Advanced Condensed Matter Physics

Prof: Saurabh Basu
Department of Physics
Indian Institute of Technology, Guwati

Lecture – 14
Feynman diagram

In the previous class, in the discussion that we had regarding the Feynman diagrams, we had just learned how to write down fully interacting greens function at any order, in terms of the non-interacting greens function. And then preliminary discussion was conducted to write down the Feynman diagrams. So, draw the Feynman diagrams and the rules were stated out now will do the same problem once again, will read on that exercise and write down each one of the terms in the expansion of the greens function in terms of the Feynman diagrams.

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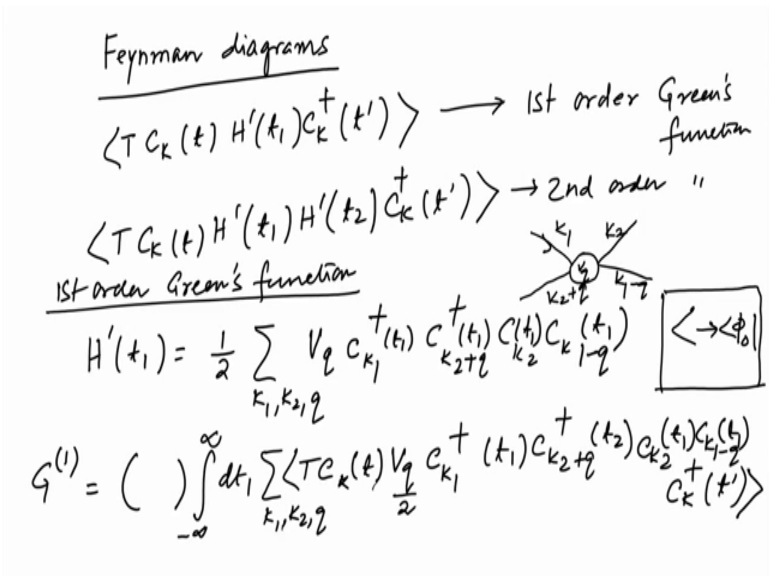

Just to let you know that we have this one C k t H prime t 1 C k t prime. This gives rise to a first order greens function. And similarly, this one gives rise to a second order and so on. This has to be dagger t prime and this is second order greens function and so on.

so, let us write down at least a first order greens function, and let us write it down the as we have seen earlier. Let us just write down a term; which is let us write down the first order itself. And so, my H prime gives some so, H prime t 1, it is equal to say half a k k 1 k 2 and q, and now I have 2 greens functions. So, it is a 2 creation and 2 annihilation operators. So, these are C k 1 plus q dagger C k 2 plus q dagger C k 2 and C k 1.

So, all are the all the creation and the annihilation operators are at time t 1 as written here the half factor is included to avoid double counting. And V q is the strength of the interaction term, which can have some dependence on q as we shall see. So, then the greens function can be written as now I will write it because it is a first order greens function I will write it with a g, with a superscript inside the bracket as one. now I will leave this bracket to be filled in later , where there will be terms such as is some powers of is that will come in, I means the square root of minus 1. And maybe other factors such as maybe minus 1 etcetera that will also come in.

But will try to take that into account. So, this is the minus infinity to plus infinity. And there is a the internal time has to be integrated over. And now I have a k 1 and k 2 and q. And I have a C k time ordered of C k. So, when I write this now, it means, there is a phi 0 which is the non-interacting ground state. So, this is implied so, it is a C k of t, then there is a V q. So, we could have written V q here as well.

so, we can write down the V q let us write it here. So, there is a V q, and then there is a C k 1 plus q dagger t 1, there is a C k 2 plus q dagger k 2 plus q. And a C k 2 p 1 and a C k 1 p 1, and the greens function the definition of the greens function will give me a C k dagger and that is it.

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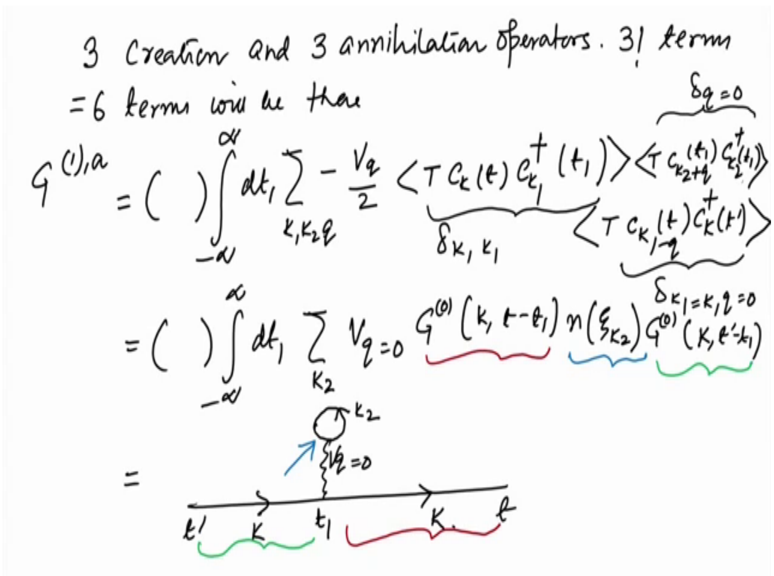

So, this is the first order greens function. And this first order greens function as we because there are 3 creation and 3 annihilation operators.

So, there are 3 factorial terms, which means 6 terms that will be there. And let us just write down the terms. So, the first term will call it as so, G 1 and this corresponds to a, as was written earlier according to certain type of combinations. And as I said that there is some bracket, that I am leaving it out and will fill in later it is minus infinity to plus infinity and a d t 1, and there is a k 1 k 2 and q, there is a minus V q by 2. We have forgotten a factor 2 here, because that came with the definition of H prime.

So, it is a V q by 2, and then there is a time order product of the non-interacting greens function. So, this is C k T C k 1 dagger t 1 and so, this is q, k 1 plus q t 1, this is written somewhat awkwardly. So, this is a q t 1. And then there are this p C k 2, and k 2 plus at t 1 and k 2 plus q at t 1.

And then there is a third term, which is t with a C k 1 t 1 and a C k t prime dagger. So, these are the 3 terms so, this tells that this is equal to so, this is equal to delta k, and k 1 plus q. this is equal to delta q equal to 0, and this equal to delta k 1 to be equal to; now there is an inconsistency that I am saying. So, will redefine and the greens function, give me a moment, let us redefine the greens function as C k k 1.

So, this could be if we redefine the greens function as the interaction term as so, this is k 1, there is a k 2 plus q and there is a k 2, and there is a k 1 minus q. So, that is coming here k 1 minus q. So, then I will have a k 1 so that, and then there is a k 2 plus q then there is a k 1 minus q ok. So, if we redefine the interaction term as this is C k 1 dagger and k 2 plus q and k 2 and k 1 minus q. So, which means that 2 particles are coming, one with a k 1 momentum and the other is k 2 plus q.

And then they interact with a vertex which is V q, and then they go over as a k 2 and k 1 minus q. So, the interaction vertex that retains you q will. So, we have just simply redefined the momentum of the creation and the annihilation operators. And in that case the first order greens function looks like this. And now I will have a term which is not so, this is equal to k 1. And then there is a k 2 plus q, and then there is a k 2 there. So, let me write it down little go neatly. So, this will be t, and then there is a C k 2 plus q t 1, and then there is a C k 2 dagger t 1, now it is fine.

And then there is a C k 1 minus q and then there is a C k. So, this is equal to k 1 equal to k and q equal to 0. This is delta q equal to 0, and this is k equal to k 1 instead of k 1 plus q. So, that is the first term, and this can be written as will again leave this bracket, and there is a minus infinity to plus infinity. And there is a d t 1, and then there is now of course, k equal to k 1. So, we have the independent variable as k 2 q is equal to 0 anyway. So, V q equal to 0, and a G 10 k t minus t 1 , now this one if you see that it is at the same time, and because q equal to 0 it become C k 2 C k 2 dagger.

Now, what can be done is that I can change the sign of this, and can write it with a negative sign and can write it with a number operator which is equal to a psy k 2. And then it is equal to a G 1 0 k. So, these zeroes were written earlier as in the so, this; and now we have k k and t prime minus t 1. So, how would the diagram look like? So, this diagram this particular diagram would look like there is a line which is from t prime to t, there is a momentum which is k, now at t equal to t 1 I have an electron bubble which is momentum k 2, and it at t equal to t 1, and then it continues as k. So, you see that this term; let us write it with a different color. So, this term is here this term is this bubble, and then this term is here ok. And the V q equal to 0 so, this is a q equal to 0, q equal to 0, and that is the so, this is V q equal to 0 is the wiggly line.

So, we have 2 greens functions as non-interacting greens functions. There is a number operator; or rather there is a bubble electron bubble which corresponds to this n psy k 2. And then there is a G 10 ; which is again the unperturbed or the non-interacting greens function, which is taking it from t 1 2. So, this is equal to so in fact, so, there is a t prime to t 1 is actually so, we should do it this labeling properly. So, this is these labeling has to be. So, this corresponds to the green term, that is the last term then this corresponds to the first term, and then there is a bubble.

So, this is the full description of this, and we will have to evaluate in order to evaluate this diagram we will have to compute the integral that is appearing which we are going to do in a short while from now, but let us all I mean write down all these terms which we see here.

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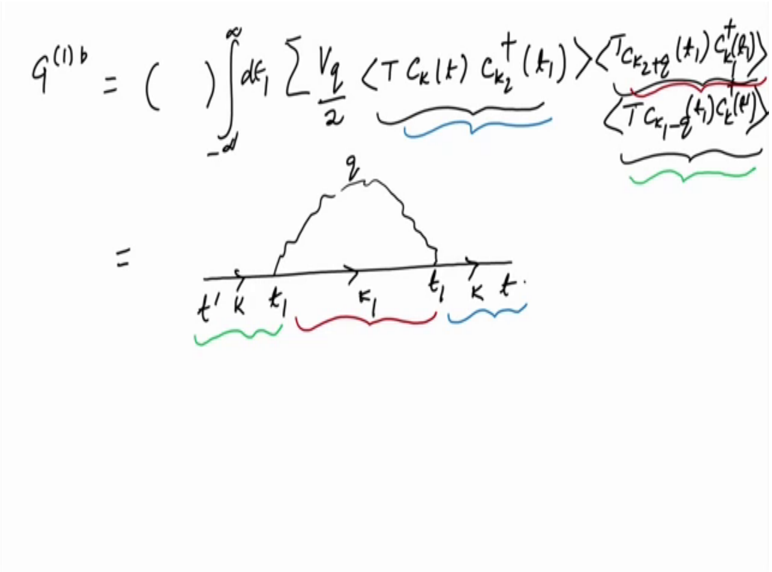

Then of course, there is a term let us call a G 1 a one G 1 now it is a b. So, that looks like again minus infinity to plus infinity d t 1, and then sum over and then we have a V q by 2 V q by 2 not sure whether we put. So, there is a so, there is a half factor that will come out.

So, there is a half there. And so, there is a V q by 2, and now there will be T C k T C k 2 dagger t 1, and T C k 2 plus q t 1, and C k 1 dagger t 1, and there is a T C k 1 minus q t 1 C k dagger t prime. So, this corresponds to a greens function, this corresponds to another non-interacting greens function. And so, there are 3 non-interacting greens function, and this term can be written as so, this term is equal to so, there is a line which goes from t prime to t.

And so, there is a so, there is a time t 1 and so, this is k, and this is equal to k 1, and this is equal to k again, and there is this term that goes from so, this is really this the term that is there. So, I mean this is really t 1; which means that these 2 t 1s though they look different they happen at the same time. So, there is an instantaneous interaction, and the vertex carries a finite momentum q.

And so, again let us do this color; so, this color the green color that we see is a non-interacting greens function that propagates from t prime to t 1 with a momentum that is case because k equal to k 1 minus q. And then there is a greens function that propagates from so, then there is a red one that is here, that is so, that talks about a greens function;

which goes from t 1 to t 1 , but with a momentum that is equal to k 1. So, k 1 equal to k 2 plus q, and similarly the other one has this one is here; which goes from t 1 to t, and the vertex V q, now carries a q a momentum q which is not equal to 0. So, this is b so, let us take the third term.

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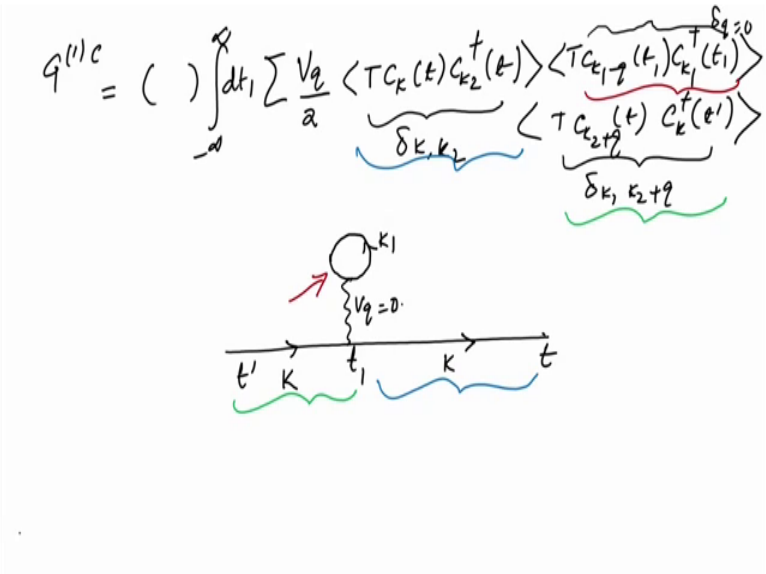

So, G 1 G 1 C that is equal to again this and it is a minus infinity to plus infinity, and we have a so, there is a d t 1. So, there is a d t 1 there is a sum over there is a V q over 2. And now I have terms such as t C k T C k 2 dagger t T C k 1 minus q t 1 C k 1 dagger t 1, and there is a T C k 2 plus q t and C k dagger t prime.

so, that tells me that this gives me delta k equal to k 2, this of course, gives me that this is delta q equal to 0. And this is delta k is k 2 plus q. So, that is that is the, those are the 3 terms. Now this is written as again a line like this and then. So, there is a this term is like the first; term where we have a t prime and a t here, and now this is propagating with k, and this is again propagating with k, this is that V q equal to 0. And this is that k 1 which is happening at t equal to t 1.

So, that is the so, again the last term corresponds to the greens function the free greens function in the. So, showing it once more this is here, this is here, and this is this one ok. So, that is the, those are the 3 terms that appear here.

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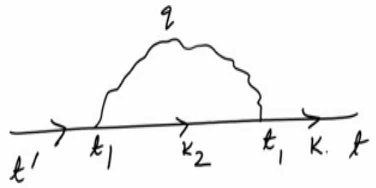
$$G^{(1)d} = (\ ) \int_{-\infty}^{\infty} dt_1 \sum_q V_q \underbrace{\langle T\, c_k(t)\, c_{k_1}^{\dagger}(t_1)\rangle}_{\delta_{k,k_1}} \langle T\, c_{k_1-q}(t_1)\, c_{k_2}^{\dagger}(t_1)\rangle\, \delta_{k_1-q,k_2} \underbrace{\langle T\, c_{k_2+q}(t_1)\, c_k^{\dagger}(t')\rangle}_{\delta_{k_2+q,k}}$$



And likewise, we can write down G 1 d we are doing it all of them (Refer Time: 22:00) to all of them separately such that you get a feel of these things happening there.

So, G 1 d which is equal to again bracket, and then minus infinity to plus infinity and a d t 1 is sum over the internal momentum. So, there is a V q and then there is a T C k t and a C k 1 dagger t 1. So, T C k 1 minus q t 1 C k 2 dagger t 1, and T C k 2 plus q t 1 and C k t prime. Again, that this gives me delta k and k 1, this gives me delta. So, k 1 so, this is equal to this gives me delta k and k 2 plus q k 2 plus q because k 1 is equal to k. So, and this is equal to delta of. So, this k 1 minus k 1 minus q equal to k 2 k 1 minus q is same as k 2. And this is delta k 2 plus q k and this is again, written as t prime t there is a vertex which is with q and this is k 2 and this is t 1 and this is t 1.

So, this the last term is a greens function on the on the left and then there is a term which is then there is a term which is the vertex is carrying a momentum q. And there is a free electron that propagates from t 1 to t 1 with momentum k 2. And then it again carries on with a momentum k from t 1 to t that is those are the 3 terms.

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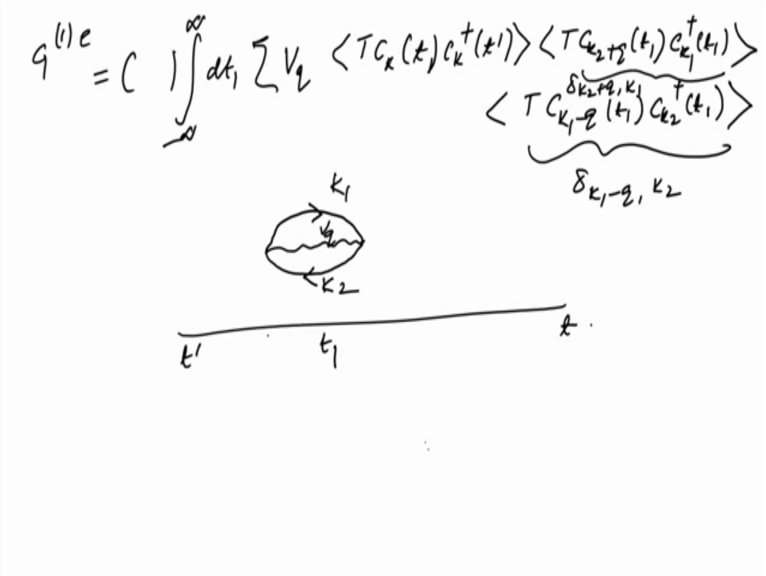

Similarly, let me write down for completeness G 1, e now this is something that is important you should see it. So, then this is equal to again that bracket, and then this d t 1 then there is a summation and then there is a V q.

Now you have a C k t, these groupings were discussed earlier. And we were going by those groupings that we have numbered them as a b C d and so on. So, this C k d dagger t prime. So, C k 2 plus q t 1 C k 1 dagger t 1 and T C k 1 minus q t 1 and C k 2 dagger t 1. So, this tells me that so, there is a free greens function that propagates with a momentum k from t prime to t. This one tells me that there is a delta k 2 plus q has to be equal to k 1.

This one says that, delta k 1 minus q has to be equal to k 2. This 2-momentum conservation means the same thing. So, this can be drawn as this. Now this is a disjoint from so, there is one that propagates from. So, it is like this, and then it is like this. So, there is one that propagates with k 1. And there is a then it propagates with we can do it. So, there is this way this propagates like this. And then there is a wiggly line V q which is equal to here. So, this is k 2, this is t prime, this is t. And this thing is happening at t 1, but these 2 are disjoint, you see that, there is a the greens function that takes from k 1 to k 2 plus q.

So, this is that k 1 to k 2 plus q, and then there is a k 1 minus q going back to k 2. So, this is that the 2 greens functions are shown on top and bottom of that shell-like structure and the wiggly line in between will be the V q which carries the momentum the interaction term that carries the momentum.

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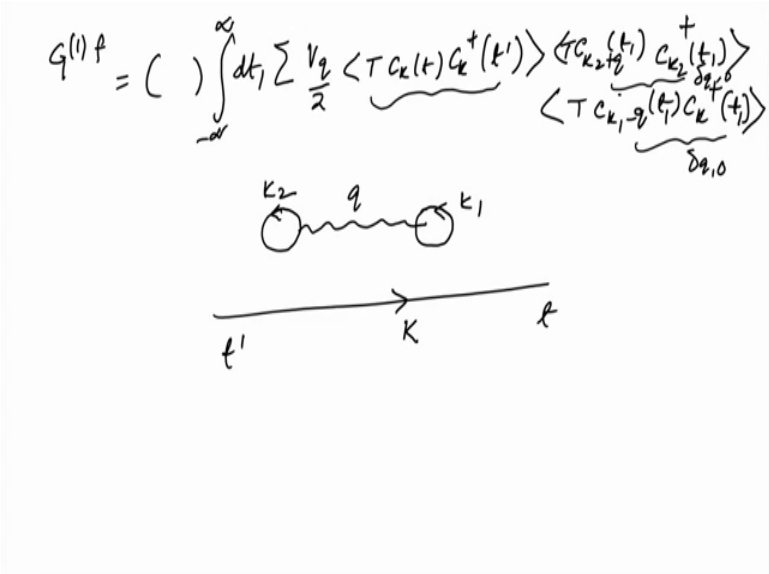

So, this is k, and the last term is G 1 and f; which is again equal to minus infinity to plus infinity. And a d t 1 sum over and then there is a V q by 2 forgetting this 2, but does not matter for drawing the diagrams.

But for getting the final answer this factor of 2 will be important T C k T C k dagger t prime, C time ordered C k 2 plus q t 1, and a C k to t 1 dagger then T C k 1 minus q t 1 , and a C k a dagger t one. So, you see again, this greens function propagates without any interaction or rather without any disturbance from t prime to t. So, will have to write that, there is a t prime to t goes with a k. Now there is there are 2 so, this corresponds to delta I mean the q equal to 0.

And this also corresponds to delta q 0. So, I have 2 electron bubbles which are like this, and they are connected with a q. So, one of them is k 1, the other is this at some intermediate time that happens and at k 2. So, these are the, this is the Feynman diagram for this particular case. So now, you see let us just go back once again. We have a greens function with a bubble, this then there is a there is a greens function.

Which there is there is an interaction vertex which carries momentum q. Then again is the greens function and bubble, then again, a greens function with a vertex carried carrying a momentum q, there is a disjoint diagram there is a disjoint diagram.

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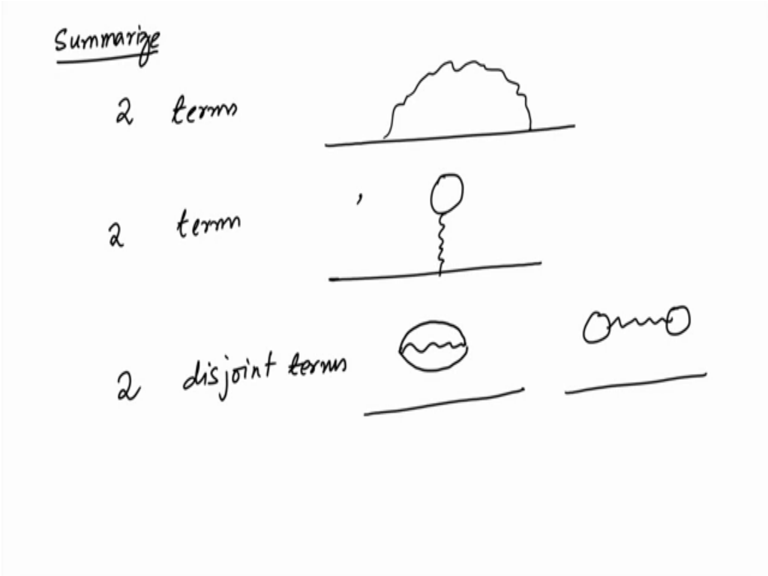

So, if you summarize, so, there are 2 terms which look like this, then there are 2 terms which look like this. And then there are 2 disjoint terms, terms one of them look like this, and so, if you summarize. The other one looks like ok.

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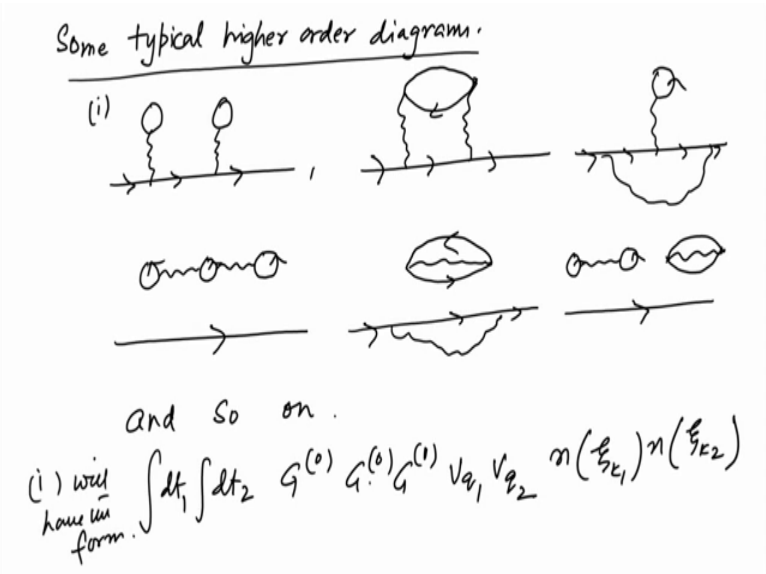

So, now let us see some higher order diagrams, some typical higher order diagrams. So, we can have a term like this. So, at higher order diagram as you understand that will have 2 vertices. So, 2 wiggly lines at least I mean that, that is the second order diagram a third order diagram will have 3 wiggly lines. So, it is one of them then there is one like this.

Then there is one like this, then there are so, then there is one term like this. So, anyway we have 2 wiggly lines and we can have a term like this; which is a disjoint diagram. So, of course, these are the last 3 are disjoint diagrams, and then there are join diagrams, and this and so on there will be many of them have just shown some typical second order diagrams.

So, we can in principle write down, any order for the diagrams, now we just have to write down the corresponding expressions for that, like the first one let us if we call it as one, will include integral, and then there will be 2 time d t 1 and d t 2, and then there will be a G 10, G 10 and 3 G 10s, and there will be a V q 1 and a V q 2, and then there are 2 electron which are probably psy k 1 and psy k 2 just roughly writing down.

So, that is the first term so, one will have the form. So, notice that these diagrams are identical, and similarly these diagrams are identical and so on. So, there are ways to actually find out such symmetries in these diagrams, and I mean calculating one diagram and then multiplying it with the number of such similar diagrams will actually one does not have to compute those many diagrams as actually that come in.

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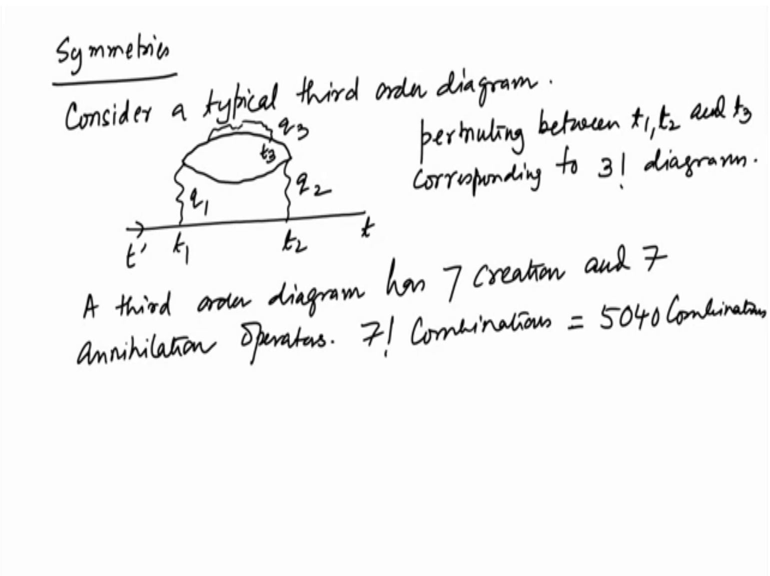

Let me give you an example the symmetry.

So, consider a typical third order diagram. So, I have 3 so, there is a q 3, there is a t 1, there is a t 2, there is a q 1, there is a q 2 and so on. and then there is a t prime, and then there is a t t and ok. So, so, these are that that is a typical diagram. So, then there is a t 3 here.

t 3 add this let me write it down neatly. So, there is a t 3 here so, now, if we permute between t 1 t 2 and t 3, they give rise to the same diagram. So, will have you know 3 factorial such diagrams, in any case I mean the; when we consider a third order diagram, we have a third order diagram has 7 creation and 7 annihilation operators. So, how many combinations we are going to get? 7 factorial combinations.

And how many of them are? So, this is equal to 5040 combinations. This tells that actually we do not have to calculate all the 5040 diagrams. This permuting between t 1 t 2 and t 3 correspond to 3 factorial diagrams. So, which means that corresponding to those 3 factorial, which is equal to 6 diagrams you have to only compute one. And similar such symmetries can actually be understood or deciphered and so, all these things will look the same. And in fact, if you think about it, that there are so, you calculate one diagram, and according to the symmetry you multiply it by n factorial of each one of those diagrams.

And, but there is also a n factorial in the denominator which cancels. So, all these will only correspond to one diagram corresponding to the symmetries that you find.

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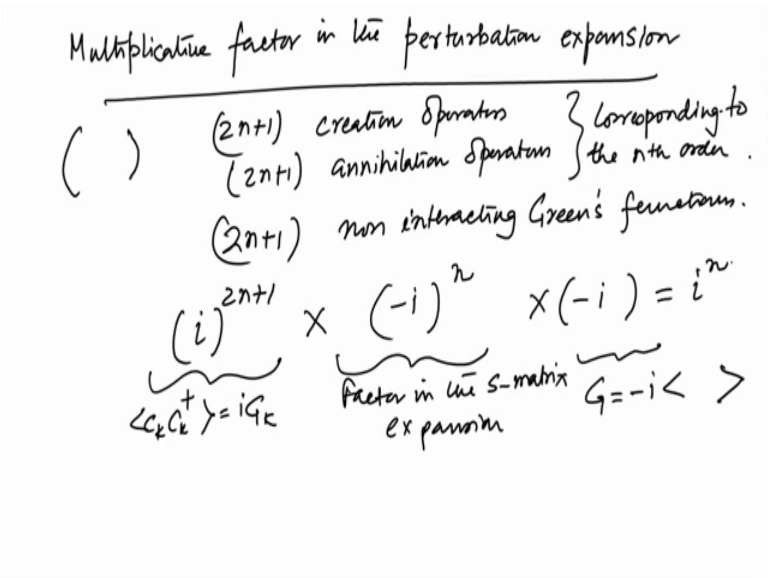

Now, we will talk about what we have been leaving out is the multiplicative factor in the in the perturbation expansion. So, how see how the multiplicative factors expansion. And what I mean is that what we were writing as this and we are leaving it out.

Actually, there will be so, there are, 2 n plus 1 creation operators in the nth order. So, there will be 2 n plus 1 creation operators, and 2 n plus 1 annihilation corresponding to the nth order. So, so, there will be 2 n plus 1 non-interacting greens function.

So, will have so, each of the greens function will have a factor of i. So, there will be a i into 2 n plus 1, because each one of those C k C k dagger it is equal to i G 1k and then will have to multiply it by a minus 1 whole to the power n, this comes from the factor in the s matrix expression. So, this you see that the s matrix expression also has a factor expansion. This has a factor of minus i to the power n. And there is a minus i that comes from the overall interacting greens function because G 1equal to minus i and then the time ordered product.

So, if you multiply all that it becomes i to the power n. So, typically in nth order greens function will have a i to the power n. Since we were talking about a first order greens function, then we have should have a factor of i in each of these cases you can put this factor of i and so on. But that is not the end of the story, there is another thing that comes in.

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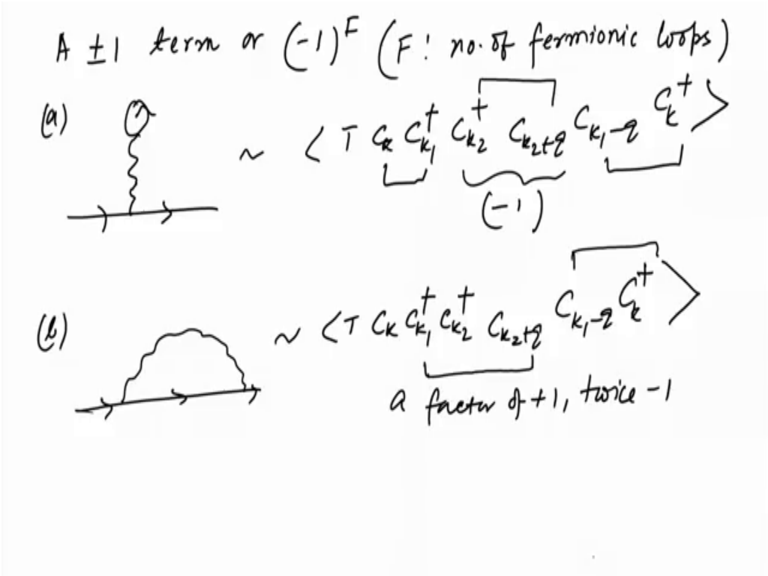

That is a plus minus 1 term or minus 1 whole to the power F where F is the number of F number of fermionic loops.

So, how do we understand that? So, we have a term which is like this, and so on so, this corresponds to t C k C k 1 dagger C k 2 dagger, and then C k 2 plus q C k 1 minus q, and then C k dagger, I am not writing the time indices explicitly. There is this, then there is this. So, if we change the order of this, this brings in of minus 1, and then of course, this remains as it is. So now, this so, basically this corresponds to delta q equal to 0, and one of the fermionic bubbles. So, we get a minus sign.

Look at this other term. So, this was the term a that we have written the number b is this. And so, this is equal to T C k C k 1 dagger C k 2 dagger C k 2 plus q C k 1 minus q C k dagger and so on, now there is a so, this one has to so, this will be one sign, this will be another sign so, a factor of plus 1 of plus 1 that is twice minus 1.

So, anyway one has to keep a track of how many swaps one makes and each time there is a bubble you are sure to get a minus sign if there are 2 bubbles then one gets a plus sign here.

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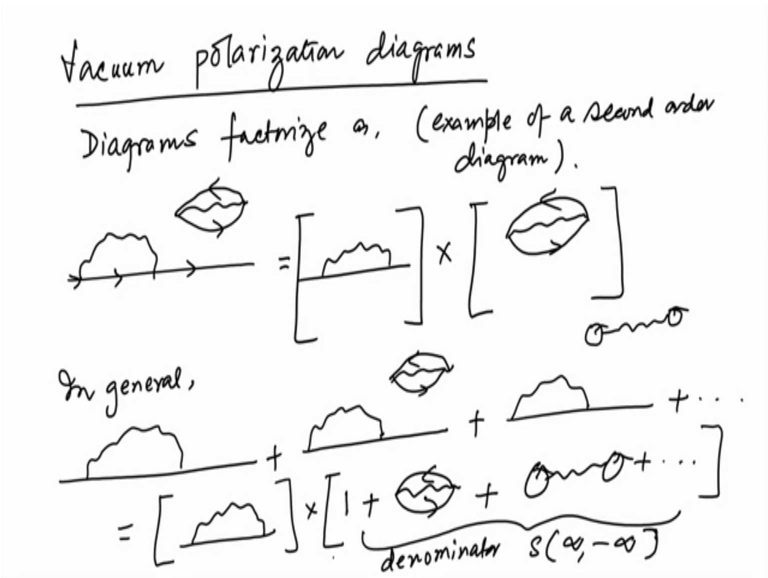

So, let us talk about the disconnected diagrams and see what they contribute. And these diagrams are called as the vacuum polarization graphs. So, at each order of the perturbation theory, all the pre-factors that we get while doing the expansion they are all multiplicative.

So, there is a numerical factor that n factorial which comes from the symmetry of all the diagrams, they cancel with the denominator, and we have a factor of you know i to the power n and a plus minus 1 depending on how many fermionic bubbles we have. And this means that the diagram actually the factorize as take an example for example of a second order diagram.

So, this is a disconnected second order diagram, and this can be easily understood that, this is does factorized as a term which is this, and multiplied by with the term which is this. So, so, in general, this plus this, and then bubble like this, and plus this, and 2 bubbles like this and so on. Can be written as so, this is equal to a term like this, and then multiplied by a term which is 1 plus this kind of a plus there are bubbles and so on.

But remember this is nothing but the denominator, which is equal to s infinity minus infinity.

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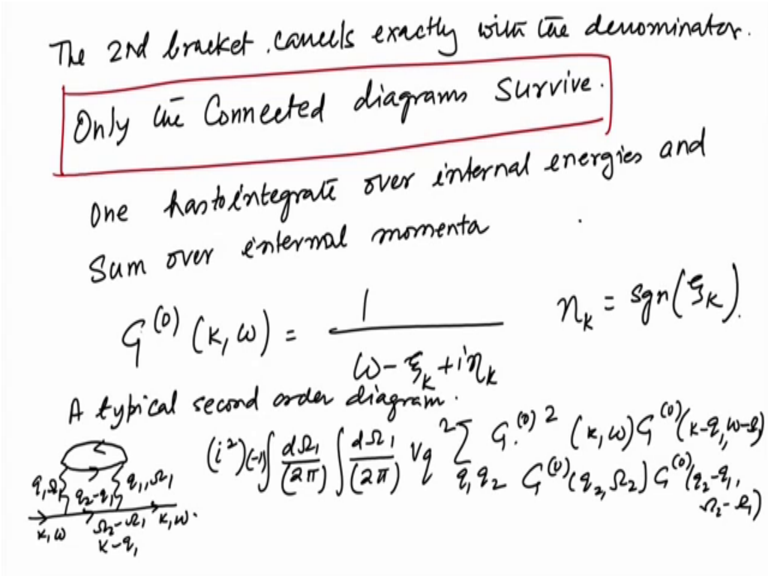

So, this cancels exactly the second bracket or the the second bracket cancels exactly with the denominators.

So, from this factorization, it is clear, that only the connected diagrams survive. Now this is a big simplification, because this simplification tells us only the connected diagrams will have to be computed, and we really do not have to compute terms such as such as these ones or these ones or these ones and so on. So, the top diagrams will be good enough to calculate and we can do away with that.

So now, how to compute we have not still yet done, the integrations over the time. So, one important thing in a is that one has to has to integrate over internal energies and sum over internal momentum. So, then each of the greens function is written as which is I'm writing it now in the k omega space, it is equal to 1 minus psy k, and a plus i eta k, where eta k was shown to be equal to a sign of this.

So, a second order a typical second order diagram, which is equal to like this and this and this. So, there is a q 1 and omega 1.

There is a q 1 omega 1. So, this is omega 2 minus omega one k minus q 1. And this is k omega and so, this is q 2 minus q 1 so, q 2 minus q 1 and so on. So, these are k omega. So, that diagram has to be written now it is a second order diagram. So, we should put a i square, and then there is a d omega 1 by there is a 2-pi normalization that one has to use.

And there is a d omega one there is again a 2 pi that has to be used, and there is a V q square, and then there is a sum over q 1 and q 2. And then there is a G 10 square k omega and a G 10 k minus q 1 omega minus omega 1 and G 10 q k omega 2 G 10 q 2 minus q 1 omega 2 minus omega one and so on.

So, this are the and there is a loop that is forming, and this loop will give rise to a minus 1. This one has to calculate with this G 1 0 etcetera. Put there and they do a method of complex integrals which you must have seen in complex mathematical physics, course a under this complex analysis and complex integrations.

And that has to be done and then, one has to evaluate these integrals accordingly. And that will give us the full greens function evaluated at a given order. For example, in this case we are evaluating it at the second order.