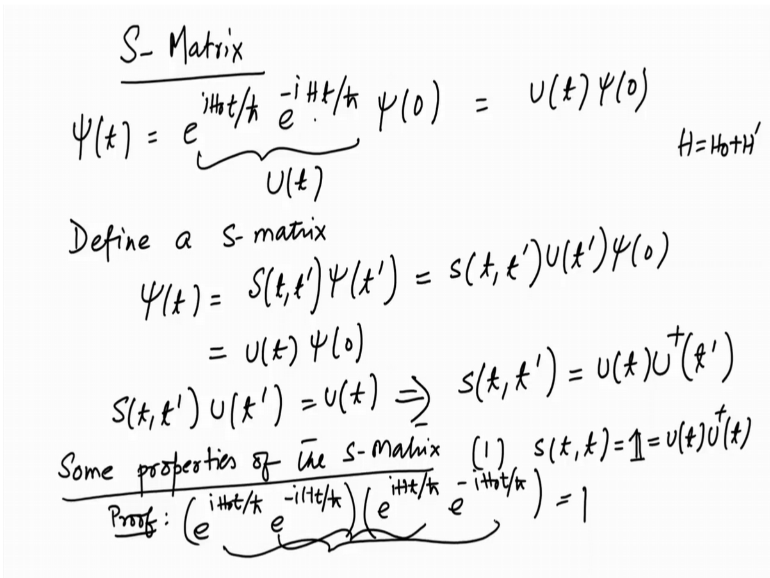
Advanced Condensed Matter Physics
Prof. Saurabh Basu
Department of Physics
Indian Institute of Technology, Guwahati

Lecture – 11
S – Matrix and free electron Green's Function

So, in the context of Green's function at 0 temperatures we have introduced the 3 representations that we would be working in, namely the Schrodinger representation Heisenberg representation. And interaction representation and we say that we would be working in the interaction representation mode and in that context we have computed the time evaluation of the wave function psi. And have introduced another operator you which was defined and it was found that the time evaluation of this operator U and the wave functions psi are similar and then finely the solution was written in an integral form.

Let us carry this discussion forward and introduce another important quantity called as the S-Matrix.

(Refer Slide Time: 01:20)



So, let us see what S-Matrix is and how it is simplifies our discussion? So, to remind ourselves the wave function in the interaction representation was written as psi of t which is equal to exponential i H naught t by H cross psi of sorry this is exponential i H t minus i H t by H cross and psi of 0 1 can refer to the discussion earlier and this was actually written as U of t. Hence this was written as U of t and psi of 0. Now so, this is how the wave function would evolve from the wave function at time t equal to 0. So, 1 is to operate by this U of t operator, which is written as exponential i H naught t by H cross and exponential minus i H t by H cross, where H is the total Hamiltonian equal to H naught plus H prime and H naught is an non interacting part of the Hamiltonian and H prime is the interaction term.

So, now one can also write it as. So, we can define in S matrix like this and by this which connects the wave function at t and the wave function at t prime. So, psi at t is connected to psi at t prime by a matrix, which is given by S t t prime. So, S t t prime connects the wave function at t and t prime. And this can still be written as S t t prime U t prime psi of 0.

Now this psi of t as we know that can be written as U of t psi of 0 hence this S t t prime U t prime equal to U of t. So, S t t prime U t prime equal to U t we can write the same thing as S t t prime, equal to U t U dragger t prime. So, this is the definition of S matrix which is constructed from the U operators, at time t and t prime it is a U t U dragger t. And now since we know the solution of U of t we can find out S t t prime see the solution of U of t is in terms of the interaction term H prime which we have shown earlier.

So, now let us some properties of the S matrix and what are the properties one of them is S t t which is equal to 1, because this is equal to U t U dragger t which is equal to 1. So, this 1 means it is a unit matrix. So, that is that can be easily shown let us show this, because U of t is like exponential i H naught t by H cross exponential minus i H t by H cross that is my U of t and U dragger t would be exponential i H t by H cross exponential minus i H naught t by H cross.

Remember that when you take a dragger you have to write the operators so, this is like taking a transpose and that has to be taken in the reverse order. So, we have written this as a so, it is a transpose and then it is taken as a conjugate and then it is written backward. So, this is off course equal to 1, because these 2 will become equal to one then farther these 2 will become equal to 1. So, that is 1 property.

(Refer Slide Time: 06:37)

(2) $S^\dagger(t,t') = U(t')\,U^\dagger(t) = S(t',t)$

(3) $S(t,t')\,S(t',t'') = S(t,t'')$

Proof: $\psi(t) = S(t,t')\,\psi(t') = \underbrace{S(t,t')\,S(t',t'')\,\psi(t'')}$

$= \underbrace{S(t,t'')\,\psi(t'')}$

EOM (Equation of Motion)

$$\frac{\partial}{\partial t} S(t,t') = \frac{\partial}{\partial t} U(t)\,U^\dagger(t') = -i\,H'(t)\,S(t,t')$$

$$S(t,t') = T\exp\left[-i\int_{t'}^{t} dt_1\, H'(t_1)\right]$$

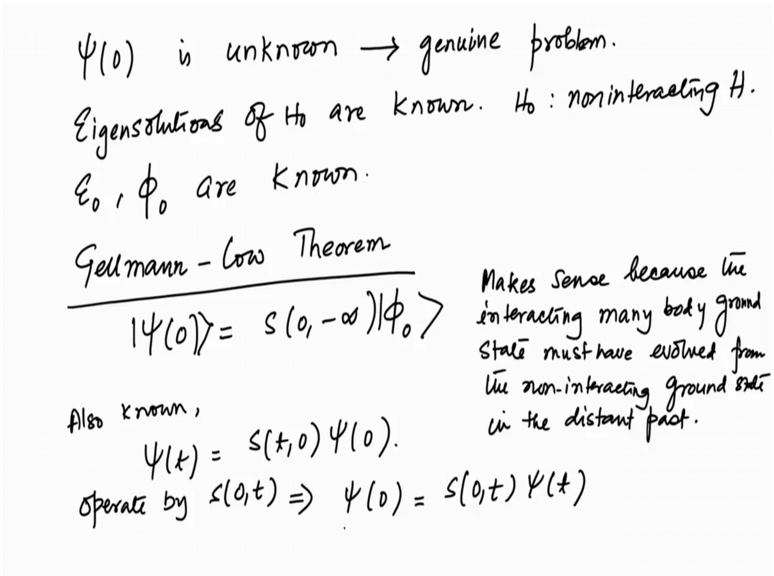
Let us see the second property of this matrix. So, property number 2, which is equal to S dragger t t prime equal to U t prime U dragger t, which is equal to S t prime t. So, which means that S dragger t t prime is same as S t prime t and the third one is S t t prime S t prime t double prime is equal to S t t double prime. So, this says that when we multiply these S matrices the resultant matrix will be the first one and the last the first time index and the last time index. And this can be you know extended to more than 2 products. And the reason that this is true is that let us give a short proof of this one as well.

So, your psi of t is equal to S t t prime psi of t prime and this is equal to S t t prime S t prime t double prime and psi of t double prime, but psi of t can be related to the psi of t double prime by simply using S t t double prime psi of t double prime. And if you compare the this one and this one it is easy to see that what is written in 3 follows from that.

So, these are the properties and now the equation of motion every time will use this word and write it as EOM. So, the equation of motion of psi is del del t of S t t prime equal to del del t of U t U dragger t prime, which is equal to minus i H prime t S t t prime. Now you see that the equation of motion exactly takes the same form as U of t or psi of t. So, the S matrix also has a solution which looks like it is a t exponential minus i t prime to t d t 1 H prime t 1. The only difference is that the S matrix is function of 2 time variables t and t prime, that is why this integral does not go from 0 to t rather it goes from t prime to t and it is the same interaction term and it is exponential related and with the time ordering operator.

This is exactly the same as we are written for the U of t or the psi of t. So, far we have introduced the 3 representations and have computed closed form for the wave functions psi the operator U and the S matrix and now the whole problem.

(Refer Slide Time: 06:37)



So, far is that that our psi of 0 is unknown and it is actually a genuine problem, because we need to know the psi of 0 because everywhere it is we are getting that the psi of t is generated from psi of 0 if psi of 0 is not known, then we cannot get psi of t even if we get the U of t or S t t prime etcetera. However, there is one thing that is known is the Eigen solutions of H 0 are known H 0 to remind you is the non-interacting Hamiltonian. So, let us write that because it will help you. So, it is a non-interacting Hamiltonian and it is assumed that we can solve it exactly and we know the wave functions and the ground state energy.

So, by Eigen solutions what we mean is that epsilon 0 and phi 0 are known where epsilon 0 is the energy of the non-interacting system and phi 0 is the wave function for the non-interacting system. These are known's these are the known quantities. And now we have to generate psi of 0 from these epsilon 0 and phi 0 in particular phi 0 because psi 0 is the t equal to 0 wave function; for the interacting system which is not known; however, the non-interacting ground state or the energy the wave function is known.

So, we need to get that now this is facilitated by a theorem called as the Gellmann low theorem. So, the Gellmann low theorem says that the interacting wave function at psi equal to 0. Now when I say wave function at the back of my mind i have ground state wave function, because that is the lowest energy state and in condensed matter physics as we have said earlier also that, we are only interested in the low energy states and which is the particularly the ground state wave function and the ground state energies. So, while psi of 0 can actually mean any state at t equal to 0, but in we actually mean that it is a ground state.

So, this is written in including the S matrix as 0 minus infinity and phi 0. So, it is clear that psi of 0, which is the wave function of the ground state wave function for the interacting system can be generated from the wave function of the non-interacting system, which is known by this S matrix, which are where the 2 time variables are 0 and minus infinity that is the theorem statement of the theorem. Why does the theorem makes the sense? It make sense because the interacting many body ground state must has evolved from the non-interacting ground state in the distant past.

So, also it is known that let me write down because this is important makes sense because the interacting many body ground state must have evolved from the non-interacting ground state in the distant past, which means at t equal to minus infinity that is how this S matrices is written as 0 and minus infinity.

Also it is true that psi of t is equal to S t of 0 psi of 0 if we operate by S 0 t and in that case we have a psi of 0, which is equal to S of 0 t psi of t. So, my interacting many body ground state at t equal to 0 is obtained from the from the many body ground state at a finite time t using this S Matrix.

(Refer Slide Time: `17:05)

For $t = -\infty$

$$|\psi(0)\rangle = S(0,-\infty)|\psi(-\infty)\rangle = S(0,-\infty)|\phi_0\rangle$$

$$|\psi(-\infty)\rangle \equiv |\phi_0\rangle$$

At $t = +\infty$

$$|\psi(\infty)\rangle = S(\infty,0)\psi(0).$$

$|\psi(\infty)\rangle$ and $|\phi_0\rangle$ may be same or differ by a trivial phase factor χ.

$$\phi_0 e^{i\chi} = \psi(\infty) = S(\infty,0)\psi(0) = S(\infty,-\infty)\phi_0.$$

$$\boxed{e^{i\chi} = \langle\phi_0|S(\infty,-\infty)|\phi_0\rangle}$$

And why is this done is because for t equal to minus infinity my psi of 0 it is equal to S 0 minus infinity psi of minus infinity i just put t equal to minus infinity there, which is nothing, but it is equal to S of 0 minus infinity and phi of 0 you want to write it with the brackets. So, psi of minus infinity is identical with phi of 0 which means that at t equal to minus infinity the interaction is not even switched on. So, the interaction is switched on only after t equal to minus infinity.

And so, this is the non-interacting ground state now there is an additional property that we would be needing at this in order to define the Green's function. So, at t equal to plus infinity we have psi of infinity it is equal to S of infinity 0 and psi of 0. Now it can also be assumed that the interaction is switched off at t equal to plus infinity. So, the interaction is switched on after t equal to minus infinity and is switched off after t equal to plus infinity. And in which case the psi infinity. So, psi infinity and phi 0 may be same and even if they are not same they should only differ by a trivial phase factor or differ by a trivial phase factor say a chi.

So, we can write which is considering the case in which it differs is phi 0 exponential i chi it is equal to psi of infinity, which is equal to S of infinity 0 psi of 0 which is equal to S of infinity minus infinity and a phi 0. I am not writing the (Refer Time: 19:57) always you can if you wish you can write it.

So, this phase factor is written as just the expectation value of the infinity minus infinity operator between the non-interacting ground state. So, this is an interesting and important property or rather and equation that you have would need while defining the Green's function. So, let us finally, the get to the Green's function at t equal to 0.

(Refer Slide Time: 20:33)

Green's function at T=0

At T=0, in Heisenberg representation,

$$G(\lambda, t-t') = -i\langle\Psi_0|T[c_\lambda(t)\, c_\lambda^\dagger(t')]|\Psi_0\rangle \quad (1)$$

λ : a relevant quantum mechanical parameter.

For an electron gas, $\lambda = (\vec{k}, \sigma)$

$|\Psi_0\rangle$ is the interacting many body ground state.

$$c_\lambda(t) = e^{iHt}\, c_\lambda(0)\, e^{-iHt} \quad (\hbar = 1)$$

Heisenberg representation.

For $t' > t$, the Green's function is defined as,

$$G(\lambda, t'-t) = i\langle\Psi_0|c_\lambda^\dagger(t')\, c_\lambda(t)|\Psi_0\rangle$$

And will write it in the Heisenberg representation. So, at t equal to 0 in Heisenberg representation there is a reason to write it initially in Heisenberg representation and then will switch over to the interaction representation.

So, the Green's function is defined as G and some parameter lambda which is a good property of the problem or the system under consideration, this is t minus t prime equal to minus i and between the interacting ground state and it is a time order C lambda t, C lambda dragger t prime. So, that is the definition which you should keep in mind lambda is any parameter it could be energy, it could be momentum, it could be some other property, which is like spin or something which is equal to minus i times the ground state expectation value of C lambda C lambda dragger at t and t prime with the time ordering involved there.

That is the definition and so, lambda will quantum mechanical parameter. Now it could be for an electron gas for an electron gas lambda could be set of 2 quantum numbers namely k and spins sigma or for other problems it could be something else. And once again to remind ourselves that psi naught is the interacting many body ground state and a c lambda and C lambda C lambda t and C lambda dragger t prime are written in the Heisenberg representation as C lambda t equal to exponential i H t C lambda 0 exponential minus i H t im dropping H cross and taking is equal to one and this is in the Heisenberg representation, this we have discussed at length.

So, this is my operator. So, for t greater t prime greater than t the Green's function is defined as G lambda sorry G lambda t prime minus t equal to i psi 0, C lambda dragger t prime C lambda t.

Since we have resorted to a particular case where t prime is greater than t i the time ordering says that we have to write down t prime on the left, in order to do that i gained an additional negative sign, because of the anti-commutation relation of the Fermi on operators, which absorbs this negative sign that written in equation one and also I now do not need the time ordering operator because I have sorted the particular case this just an example that we are wanted to site.

So, let us just look at things such as.

(Refer Slide Time: 25:46)

$$|\psi_0\rangle = S(0,-\infty)|\phi_0\rangle$$

$$c_\lambda(t) = e^{iH_0 t} c_\lambda(0) e^{-iH_0 t}$$

$$c_\lambda(0) = e^{-iH_0 t} c_\lambda(t) e^{iH_0 t}.$$

These operators have to be converted to interaction representation,

$$c_\lambda(t) = e^{iHt} \underbrace{e^{-iH_0 t} c_\lambda(t) e^{iH_0 t}} e^{-iHt} \qquad e^{iHt} c_\lambda(0) e^{-iHt}$$

$$= U^\dagger(t) c_\lambda(t) U(t). \qquad U(t) = e^{iH_0 t} e^{-iHt}$$

$$= S(0,t) c_\lambda(t) S(t,0)$$

So, what is my psi 0, once again it is equal to S 0 minus infinity phi 0 it is already written earlier. And now we have to go to the interaction representation, because this is what we had planned now in order to go to the interaction representation we write down the operators in the interaction representation as this. From now on we would take H cross equal to 1 so, not put that in exponent the denominator of the exponent.

So, this is how the time evolution of the operators are written with the H naught and the minus H naught t. And so, if i take if i want to write the c lambda of 0 so, that then i can write it as i H 0 t, C lambda of t exponential i H 0 t. So, what I did is that i left multiply a exponential i H 0 t and right multiply a exponential i H naught t.

And that gives C lambda at t equal to 0 in terms of C lambda at t. Now these operators' operators have to be converted to interaction representation. And in order to do that we can write down a C lambda t, which is exponential i H t exponential minus i H 0 t c lambda t, exponential i H 0 t, exponential minus i H t. Remember the time evolution of the operators in the Heisenberg representation is given by exponential i H t and C lambda 0 exponential minus i H t. And now my C lambda 0 is same as this, which is from the step earlier. So, I will have to write a in a interaction representation my C lambda as this and this can be written as U dragger t C lambda t U of U of t and remember that write it neatly.

So, these U of t S are the operators that we have written earlier where U of t is written as exponential i H 0 t exponential minus i H t. So, this is how the interaction that operators in interaction representations are written and this can be written as S of 0 t C lambda t S of t is 0.

So, I introduce the S matrices which we have learned and written down the operators and this was precisely the reason that we have introduced the S matrices and U in matrices, and now we know the complete solution of the S matrices and the U matrices which would be used in order to define the Green's function.

So, let us write down. So, this will be slightly bigger expressions, but nothing to worry these are well established and they can how they can be used to deal with physical problems we shall see.

(Refer Slide Time: 30:08)

$$G(\lambda, t-t') = -i\,\theta(t-t')\,\langle \phi_0 | S(-\infty,0)\,S(0,t)\,c_\lambda(t)\,S(t,0)\,S(0,t')\,c_\lambda^\dagger(t')\,S(t',0)\,S(0,-\infty)\,|\phi_0\rangle$$
$$+\,i\,\theta(t'-t)\,\langle \phi_0 | S(-\infty,0)\,S(0,t')\,c_\lambda^\dagger(t')\,S(t',0)\,S(0,t)\,c_\lambda(t)\,S(t,0)\,S(0,-\infty)\,|\phi_0\rangle$$

Extreme left end of the terms, can be replaced by,

$$\langle \phi_0 | S(-\infty,0) = e^{-i\chi}\langle \phi_0 | S(\infty,-\infty)\,S(-\infty,0)$$
$$= \frac{\langle \phi_0 | S(\infty,0)}{\langle \phi_0 | S(\infty,-\infty)|\phi_0\rangle}$$

So, G lambda t minus t prime i t minus t prime this is what we have learned that if t is greater than t prime the theta function. So, this is the theta function which will take a value equal to 1 and if not the it will take a value equal to 0. Now I am introducing the non-interacting ground states and writing it as minus infinity 0 S 0 t C lambda S t 0 S 0 t prime C lambda dragger t prime.

Now the entire thing has to be written for the right part, which is S t prime S t prime 0 S 0 minus infinity and a psi 0 of this only term there is another term, which will have to write for t prime greater than t, which will come by sign opposite to the first one and hence will have to write it as plus i theta t prime minus t. The now it will be phi 0, now everything that we have written on the other side will come here it will be S of minus infinity 0 S of 0 t prime and C lambda dragger t prime and S prime t 0 and S 0 t and C lambda t. So, everything we have written on the right now will come in the left, but in reverse order and S of 0 S of t 0 S of t 0 and S of 0 minus infinity and a phi 0.

So, the total Green's function it is big no doubt, but you do not have to worry because ultimately we are going to bring it a very simple form simple looking form and the utility to physical systems will be shown. So, this a these are the 2 terms which are taken for t greater than t prime t less than t prime. Now what we can do is that the extreme left and back head for the first term left end of the first term or rather it is both the terms of the terms can be replaced by phi 0, S minus infinity 0 S equal to exponential i chi of phi S infinity minus infinity by the property of the S matrix that we have written is this. And now exponential i chi as we have shown that this is equal to the this expectation value of S infinity minus infinity.

(Refer Slide Time: 34:58)

The Greens function becomes,

$$G(\lambda, t-t') = \frac{-i}{\langle\phi_0|S(\infty,-\infty)|\phi_0\rangle}\Big[\Theta(t-t')\langle\phi_0|S(\infty,t)C_\lambda(t)S(t,t')C_\lambda^{\dagger}(t')S(t',-\infty)|\phi_0\rangle$$

$$-\Theta(t'-t)\langle\phi_0|S(\infty,t')C_\lambda^{\dagger}(t')S(t',t)C_\lambda(t)S(t,-\infty)|\phi_0\rangle\Big]$$

First term can be simplified as,

$$\Theta(t-t')\langle\phi_0|S(\infty,t)C_\lambda(t)S(t,t')C_\lambda^{\dagger}(t')S(t',-\infty)|\phi_0\rangle$$

$$= \Theta(t-t')\langle\phi_0|T\, C_\lambda(t)C_\lambda^{\dagger}(t')S(\infty,-\infty)|\phi_0\rangle$$

$S(\infty,-\infty)$ involves operators in 3 time intervals,

$(\infty,t),\ (t,t'),\ (t',-\infty)$.

So, this can be written as a phi a S infinity 0 these 2 can be combined as S infinity 0 and then we can write it with the denominator replacing the exponential i chi, which is equal to sorry there is a exponential minus i chi with this, minus i divided by this exponential i chi term S infinity minus infinity phi 0 and there is this theta function with the t minus t prime and a phi 0 S infinity t C lambda t S t t prime C lambda dragger t prime and S t prime and also there is another term in the bracket with minus, theta function t prime minus t phi 0 S infinity t prime C lambda dragger t prime S t prime C lambda t t minus infinity phi 0 and the bracket.

So, these are the 2 terms and we have introduced the S matrices properly and so, the let us see the first term. So, this term is theta t minus t prime phi 0 S infinity t C lambda t S t t prime, C lambda t prime dragger S t prime minus infinity and a phi of 0 this can be written as theta t minus t prime phi 0 t C lambda t C lambda dragger t prime S infinity minus infinity phi 0, this is an important step have a look at this and this says that C S infinity and minus infinity involves operators in 3 time intervals. So, intervals and these are infinity to t t to t prime in this case it is t is greater than t prime and it is t prime to minus infinity. So, the time ordering operator automatically will sort these time intervals, because if we put the time that is largest will to the left will put it to the left.

So, proper sequencing of the time will be done by the time ordering operator and in which case we can write it and the second term also can be written it in the similar manner. So, this term also can be written it for the similar manner and again it will be 3 intervals, but; however, we have t prime is greater than t. So, there will be t t prime instead of that will have t prime t and there will be a t minus infinity.

(Refer Slide Time: 39:34)

$$G(\lambda, t-t') = -i \frac{\langle \phi_0 | T c_\lambda(t) c_\lambda^\dagger(t') S(\infty,-\infty) | \phi_0 \rangle}{\langle \phi_0 | T S(\infty,-\infty) | \phi_0 \rangle}$$

$$\boxed{G(\lambda, t-t') = -i \langle \phi_0 | T c_\lambda(t) c_\lambda^\dagger(t') | \phi_0 \rangle}$$

Final form of the $T=0$ Greens function.

So, if we combine both my Green's function will be written as lambda t minus t prime minus i phi 0 T C lambda t C lambda dragger t prime and S infinity minus infinity and a phi 0, and divided by this phi 0 S infinity minus infinity phi 0. In fact it is not required to put the S infinity minus infinity, because the time ordering will take care of the time that are appearing in the time variables that are appearing in the problem.

And in which case we can drop the S infinity and minus infinity and we can also sort of write the wave function without the write the Green's function without the denominator as G lambda t minus t prime, which is equal to minus i phi 0 T C lambda t C lambda, it is just pure applications there is no comma there. So, it is C lambda t prime dragger a phi 0 there. So, this is the final form of the T equal to 0 Green's function.

So, it says that the operators C lambda t and C lambda dragger t prime are taken the expectation is taken the time ordered expectation is taken with respect to the non-interacting ground state, which are known. And one has to multiply it with a minus i factor and the physical consequence or the physical situation that would it would correspond to is that if we create a particle given by C lambda dragger at time t prime and later take out the particle from the assembly of this many particle system at T equal to t.

The particle are rather the system will have the information that what kind of interactions are gone into the inside the system and the Green's function will provided information about that. It is more important, now to relate physical quantities to this Green's function which is what we will do. We will now do one example of this Green's function and the most simple case that is the non-degenerate electron gas at which means it is a non-interacting system.

(Refer Slide Time: 43:09)

Example

Degenerate, non-interacting electron gas

Simple metal at $T=0$. All electronic states for $\epsilon_k \left(= \frac{\hbar^2 k^2}{2m}\right) > \mu$ are empty and $\epsilon_k < \mu$ are filled.

Define $\xi_k = \epsilon_k - \mu$

$$\langle \phi_0 | c_k^{\dagger} c_k | \phi_0 \rangle = \langle n_k \rangle = \Theta(k_F - k) \underset{\beta \to \infty}{=} \frac{1}{e^{\beta \xi_k} + 1}$$

$$\langle \phi_0 | c_k c_k^{\dagger} | \phi_0 \rangle = 1 - \langle n_k \rangle = \Theta(k - k_F)$$

So, example so, it is a degenerate non interacting electron gas and let us take a simple metal at T equal to 0. So, all electronic states for epsilon k, which is equal to H cross square k square over 2 m and being the electronic mass for epsilon k greater than mu are empty and for epsilon k less than mu are filled.

This you must have studied in the first course of solid state physics, which is the model for a metal and we do not need to have t equal to 0 we will relax this condition. That since we are talking about the chemical potential we are restricting ourselves to t equal to 0, when we rather we can talk about the Fermi energy when we talk about t equal to 0, but a in principle this the illustration will carry over to t naught equal to 0.

Now since all the energies are measured from the Fermi level let us define a variable Define rather curly epsilon which is equal to epsilon k minus mu and we have for a uniform spherical Fermi surface, we have phi 0 C k draggers C k. Now your lambda is equal to simply equal to k we do not need to put the spin in the less there is term let us the spin. So, this is equal to nothing, but equal to n k and this is equal to k f minus k, which is limit beta tending to infinity, which means t tending to 0 of this exponential beta psi k plus one that is the Fermi distribution.

So, this is the definition that we have the ground state expectation non interacting ground state expectation value of the C k up C k draggers C k is nothing, but expectation value of n k which is nothing, but the theta functions are all k is greater than k that is k less than k is which is nothing, but the Fermi function.

And for the other term because we have 2 terms in the Green's function, because of the time ordering and this is equal to C k C k dragger phi 0, which is nothing, but equal to 1 minus n k and that is equal to theta of k minus k, f the states which are all empty at t equal to 0 or t tends to 0.

(Refer Slide Time: 47:09)

So the Green's function (unperturbed) is

$$G^0_\lambda(t-t') = -i\,\theta(t-t')\langle\phi_0|c_\lambda(t)c_\lambda^\dagger(t')|\phi_0\rangle + i\,\theta(t'-t)\langle\phi_0|c_\lambda^\dagger(t')c_\lambda(t)|\phi_0\rangle$$

$$= -i\left[\theta(t-t')\theta(\xi_k) - \theta(t'-t)\theta(-\xi_k)\right] e^{i\xi_k(t-t')}$$

Fourier transform of this,

$$G^0(\vec{k},\omega) = -i\left[\theta(\xi_k)\int_0^\infty dt\, e^{it(\omega-\xi_k+i\eta)} - \theta(-\xi_k)\int^\infty dt\, e^{it(\omega-\xi_k-i\eta)}\right]$$

$$= \frac{\theta(\xi_k)}{\omega-\xi_k+i\eta} + \frac{\theta(-\xi_k)}{\omega-\xi_k-i\eta}.$$

So, the so the Green's function, now this Green's function, because we are talking about a non-interacting problem will call it an unperturbed Green's function this word will be clarified more as we go along is G lambda t minus t prime, it is equal to minus i theta t minus t prime and a phi 0 C lambda t this I am writing it in the form that we have introduced and phi 0 there is a general form and the plus i theta t prime minus t phi 0 and the C lambda dragger t prime and C lambda t phi 0 the sign change occurs, because the operators swap the positions. And this is equal to minus i theta of t minus t prime and a theta of xi k minus theta t prime minus t and theta of minus xi k and exponential i xi k t minus t prime.

So, the Fourier transform of this of this is G now let us write it because we are talking about the unperturbed principle let us write it with 0. So, G is 0 k omega it is equal to minus i, xi k and using the definition of the theta function for the time x equal to 0 to infinity and the d t exponential i t, omega minus psi k plus i eta and for the other one will write it with minus minus psi k 0 to infinity d t exponential i t omega minus psi k minus i eta and this tells me that this becomes equal to this xi k omega minus xi k plus i eta and the plus theta minus xi k divided by this minus sign.

Let me write it a little more clearly and it is a omega minus xi k minus i eta.

(Refer Slide Time: 50:45)

In a compact form,

$$G^0(k,\omega) = \frac{1}{\omega - \epsilon_k + i\eta_k}$$

where $\eta_k = \eta \, sgn(\xi_k) = \eta \, sgn(\epsilon_k - \mu)$

$= +\eta$ when $\epsilon_k > \mu$ (1st term)

$= -\eta$ when $\epsilon_k < \mu$) (2nd term).

Introducing additional properties, such as, spin etc.

$$G^0_{\alpha\beta}(\vec{k},\omega) = \delta_{\alpha\beta}\left[\frac{\theta(k-k_f)}{\omega-\xi_k+i\eta} + \frac{\theta(k_f-k)}{\omega-\xi_k-i\eta}\right].$$

So, this can be written in a compact form as a G 0 k omega equal to 1 divided by minus plus i eta k, where eta k equal to some eta S g n of this where this is equal to eta S g n of

S g n of epsilon k minus mu. Which says that whether epsilon k is greater than mu or epsilon k is less than mu eta will take form, which is plus or minus where eta is positive number.

So, will have a i eta or i minus i eta depending on whether we are talking about the particle or the whole will clarify this more as we go along, but this term the term the first term that is the term on the left is called as the electron propagator and whereas the term on the right would be called as the whole propagator, because this corresponds to filled states and this corresponds to empty states.

So, this is equal to plus eta when epsilon k is greater than mu. So, that is the first term and x equal to minus eta, when epsilon k is less than mu that is the second term. So, finally, the Green's function if we introduced spin so, introducing additional properties such as spin etcetera. One can write the G alpha beta 0 k omega will be equal to delta alpha beta theta k minus k f divided by omega minus plus i eta, and plus theta k f minus k omega minus. So, in fact, this correspond to the electron particle term the second term, sorry for messing that up earlier and the first term corresponds to the whole term which are all empty where k is greater than k f.

So, at t equal to 0 none of these states are occupied and all the states are occupied, which are called as the particle states for k f greater than k.

So, this is the Green's function that we compute for a degenerate free electron gas and so, it gives the entire energy spectrum of the poles of the Green's function gives the energy spectrum for both the particles and the holes and they just differ by of course, the numerator and in the poles there will fine, whether the complex term actually pushes the pole up of the real line or it pushes down of the real line we will clarify all these things in our sub sequent discussion.