a frequency range that is given by. So, if you plot this then it looks like that it is so.

So, this is a V S q omega and this is omega and it is in this range. So, it is in this range that it is negative and one can have an attractive interaction. So, once again to repeat it is that this is possible for the two electrons to bind, if they can construct a wave function which selectively chooses the frequency the frequency range that is attractive. In fact, Alkali metals like sodium potassium etcetera alkali metals have very weak electron phonon interaction and hence do not give rise to superconductivity ok.

So, now so, this the interaction has to be in this regime that is shown here and it is usually is as a competition of two terms; one term being necessarily positive and repulsive which is the screen coulomb term and the other term which arises because of the electron phonon interaction the strength of that being M q, but because of a denominator which is the competition between the energy of the pair versus the energy of the Debye energy.

So, and then since omega is close to omega D omega is measured from the field Fermi sea omega is close to omega D. If it can happen that omega is less than omega d then this term the denominator becomes small and negative and then we can have an attractive interaction and that attractive interaction will actually give rise to super conductivity, but we have not still derived or at least explain that how this interaction term comes from and that can be understood in 2 different ways; one is that rigorously taking an electron photon matrix element I draw all the diagrams and calculate this term the second term particularly. There is another way of doing that which is what we are going to do. So, let us do the origin of ok.

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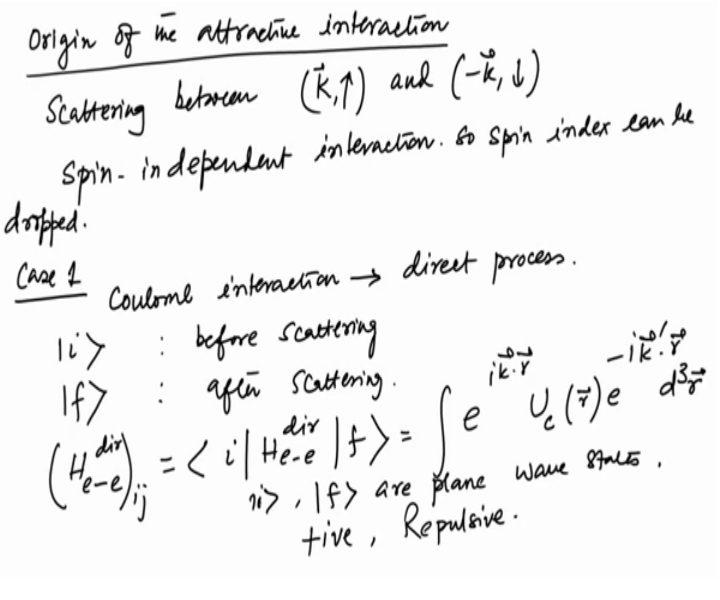

Origin of the attractive interaction

Scattering between $(\vec{k}, \uparrow)$ and $(-\vec{k}, \downarrow)$

Spin-independent interaction. So spin index can be dropped.

Case 1 Coulomb interaction → direct process.

$|i\rangle$: before scattering

$|f\rangle$: after scattering.

$$\left(H_{e-e}^{dir}\right)_{ij} = \langle i | H_{e-e}^{dir} | f \rangle = \int e^{i\vec{k}\cdot\vec{r}} U_c(\vec{r}) e^{-i\vec{k}'\cdot\vec{r}} d^3\vec{r}$$

$|i\rangle$, $|f\rangle$ are plane wave states.

+ive, Repulsive.

Ah. So, we have a scattering between case K up and minus K down electron ah. So, one could ask the question that why is it that one has one electron has a momentum K the other electron has the momentum minus K ah. It can be shown that with the momentums being chosen in this particular fashion a 2 particle state yields the minimum energy which is the most stable configuration.

And for a bound pair to happen because they are in the same state we cannot expect a both up pair to occur or an up down pair to occur. So, the pairing has to be between up and down. So, these are the momentum and spin of the participating electrons which are participating in the bound pair. So, let us see that how these pairing takes place just to say that it is a spin independent interaction and so, spin index can be dropped ok. So, let us say the case one the case one is that that there is a direct electron interaction which is facilitated by the coulomb forces.

So, let us consider an initial state. So, this case one is coulomb interaction and it is a direct process what do I mean by direct process is that there is an initial state I before scattering and there is a final state f after scattering. So, the matrix element of such a electron interaction in such a direct process will just write e and dir let us just give a little of bit of space here and this is written as is i H e e dir and f and this can be written as exponential i K dot r, which is a plane wave initial state and you see that is what we write as coulomb and then exponential minus i K prime dot r and then d cube r.

So, this is the matrix element and this matrix elements is the incoming state momentum. So, the electron comes with a momentum K and leaves with a momentum minus I mean leaves with a momentum K prime this minus sign is coming, because you are taking a ket here whereas, this is a bra here. So, this is exponential I K dot r.

So, we have assumed that both i and f are plane wave states i and f are plane wave states. So, this term cannot give any negative contribution and hence it cannot account for the 2 electrons to form a bound pair. So, this is positive and hence it is repulsive. So, we are not yet there for getting the attractive interaction.

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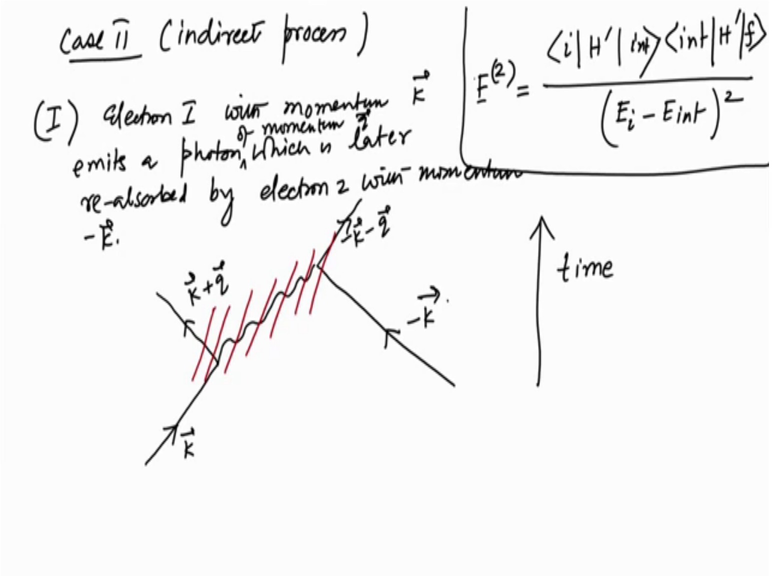
Case II (indirect process)

(I) electron I with momentum $\vec{k}$ emits a photon, of momentum $\vec{q}$ which is later re-absorbed by electron 2 with momentum $-\vec{k}$.

$$E^{(2)} = \frac{\langle i|H'|int\rangle\langle int|H'|f\rangle}{(E_i - E_{int})^2}$$

Now, let us look at a case 2 which is an indirect process it takes place via exchange of phonons and this there is an intermediate process involved and it is a that is why it is a second order process remember your second order perturbation theory, where the way it is written is that there is an intermediate there is an initial state, and then there is a perturbation, and then there is a final state I am sorry there's an intermediate state here, and then there is another matrix element and the final state and then here one gets the energy so, I minus the intermediate energy square.

So, this is the correction let us call it as E 2 in the second order perturbation theory this is just a result that we are going to use. So, there are 2 ways that this indirect thing can take place; one is number one in which the electron one let us just label them even though they are indistinguishable this is for our convenience. So, with momentum K emits a photon which is later reabsorbed by electron 2 with momentum minus K. Let us just say this process once more. So, there is an electron called electron one which is a momentum K it emits a photon, which later at a later time the electron 2 absorbs this which had a momentum minus K and this process would be looked upon as this.

So, this is the K which is electron number one and this is an electron minus K this is electron 2 this one m it is K m it is a photon of momentum q. So, this becomes K plus q and this becomes minus K minus q and because 2. So, this is time axis. So, this is how time increases and we are particularly you know talking about that. So, this is the this is how the time grows.

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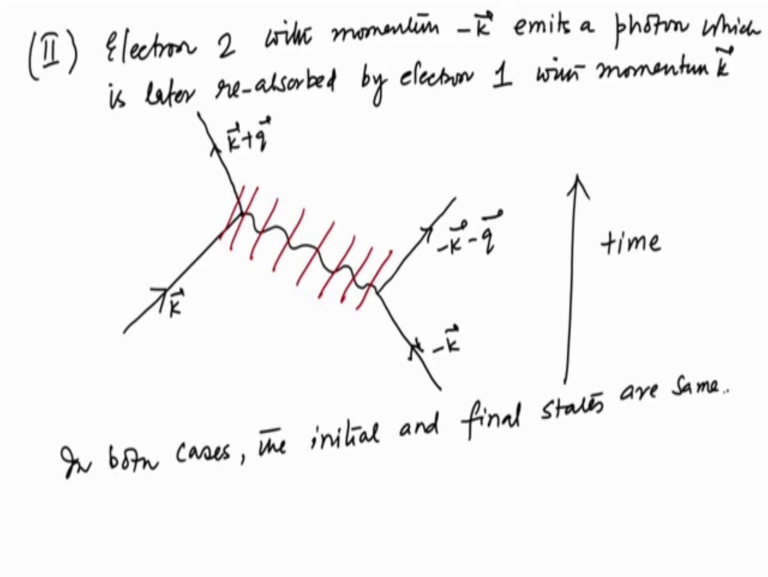

(II) Electron 2 with momentum $-\vec{k}$ emits a photon which is later re-absorbed by electron 1 with momentum $\vec{k}$

In both cases, the initial and final states are same.

So, later on when after it emits it gets reabsorbed. Now the same thing can be thought of for the reverse case that is to is that electron 2 with momentum K, minus K emits a photon which is later reabsorbed by electron 1 with momentum K.

So, this is a similar thing it is just that the alignment of the figure will be a little different. So, they will be K here and it goes like this and now because the process happens in the reverse order. So, we will draw it this way. So, this is electron 1 with K again it is K plus q here and it is minus K and this is minus K minus q.

And we are talking about. So, this is the again the time. So, these are the 2 possibilities of the indirect process that one emits the photon and the other absorbs it and there is a time delay between that and in principle both can occur and both will have to be considered. Now you see that in both cases the initial and final states are same.

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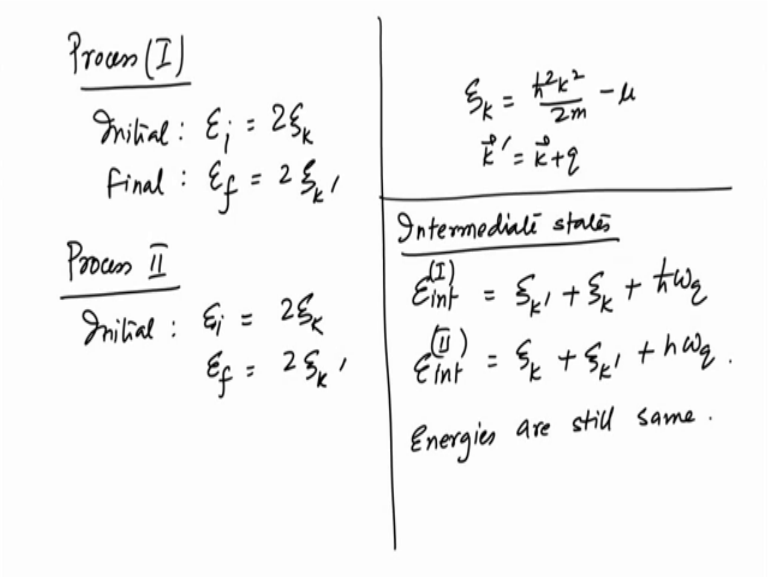

Process (I)

Initial : $\varepsilon_i = 2\xi_k$

Final : $\varepsilon_f = 2\xi_{k'}$

Process II

Initial : $\varepsilon_i = 2\xi_k$

$\varepsilon_f = 2\xi_{k'}$

$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu$

$\vec{k}' = \vec{k} + q$

Intermediate states

$\varepsilon_{int}^{(I)} = \xi_{k'} + \xi_k + \hbar\omega_q$

$\varepsilon_{int}^{(II)} = \xi_k + \xi_{k'} + h\omega_q$.

Energies are still same .

So, let us do this that process 1 process one had initial is epsilon i which is 2s xi K which is epsilon K minus mu ah. So, xi K is equal to h cross square K square over 2 m minus mu and final is epsilon f which is 2s xi K prime and K prime equal to K plus q. So, that is the that is for the process 1 and similarly for the process 2 it is initial is xi 1 equal to 2 xi K and f is equal to 2 xi K prime well I will you know I mean this k’s are vector, but we are ignoring it at the moment. And so, even if the initial and the final states are say I mean the energies are same ah, but the inter I mean the initial and the final states are same, but the intermediate states are different and let us say the intermediate step.

So, for process one the intermediate is equal to xi K prime plus xi K plus h cross omega q and for the second one it is equal to xi K plus xi K prime plus h cross omega q. So, energies are still same of the intermediate state now calculate the matrix elements including these intermediate states.

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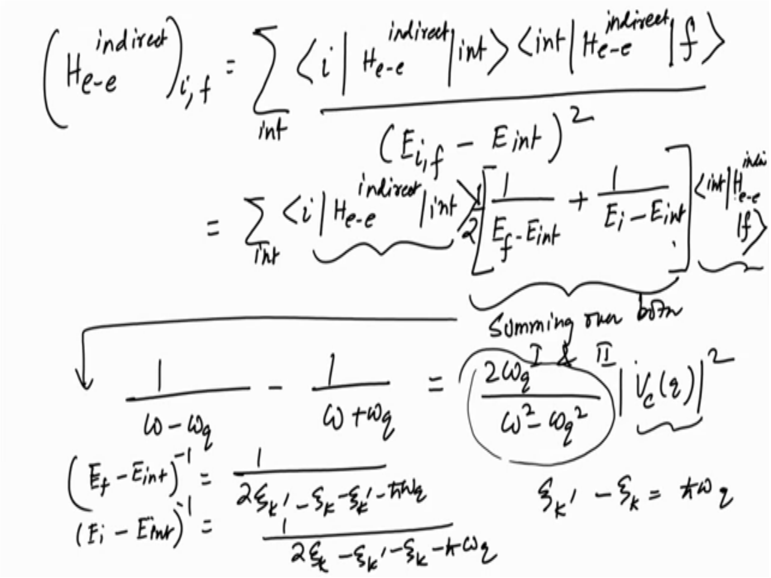

$$\left(H_{e-e}^{\text{indirect}}\right)_{i,f} = \sum_{\text{int}} \frac{\langle i | H_{e-e}^{\text{indirect}} | \text{int} \rangle \langle \text{int} | H_{e-e}^{\text{indirect}} | f \rangle}{(E_{i,f} - E_{\text{int}})^2}$$

$$= \sum_{\text{int}} \underbrace{\langle i | H_{e-e}^{\text{indirect}} | \text{int} \rangle}_{} \underbrace{\frac{1}{2}\left[\frac{1}{E_f - E_{\text{int}}} + \frac{1}{E_i - E_{\text{int}}}\right]}_{\text{Summing over both I \& II}} \underbrace{\langle \text{int} | H_{e-e}^{\text{indi}} | f \rangle}_{}$$

$$\frac{1}{\omega - \omega_q} - \frac{1}{\omega + \omega_q} = \left(\frac{2\omega_q}{\omega^2 - \omega_q^2}\right) \underbrace{|V_c(q)|^2}_{}$$

$$(E_f - E_{\text{int}})^{-1} = \frac{1}{2\xi_{k'} - \xi_k - \xi_{k'} - \hbar\omega_q}$$

$$(E_i - E_{\text{int}})^{-1} = \frac{1}{2\xi_k - \xi_{k'} - \xi_k - \hbar\omega_q}$$

$$\xi_{k'} - \xi_k = \hbar\omega_q$$

So, H r e in direct using the same notation as earlier between i and f you have to sum over the intermediate states and it is i H e indirect, I will just write it intermediate and intermediate H e e indirect indirect means the exchange of phonons. So, they do not directly interact and f and then there is a square of the matrix element now it is either i or f both of them being the same and the square of that. So, this is gives rise 2 and intermediate I H e e indirect intermediate and half of half and one divided by E f minus E int plus 1 by E i plus minus E int and then we have intermediate H e e indirect and f and so, on.

So, these are simply numbers because these are matrix elements. So, this is summing over over both 1 and 2. So, this is written here. It is equal to 1 divided by omega minus omega q, 1 divided by omega plus omega q. So, this becomes equal to 2 omega q divided by omega square minus omega q square and V C q square which is so, in in a essence this exactly looks like the second term that we have gotten with an energy denominator, which is here given by this.

And if you want how this comes then you can look at this thing that E f minus E int inverse it is equal to one divided by 2 xi K prime minus xi K minus xi K prime minus H cross omega q and E i minus E int inverse it is equal to one divided by 2 xi K minus xi K prime minus xi K minus H prime omega q and also xi K prime minus xi K has been used as H cross omega q.

So, this matrix elements are positive definite this one; however, because of this energy denominator and in that narrow range that, we have talked about this could be attractive and hints would give rise to the attractive interaction. Now let us look at the a 2 particle wave function BCS gap equation ok.

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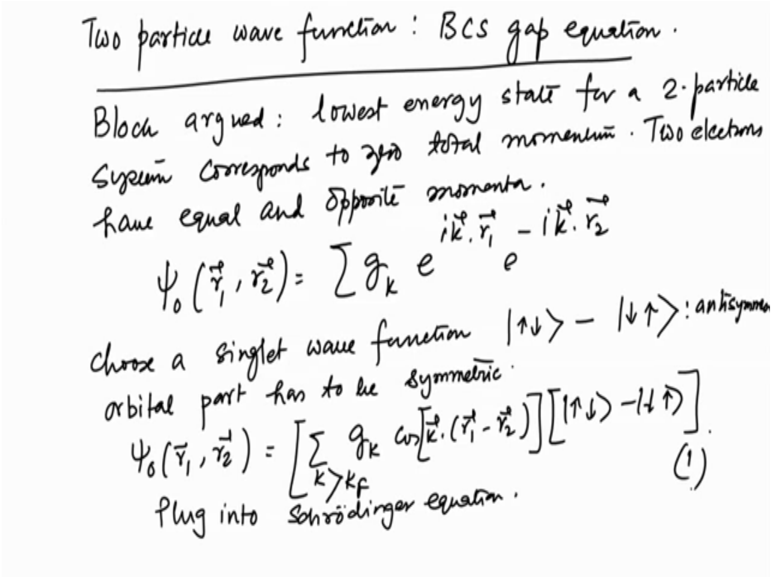

Two particle wave function : BCS gap equation.

Bloch argued : lowest energy state for a 2-particle System corresponds to zero total momentum. Two electrons have equal and opposite momenta.

$$\psi_0(\vec{r}_1, \vec{r}_2) = \sum g_k \, e^{i\vec{k}\cdot\vec{r}_1} \, e^{-i\vec{k}\cdot\vec{r}_2}$$

choose a singlet wave function $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$: antisymm

orbital part has to be symmetric.

$$\psi_0(\vec{r}_1, \vec{r}_2) = \left[\sum_{k>k_F} g_k \cos\left[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)\right]\right]\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]. \qquad (1)$$

Plug into Schrödinger equation.

So, block actually argued lowest energy state for a 2 particle system corresponds to 0 total momentum. So, the 2 electrons are equal and opposite momentum 2 electrons which are taking part in binding. So, we can write down the orbital wave function it is only the orbital part that we are writing it down to have r 1 and the r 2 equal to g K exponential i K dot r 1 exponential minus i K dot r 2 where g K is some weight or the amplitude of the wave function.

Now, when we write the full wave function the orbital part would be symmetric or anti-symmetric depending on the spin part beings anti-symmetric or symmetric, because the full wave function consisting of the orbital part and the spin part will have to be anti-symmetric. So, if we choose symmetric for this then the spin part would be anti-symmetric and if we choose anti-symmetric part for this then the spin part of the symmetric, but remember that we have done this earlier for 2 spins we can have 4 states one of them is called as a singlet state which is an anti-symmetric.

And the other 3 are called triplet states which are symmetric which means that if you interchange the particles then the wave function does not change. So, because the spin part for a singlet wave function is anti-symmetric. So, the orbital part will choose will be chosen as symmetric.

Now, this is also shown that it is generally true for a 2 particle system that the singlet one has a lower energy. So, we take a singlet choose a singlet wave function and BCS theory corresponds to singlet pairing. So, we have the singlet wave function as up down or down up and this is anti-symmetric and because this is anti-symmetric the orbital part has to be symmetric.

So, we take this orbital part as and also we will have to take that K less than I mean K greater than K because we are writing down the 2 particle wave function which is residing just outside the Fermi surface. So, it is g k and the cosine K dot r 1 minus r 2 and then up down minus down up. Now this can be plugged into the call this as equation 1 and this is plugged into the Schrodinger equation.

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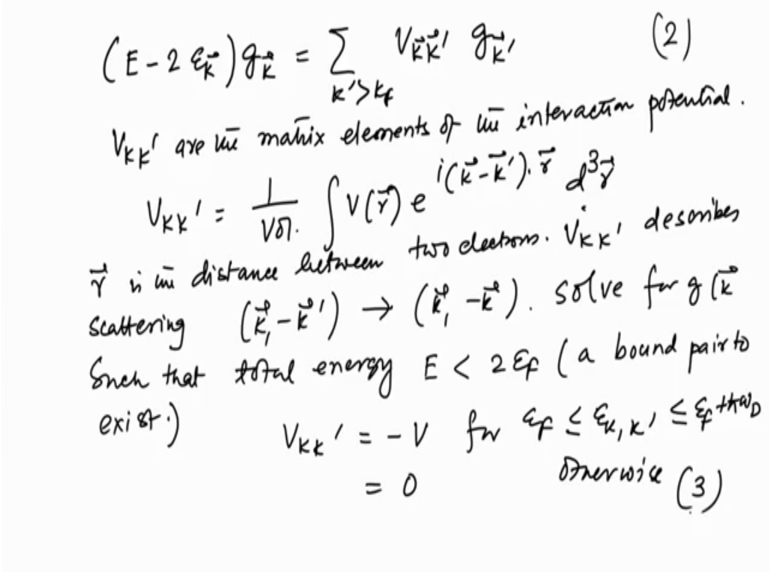

$$(E - 2\varepsilon_{\vec{k}})g_{\vec{k}} = \sum_{k'>k_f} V_{\vec{k}\vec{k}'}\, g_{\vec{k}'} \qquad (2)$$

$V_{kk'}$ are the matrix elements of the interaction potential.

$$V_{kk'} = \frac{1}{V\Omega}\int V(\vec{r})\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, d^3\vec{r}$$

$\vec{r}$ is the distance between two electrons. $V_{kk'}$ describes scattering $(\vec{k}_1, -\vec{k}') \rightarrow (\vec{k}_1, -\vec{k})$. Solve for $g(\vec{k}$ such that total energy $E < 2\varepsilon_f$ (a bound pair to exist.)

$$V_{kk'} = -V \quad \text{for } \varepsilon_f \le \varepsilon_{k,k'} \le \varepsilon_f + \hbar\omega_D$$
$$= 0 \quad \text{otherwise} \qquad (3)$$

So, this is E minus 2 ek and a g K equal to K prime greater than K f. So, used V K K prime and a g K prime. So, all these are K's are of course, vectors. So, V K K prime are the matrix elements of the interaction potential. And the V K K prime is something that we have seen just a while back, but just to keep our discussion simple we will take this as coulomb term or rather you can consider this V r to be arising from that kind of a an interaction which is mediated by phonons.

So, r is the r is the distance between two electrons with momentum K and distance between two electrons and the V K K prime of course, it ah. So, V K K prime describes scattering of K prime minus K prime to K and minus K.

So, we are to solve for this is equation number 2 solve for g K such that such that total energy E is less than 2 E F is basically that is the condition for a bound pair to exist. It is very hard to do this calculation for general potential. So, what you can do is that a V K K prime can be taken as a minus V for epsilon f to be smaller than equal to epsilon K K prime and smaller than equal to epsilon f plus h cross omega D.

So, this is as we have said earlier that it will be overall negative because mediated by phonons of course, that has to be satisfied by the participating electrons, they have to choose the energy range where it becomes negative, but suppose it becomes negative. And the K and K prime for both the electrons would have to lie between epsilon f 2 epsilon F plus h cross omega t it is equal to 0 otherwise.

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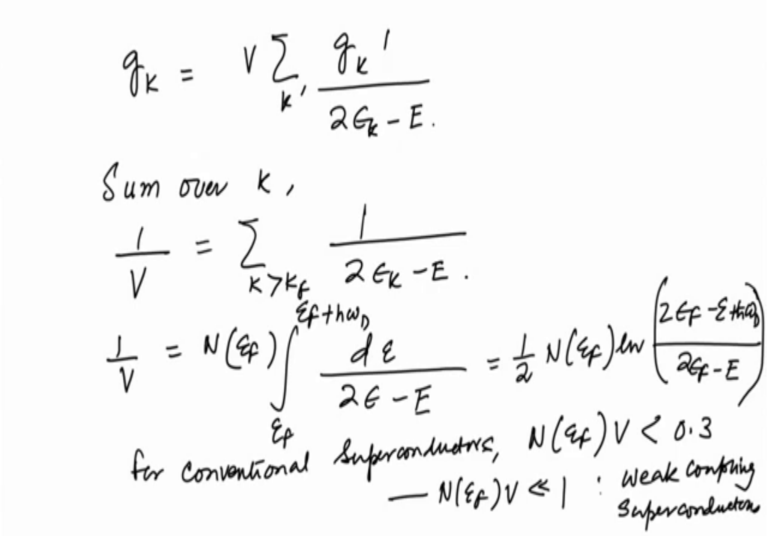

$$g_K = V\sum_{K'}\frac{g_{K'}}{2\epsilon_K - E}.$$

Sum over K,

$$\frac{1}{V} = \sum_{K>K_f}\frac{1}{2\epsilon_K - E}.$$

$$\frac{1}{V} = N(\epsilon_f)\int_{\epsilon_f}^{\epsilon_f+\hbar\omega_D}\frac{d\epsilon}{2\epsilon - E} = \frac{1}{2}N(\epsilon_f)\ln\left(\frac{2\epsilon_f - E + \hbar\omega_D}{2\epsilon_f - E}\right)$$

for conventional Superconductors, $N(\epsilon_f)V < 0.3$

— $N(\epsilon_f)V \ll 1$: Weak coupling Superconductors

Now, with this we have g K if you put it there it becomes equal to a K prime and g K prime and 2 epsilon K minus E. Now sum over K so, it becomes one over V this V is the strength of the interaction and we just reiterate that this V could be infinitesimally small a super conductivity will still occur.

So, this is equal to sum over K greater than K 1 by 2 epsilon K minus E, now we can replace the summation by integration and we had to just invoke the density of states, but as we have told several times that the super conductivity is a phenomena that happens very near to the Fermi surface. So, while we convert the sum into an integral we do not need to take the energy dependence of the density of states rather we can take the density of states at the Fermi level.

So, this is equal to n epsilon f and the integration has to be done not the overall energies, but what is the relevant scale is this and then it is 2 epsilon minus E this epsilon is a running variable this capital epsilon is the 2 particle energy it is N e f once we do the integration it becomes log of 2 epsilon f minus epsilon plus h cross omega d and we have 2 epsilon f minus E.

So, that is the equation that one gets and now for conventional superconductors N epsilon f into V is usually of the order of 0.3 or less than 0.3. So, this is what is meant by the weak coupling approximation which demands that N epsilon f V has to be less than 1 or may be much lesser than 1. So, this is weak coupling superconductors.

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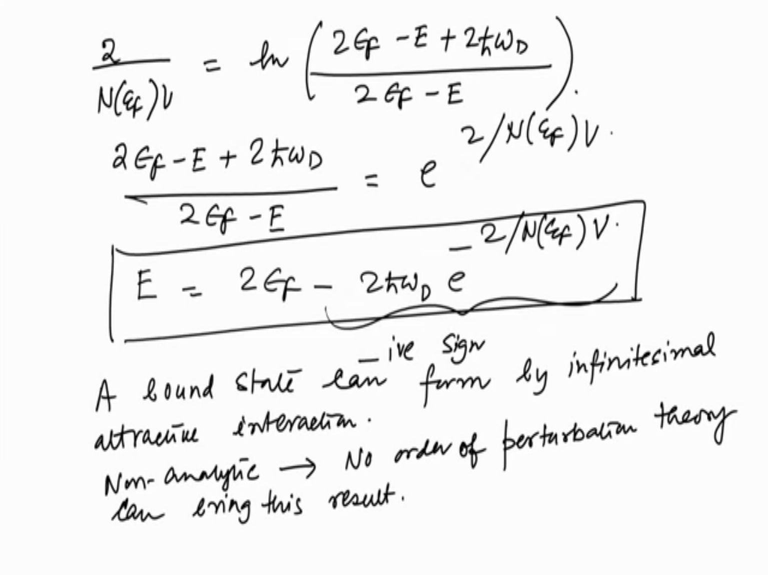

$$\frac{2}{N(\epsilon_f)V} = \ln\left(\frac{2\epsilon_f - E + 2\hbar\omega_D}{2\epsilon_f - E}\right).$$

$$\frac{2\epsilon_f - E + 2\hbar\omega_D}{2\epsilon_f - E} = e^{2/N(\epsilon_f)V}.$$

$$\boxed{E = 2\epsilon_f - \underbrace{2\hbar\omega_D e^{-2/N(\epsilon_f)V}}}$$

−ive sign

A bound state can form by infinitesimal attractive interaction.

Non-analytic → No order of perturbation theory can bring this result.

So, then we have 2 divided by N e f into V equal to log of 2 epsilon f minus E plus 2 h cross omega D divided by 2 epsilon f minus E. And now if I want to free this log we have a 2 epsilon F minus E plus 2 h cross omega D divided by 2 epsilon F minus E and this has to be exponential 2 divided by N E F into V. And if you solve for E it becomes equal to. So, I can I can multiply this here and just doing one more step I would land up with this 2 h cross omega D exponential minus 2 by N E F into V.

So, this is the energy of total energy of the 2 particle system. So, this is with respect to if we take it with respect to the Fermi surface of the 2 electrons this is comes with a negative sign. And this negative sign talks about that the, they have a bound their bound they are in bound state. So, a bound state can form off infinitesimally this is not committed to the magnitude of V any magnitude of V would do.

So, by an infinitesimal attractive interaction also you see that the binding energy is non-analytic it comes as an exponential and hence no orders of perturbation theory, can bring this result. Our perturbation theory in orders or in terms of V the strength of the attractive interaction between the electrons just one more thing that we need to do here is that ah. So, for the wave function we have.

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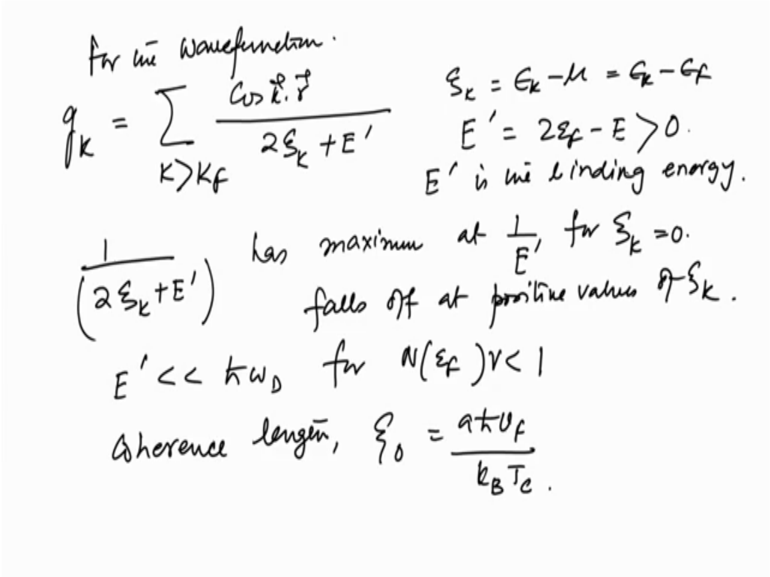

for the wavefunction.

$$g_k = \sum_{k>k_f} \frac{\cos \vec{k}\cdot\vec{r}}{2\xi_k + E'}$$

$\xi_k = \epsilon_k - \mu = \epsilon_k - \epsilon_f$

$E' = 2\epsilon_f - E > 0.$

E' is the binding energy.

$\frac{1}{(2\xi_k + E')}$ has maximum at $\frac{1}{E'}$, for $\xi_k = 0$.

falls off at positive values of ξ_k.

$E' << \hbar\omega_D$ for $N(\epsilon_f)V < 1$

Coherence length, $\xi_0 = \frac{a\hbar v_f}{k_B T_c}$.

The amplitude of the wave function which is g K this is equal to a cosine K dot r divided by 2 xi K plus E prime where xi K equal to epsilon K minus mu, and the which is equal to epsilon K minus epsilon F at t equal to 0 and E prime equal to 2 epsilon F minus E which is of course, greater than 0.

Ah. So, E prime can now be called as a binding energy and this K has to be greater than K F, just a few comments this weighting factor which is given by 1 divided by xi K plus has a maximum at 1 over e prime for xi K equal to 0, which are valid or which are applicable for electrons at the Fermi surface and it falls off at positive values of. So, if you note that the E prime the binding energy is much smaller than the h cross omega D for N epsilon F to be into V to be less than 1 ah.

So, these make sure that the detailed behavior of V K K prime is not important. Also the small energy range allows the estimation of the range of the cooper pairs, which we know is called as the coherence length and that coherence length is given by, this is what has been told earlier and this is of the order of one in BCS theory it is of the order of 0.8 and that gives the good estimation of the coherence length of the cooper pair wave function and that matches with the experimental values which are of the order of maybe 3 to 5000 angstrom.

So, this is a theory which gave the calculation had given rise to a gap the main features of the gap is that it is valid for any strength of the electron electron interaction. So, long is attractive it is a non perturbative the result is non perturbative and hence you cannot do a perturbation theory of any order in order to get this result. And also it allows the estimation of the extent of the cooper pair wave function which is called as the coherence length.