Advanced Condensed Matter Physics
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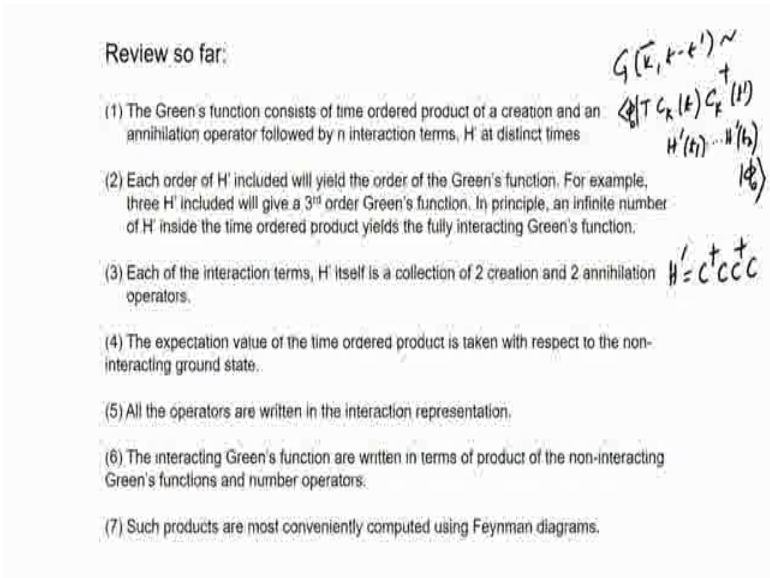
Lecture – 13
Green's function and Feynman diagrams



Review so far:

(1) The Green's function consists of time ordered product of a creation and an annihilation operator followed by n interaction terms, H' at distinct times

(2) Each order of H' included will yield the order of the Green's function. For example, three H' included will give a 3rd order Green's function. In principle, an infinite number of H' inside the time ordered product yields the fully interacting Green's function.

(3) Each of the interaction terms, H' itself is a collection of 2 creation and 2 annihilation operators.

(4) The expectation value of the time ordered product is taken with respect to the non-interacting ground state.

(5) All the operators are written in the interaction representation.

(6) The interacting Green's function are written in terms of product of the non-interacting Green's functions and number operators.

(7) Such products are most conveniently computed using Feynman diagrams.

$G(k, t-t') \sim \langle \phi_0 | T\, C_k(t)\, C_{k'}^{\dagger}(t')\, H'(t_1) \cdots H'(t_n) | \phi_0 \rangle$

$H' = c^{\dagger} c^{\dagger} c\, c$

So, let us have a quick review on the topics that have been discussed, so, far in the context of Green's functions. So, the Green's function consists of time order product of a creation and an annihilation operator followed by n interaction terms, which are all written at distinct times that is the first point. So, this was just to say that it is of the form of a phi 0 and T and then a C k t and a C k prime t prime dagger, and then there are H primes or H interaction terms that are at T 1 and all the way up to h prime t n and this is the expectation value of the time order product is taken.

So, this is 1 annihilation term and the there is a creation term and this thing equal to. So, this along with a minus i is same as is same as G k let us say that let us make this same. So, it is G k a and p minus t prime.

So, this is what it is proportional to with a factor of minus i will see all that in a short while. Now each order of H prime included here in this expression will give a sort of that order of the Green's function. So, if I include 1 H prime I will get a first order Green's function, if I include 2 h primes I will get a second order Green's function and so, on.

So, these are in principle we have infinite number of these H primes. So, a full Green's function contains infinite number of H primes, what are the implications of a each of these H primes is what we are going to come next?

So, a fully interacting Green's function contains a 1 creation and 1 annihilation and n of those H primes. So, each of the third point is each of the interaction terms H prime itself is a collection of to creation and to annihilation operators. This is what we have learned when we are dead second quantization, that each interaction term at least as 2 creation and 2 annihilation operators hm.

So, there could be higher order of interaction terms, but we are mostly going to be talking about interaction terms, where 2 particles come and interact and they exchange their you know spin momentum etcetera and they go on as 2 different particles there could be you know production of particles as a result of scattering, but let us not talk about that immediately will sort of only be constrained with each of these H primes having 2 creation and to annihilation operators.

So, H prime has C dagger C and C dagger C with a momentum and spin indices as case requires. So, the expectation value of the time ordered product is taken with respect to the non-interacting ground state, this is what we have learned it is actually the ground state of the many body system, but then we learned how to construct a non-interacting ground state from an interacting many body ground state, and we have shown that there is a phase factor that comes, which is which connects the many body ground state as well to the to that of the non interacting ground state. And properly done we can simply take the time ordered product to be taken between or the expectation to be taken between the non-interacting ground states.

All the operators point number 5 all the operators are written in the interaction representation. This is important, because these Green's function is fully in the interacting interaction representation, in which both the wave functions and operators they carry the time dependence. Somehow it is only important for us that the operators carry the time dependence, but nevertheless we should write those operators the creation and the annihilation operators to be in the interaction representation.

In fact, the difference between the interaction representation and the Heisenberg representation is that that the Heisenberg representation has similar time evolution as that of the non-interacting system for the interaction representation.

So, now the non-interacting so, the interacting Green's functions are written in terms of the products of non-interacting Green's function at the number operators. This is done by doing a normal order product and each of the or rather the interacting of the full Green's function is split up into several of the non-interacting Green's functions. And all the number operators and then of course, the interaction vertex what we mean by that is, what we are going to come next the interaction vertex, will appear and or the strength of the interaction will appear and the fully Green's function will contain information about the nature of the interaction vertex or their strength of the interaction?

Such products are most conveniently computed using Feynman diagrams. So, we are next going to discuss Feynman diagrams, but before that let us write down 1 example of a Green's function and see that what it means or how it can be split up decomposed into the non-interacting Green's function?

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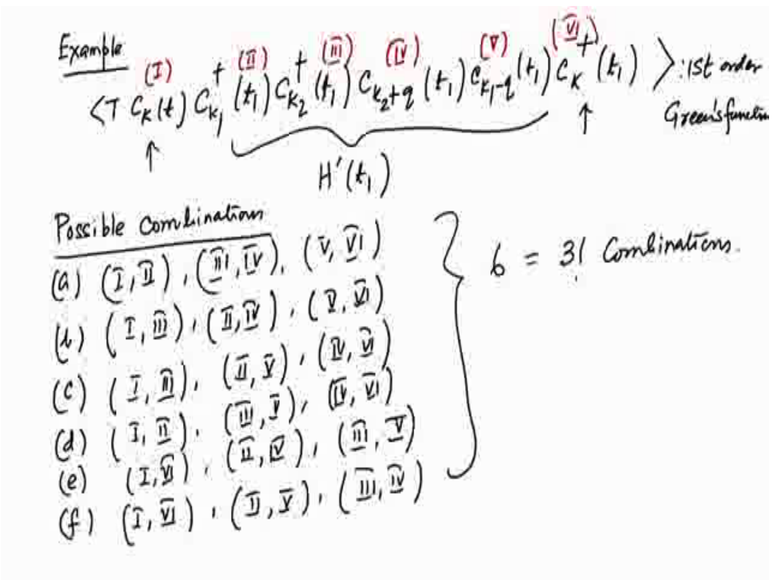

So, we start with an example and let us write down apart without those factors of minus i etcetera will simply write down a time order product and this means this left angular bracket means that there is it is taking with respect to phi naught 1 can write it also explicitly, but it is understood.

So, it is a C k t C k 1 dagger t 1 C k 2 dagger t 1 and C k 2 plus q plus C k 2 plus q t 1 and C k 1 minus q t 1 and C k dagger t prime and so, this is the 1 that we are going to have.

So, let us just write it as so, there will be nothing here and there will be nothing here, but as I said the angular brackets denote that we are taking the expectation with respect to the non-interacting Green's function. Now what we have done is that we have simply written this as the simply this is the H prime t 1. So, we have written down a first order Green's function containing 1 H prime 1 1 annihilation operator here and a creation operator here.

So, that is our, that is the term that we need to calculate. So, let us write down or rather number these operators so, that we can take the proper combination in order to write the non-interacting Green's function. And so, this is 1 this is 2 let us let me write a different color for that. So, that. So, this is 1, this is 2, this is 3, this is 4, this is 5, and this is 6.

So, what are the combinations allowed? So, possible combinations so, a will be a is 1 2 3 4 and 5 6 that is combination number 1. So, we can take a combination of 1 and first and second term third and fourth term and 5th and 6th are also we can take first and third term.

So, the combinations are being formed by 1 creation and 1 annihilation operator. So, 1 3 2 4 and 5 6 so, we have a third combination, which is 1 3 2 5 and 4 6. So, that is the third combination and d to be 1 2 3 4 or rather 3 5.

1 2 3 5 and 4 and 6 fifth combination is 1 and 4 3 and or 2 and 4 1 and 6 sorry 1 and 1 and 6 2 and 2 and 5 and 3 and 3 and 4, so, 1 and 6 2 and 4 and 3 and 5.

So, this is the fifth combination and there will be another 1 there will be another one, which is 1 6 and 2 5 and 3 4 ok. So, these are the 6 combinations and the 6 combinations are coming, because of we have 3 operators 3 creation operators and 3 annihilation operators. So, there are 3 factorial combinations possible. So, all these 6 combinations for just for the first order Green's function have to be evaluated.

So, let us first write down each one of them and specifically we shall take some time and treat this as a tutorial such that you learn, these decoupling and writing it writing the full brains function the first order Green's function as in terms of the non-interacting Green's function.

So, the first term is according to the combinations prescribed 1 2 3 4 and 5 6. So, will take a combination of C k t C k 1 t 1 then C k t 2 dagger and C k a 2 and then C k 1 minus k, and then C k dagger remember that that is not going be the Green's function unless you multiply it is going to be a non-interacting Green's function if you multiply it with a factor of i, because the Green's function itself is defined with a factor of minus i.

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$$(a)\quad \underbrace{\langle T c_k(t)\, c_{k_1}^{\dagger}(t_1)\rangle}_{\delta_{k,k_1} G^{(0)}(k,\, t-t_1)} \underbrace{\langle T c_{k_2+q}(t_1)\, c_{k_2}^{\dagger}(t_1)\rangle}_{\delta_{q=0}\, n(\xi_k)} \underbrace{\langle T c_{k_1-q}(t_1)\, c_k^{\dagger}(t')\rangle}_{\delta_{q=0}\, G^{(0)}(k_1,\, t_1-t')}$$

$$(b)\quad \underbrace{\langle T c_k(t)\, c_{k_2}^{\dagger}(t_1)\rangle}_{\delta_{k_1 k_2} G^{(0)}(k,\, t-t_1)} \underbrace{\langle T c_{k_2+q}(t_1)\, c_{k_1}^{\dagger}(t_1)\rangle}_{\delta_{k_1=k_2+q}\, n(\xi_{k_1})} \langle T c_{k_1-q}(t_1)\, c_k^{\dagger}(t')\rangle \quad \delta_{k_1-q,k}\, G^{(1)}(k,\, t_1-t')$$

$$(c)\quad \langle T c_k(t)\, c_{k_2}^{\dagger}(t)\rangle \langle T c_{k_1-q}(t_1)\, c_{k_1}^{\dagger}(t_1)\rangle \underbrace{\langle T c_{k_2+q}(t)\, c_k^{\dagger}(t')\rangle}$$

$$\delta_{k,k_2}\, n(\xi_k) \quad \delta_{q=0}\, n(\xi_{k_1}) \quad \delta_{k,k_2+q}\, G^{0}(k;\, t-t')$$

$$(d)\quad \underbrace{\langle T c_k(t)\, c_{k_1}^{\dagger}(t_1)\rangle}_{\delta_{k,k_1} G^{(0)}(k,\, t-t_1)} \langle T c_{k_1-q}(t_1)\, c_{k_2}^{\dagger}(t_1)\rangle \langle T c_{k_2+q}(t_1)\, c_k^{\dagger}(t')\rangle$$

$$\delta_{k_1-q,k_2}\, n(\xi_{k_2}) \quad \delta_{k,k_2+q}\, G^{(0)}(k,\, t_1-t')$$

So, a becomes equal to T C k t and C k 1 dagger t 1 right and so, this one and then it is t C k 2 plus q t 1 C k 2 dagger t 1 and a C time ordered of C k 1 minus q t 1 and C k dagger t prime and these are the 3 combinations that I get.

And so, I will have to multiply each one of them by a negative sign and also do not forget to bring in the negative sign that comes, because of the swapping of the operators that is very important. So, setting that aside for the moment we would write this 1 as delta k k 1 and G 0 k t minus t 1 ok. So, just remind you that a factor of I is what we have dropped so, far, but will get it back once when we do more thorough calculations. So, this is the first term. So, your k has to be equal to k prime that is the conservation says and this is a G 0 with order Green's function which is a function of k and t minus t 1.

Now, this one necessarily demands that because k 2 plus q has to become equal to k 2, which means that this is necessarily a delta q equal to 0 and a delta q equal to 0. And now this is going to be a number operator, because it is at same time at time t equal to t 1. So, it is not a propagator or a Green's function that propagates the system from sometime t 2 sometime t prime. So, it just loops back to the same time. So, will write it with a n and this ispsi k, where xi k could be epsilon k minus mu where epsilon k is this other single particle energies.

Similarly so, this is the second term and similarly the third term is here. So, that is again equal to delta q equal to 0 and a G 0 k 1 and t 1 minus t prime. This is how we have now written it the full Green's function at the first order in terms of the non-interacting Green's function G 0 and a number operator which is n of xi k and another G 0, which is there? And how to compute this is what we will see and that is where the Feynman diagrams play a role, but before that let us writes down all the terms. The second term so, again I want to remind you that I am treating it as a tutorial you should do each and every term by hand and satisfy yourself that this is what is coming?

Because this forms the basis of what we are going to do next with the Feynman diagrams. Similarly for b we have T C k t C k to dagger t 1 T C k 2 plus q t 1 C k 1 dagger t 1 and T C k 1 minus q t 1 C k dagger t prime.

So, this tells that it is equal to delta k k 2 sorry this is delta k and k 2 naught k prime and G 0 k t minus t 1. And similarly this one will have delta of k 1 equal to k 2 plus q and this will be simply a G 0 and k 1 and again this will give me a so, this is from t 1 2 t 1, which means it is time that if the time is repeated provided I have written it correctly yeah that I have written it correctly.

So, this will be, but this is not at and given q equal to 0. So, this will be like G 0 and k 1 and this will be like a t 1 k 2 plus q will come back to this. And will have a delta k 1 minus q is equal to k and there is a G 0 and there is a k and there is a t 1 minus t prime. And so, these ones so, let us write down the number C, T C k t C k 2 dagger t C k 1 minus q t 1 C k 1 dagger t 1 and T C k 2 plus q t C k dagger t prime and so, on.

So, this tells me that this is equal to delta k k 2 again this is at the same time. So, this is equal to a number operator and apsi k say for example, and again this is this is for. So, this is k equal to k 2 and this is for. So, this is actually k 1.

So, this is delta k q equal to 0 and there is a number operator psi k 1 and this will be delta k k 2 plus q and there is a Green's function, which propagates k t minus t prime. So, these are the 3 terms there and similarly for d, we have t C k t C k 1 dagger t 1 and t C k 1 minus q k 1 minus q t 1 and C k 2 dagger t 1 and similarly T C k 2 plus q t 1 and C k dagger t prime and so, on.

So, this is equal to delta k k 1 and G 0 once again to remind you that this G 0 will have to add an i. So, that without that it does not become G 0 we are writing it loosely without that factor of i. So, this is k t minus t 1 and now this is of course, will have a delta k 1 minus q, which is equal to k 2. So, this is k 2 and this will give me a so, just like the first 1 it will give me a so, it will be like a n xi k 2.

Similarly this will also be k 2. So, this will also be let me correct it here this was written mistakenly. So, this will be like n and xi k 1. So, similarly it will be xi k 2 and then this will be like delta of k k 2 plus q G 0 and k t 1 minus t prime. So, that is the fourth 1 and similarly the fifth 1 can be written as something interesting about the fifth 1 which will come.

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(e) $$\underbrace{\langle T c_k(t) c_k^\dagger(t')\rangle}_{G^{(0)}(k,\, t-t')} \langle T c_{k_2+q}(t_1) c_{k_1}^\dagger(t_1)\rangle \langle T c_{k_1-q}(t_1) c_{k_2}^\dagger(t_1)\rangle$$
$$\delta_{k_1, k_2+q}\, n(\xi_{k_1})\, \delta_{k_2, k_1-q}\, n(\xi_{k_2}).$$

(f) $$\underbrace{\langle T c_k(t) c_k^\dagger(t')\rangle}_{G^{(0)}(k,\, t-t')} \langle T \underbrace{c_{k_2+q}(t_1) c_{k_2}^\dagger(t_1)}_{\delta_{q=0}\, n(\xi_{k_2})}\rangle \langle T c_{k_1-q}(t_1) c_{k_2}^\dagger(t_1)\rangle$$
$$\delta_{q=0}\, n(\xi_{k_1})$$

A second order Green's function. will contain $5!$ terms.
$= 5 \times 4 \times 3 \times 2 = 120$ terms.
A 3rd order will have $7!$ number of terms.

So, we have a C k t and C k dagger t prime C k 2 plus q t 1 C k 1 dagger t 1 t C k 1 minus q t 1 and C k 2 dagger t 1. So, these are the 3 terms this term will give me a delta k. So, this is anyway k. So, delta k k and there is nothing, but there is a G 0 which is at k 2 t minus t prime.

Similarly this will be k 1 to k 2 plus q and again it will be like this k 1 and this will be like delta of k 1 or k 2 k 2 k 1 minus q and a G 0 and k 2 and this is at the same time. So, this will actually be a so, this will actually be a n which is xi k 2 anyway. So, these are 5 terms and to write down the sixth term we have T C k t C k dagger t prime T C k 2 plus 2 a t 1 C k dagger t 1 and C k 1 minus q t 1 and C k dagger C k dagger t 1 and again this is equal to a G 0 k t minus t prime there is nothing to write here. So, let me it is this part.

And right this one has to be written as a delta this is k 2. So, this is equal to q equal to 0 this and a k 2 and similarly this one is also delta q equal to 0 psi k 1 and so, on.

So, it is 3 5 3 and 5 k 2 and k 1 minus q. So, this is actually k 2. So, this is the last 1 is 3 5 3 is t 1 this t 1 this k 2 and k 1 minus k 2 3 is k 2 and this k 1 minus q this 2 ok. So, this is k 1 minus k 2 equal to. So, so this is anyway.

So, these are the terms that are that need to be written down including the 0th order Green's functions. And now if you write down second order Green's function you will have many more terms and we can write down and then there will be a second order Green's function will have how many terms a second order. So, how many creation and annihilation operator to H prime will give me 4 4 8 4 creation operators and for annihilation operators and 1 will come from the Green's function definition. So, there are 5 creation and 5 annihilation operators. So, this will contain 5 factorial terms.

Which means that equal to 5 into 4 into 3 into 2, which is equal to 20 into so, it is 120 terms that will be there a second ordered itself and the third order you can understand that it will go as. So, third order will have. So, 67 so, 7 factorial number of terms so, the question is that how actually to compute all these terms, even if we say want to write down the first order Green's function that is containing 1 H prime.

We got 6 terms and maybe some terms are 0 we do not know as yet, but even then the nonzero terms how to compute them that is the question and the computation of these terms are done by a Feynman diagrams.

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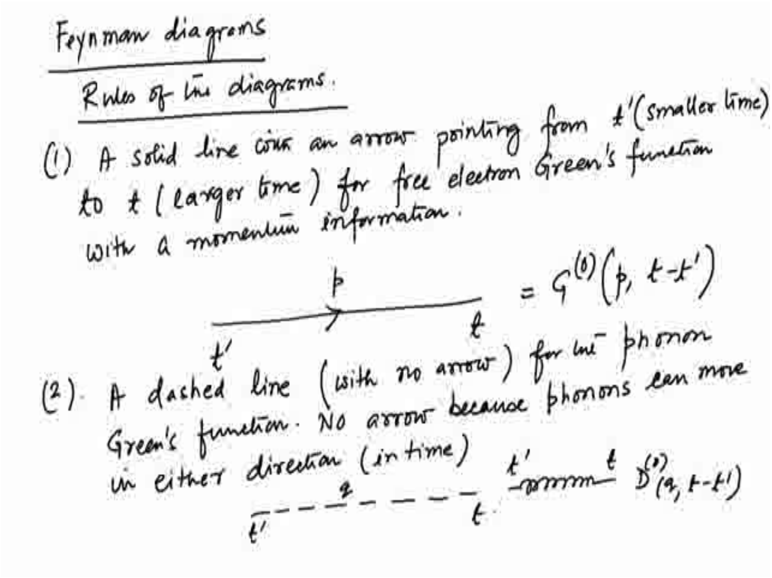

So, this is will introduce Feynman diagrams here and tell you the basic features of the Feynman diagram. So, so Feynman introduced this idea of representing the terms in these above discussions or above expressions by diagrams and these diagrams are actually very very important in a number of physical processes, which are themselves I mean interesting scattering problems so, the rules of the diagram.

So, rule number 1 is that a solid line with an arrow pointing from smaller time t prime, which is a smaller time to t, which is larger time for free electron Green's function with a momentum information ok.

So, basically that tells that this is like this with a t prime going towards a t and there is a momentum information. So, this line carries a momentum p. So, the system or rather the free electron Green's function propagates the free system from a smaller time t prime to a larger prime t and a corresponding to a given momentum p.

So, this is written as this is equal to G 0 p t minus t prime, this is the notation for that this is for the electron Green's function, what about the phonon or the photon Green's function? Let us just write it for the photon or phonon. So, a dashed line with no arrow for the phonon Green's function and no arrow; because phonons can move in either direction on the line, that is in time.

Sometimes this so, it is this line and there is a t prime to t and similarly there is a say a q momentum will not put a arrow, sometimes this is also written it with a springy line like a spring.

So, this is t prime and t it is also written like that and it is written as d 0 q t minus t prime. So, these are the electron and phonon non-interacting Green's function third very importantly.

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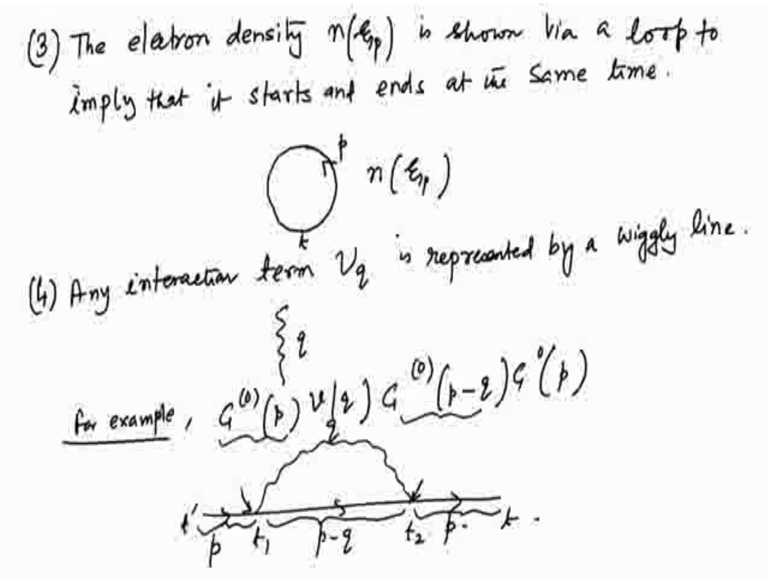

The electron density n of xi p is shown via a loop to imply that it starts and ends at the same time.

So, it just shows like this we show it by an arrow this is p and this is time it starts from the same time, I mean it starts from time t and comes back to the same time and this is written aspsi p this is what we were writing earlier.

Forth any interaction term v q is represented by a wiggly line so, such as this. So, this is it may carry a momentum q. So, this is the v of q v as a function of q it could have a energy information, which means that it will be a v q omega. So, for example, we can write down 1 of the terms in this which had, so a. G 0 p v of q which comes from an int and there is a G 0 p minus q and G 0 p will be shown as a line and then there is a wiggly line.

So, it comes from we have not written the time information, but the time is actually t 2 t minus t 1 and t 1 minus t 2 etcetera. So, these are this and so on. So, this is say p this is p minus q and this is again p. So, an electron was coming with momentum p and it is scatters of by a potential v of q, which takes away a momentum q here. So, that is why the wiggly line is represented by wiggly line is the v q which carries a momentum.

Then so, this is the this the leftmost side is the propagated here the v q is by the wiggly line and p minus q is the propagated here and then there is a propagator there.

So, the total Green's function or the total expression for the interacting Green's function at the first order is written as a G 0 and then there is a wiggly line and then there are you know sort of is a propagator from p sometime, t 2 t 1 or. So, this is like t 2 or t prime, which we have not put it here so, maybe t 1 and then maybe t 2 and then maybe t and so on. So, these are the way to write down this each one of those terms, that we have got.

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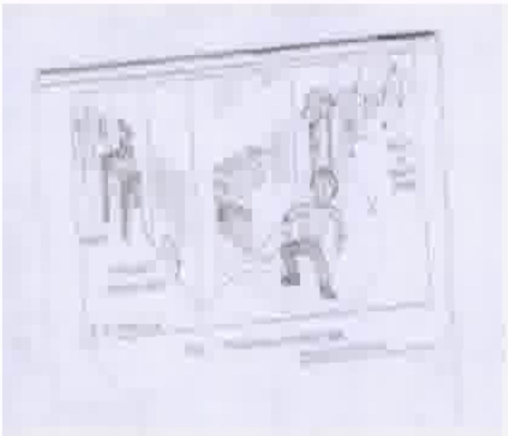

Propagator and the drunken man

And. So, rule number 5 is that at the vertex at the vortex, there is a momentum and energy conservation.

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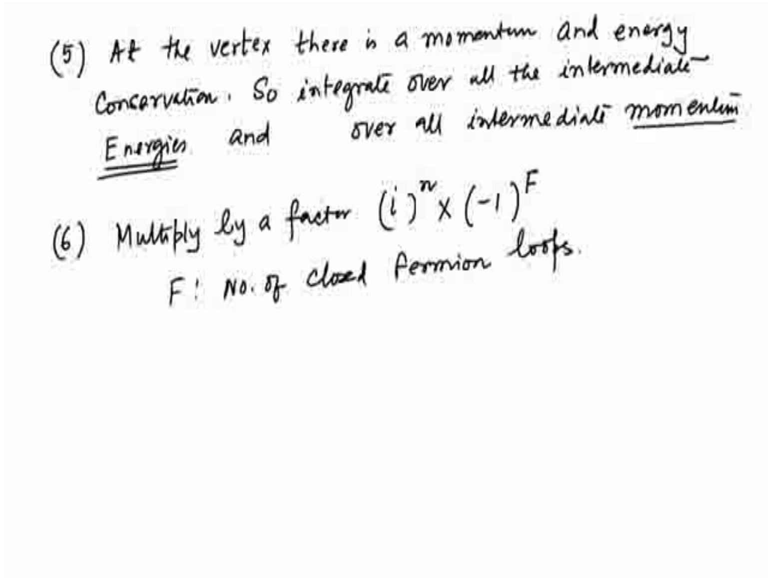

So, what I mean by vertex is these points that is a vertex and this is a vertex there are energy and momentum conservation. So, what we have to do is that? So, integrate over all the intermediate momenta and some over all integrate over all.

So, integrate over all intermediate let us not write it as momenta as energies and some overall intermediate. So, basically it is a; you have to integrate over all let us not introduce this word some and over all intermediate momentum.

So, that is the rule number 5, that is rule number 5 and the rule number 6 immediately it will not be clear, but nevertheless will put it any way multiply by a factor of i to the power n, because each Green's function to convert a line into a Green's function or each of this time ordered product into a Green's function we need to multiply it with i.

So, each of the terms will have to be multiplied with i and hence n of those terms will have to be multiplied with i to the power n. And then we will have to also multiply it with a minus 1 whole to the power F, where F are the number of closed for me on loops. So, these are the number operators. So, if we have 1 number operator then will have to multiply it by minus 1 to the power 1, if there are 2 of them then it is there will be no sign because minus 1 whole to the power 2 it will have to be multiplied by and so, on.

So, these all these rules put together from 1 to 6 as written there will help us in writing down these Green's functions the fully interacting Green's function in terms of the non-interacting Green's function and then by summing over the all the intermediate momentum and energy will get the finally, the interacting or fully interacting Green's function.

So, let us write down at least 1 term we have written down 1 term already let us write down another term, which is which was written down by us.

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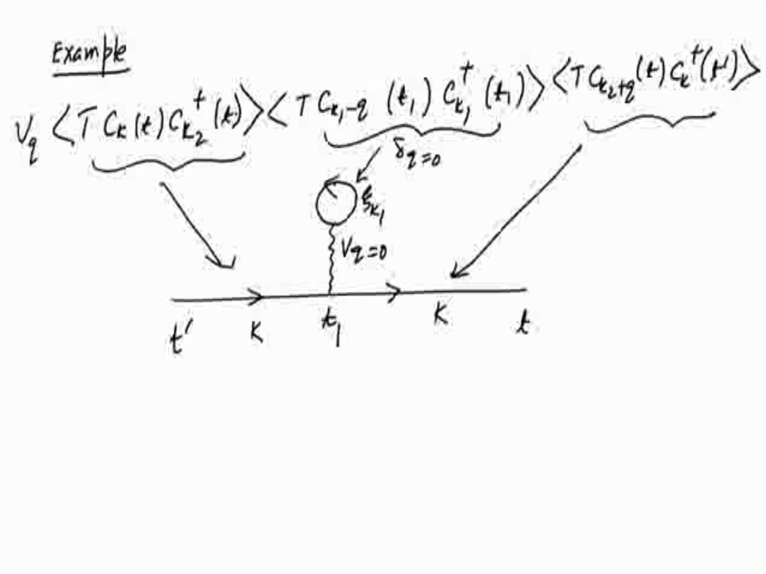

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So, example is time ordered C k t C this is the third term in the example that we have done T and we have T C k 1 minus q t 1 and C k 1 dagger t 1 and C k 2 plus q t and C k dagger t prime.

So, this term can be written as a line which will go like this from a t prime to t and then there will be a term which is and there is an interaction term. Of course, that we are not writing it explicitly, but the agent will come with a v q here and this. So, this will be here. So, this is a because this will correspond to delta q equal to 0. So, this interaction term corresponds to v q equal to 0 and this corresponds to a. So, this is 1 Green's function, taking it from some t prime to some intermediate t 1 and this another.

So, this is that Green's function. So, this is that Green's function here, this is that again that Green's function which is here and this is that term which is here ok? So, this is the so, this is a fermionic bubble. So, there is 1 fermionic bubble. So, will have to multiply it by minus 1 to the power 1 and since there are 3 such terms will have to multiply it by i to the power n which is what we have prescribed I to the power n. So, it is i to the power 3 and rather there are 2 Green's functions. So, will have to write it with i square and there will be a minus sign coming because of this.

So, this is momentum is k here momentum is. So, this does not carry any momentum. So, this is and this has momentum ahpsi k 1 and this is this continues with the momentum k. So, a free particle Green's function propagates from minus t, I mean from t prime to t 1 with a momentum k, then there is a interaction vertex which carries no momentum it is connected to a fermionic bubble and then it carries on with.

So, the Green's function or the propagator carries on with a momentum k from time t 1 2 t, will do more examples of Feynman diagrams and will in the next discussion will try to compute them by computing the internal summing over the internal moment and the energies will calculate and for a specific case.