Nuclear and Particle Physics
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Module - 10
Gauge Symmetry
Lecture - 04
Dirac Lagrangian U (1) Gauge Symmetry

So, we discussed the Lagrangian and how to get the equations of motion from that a Lagrangian in particular we considered the Lagrangian, Lagrangians of fields that will give us the equations of motion corresponding to a Fermionic particle which is basically the Dirac equation.

And similarly for the Klein Gordon equation we had written down the Lagrangian and also for the Lagrangian that would give us the Maxwell's equation, what we studied was the free particle. Now if you want to now look at interactions from the same framework or in the same framework, then how do we actually take that into account this what we will discuss today.

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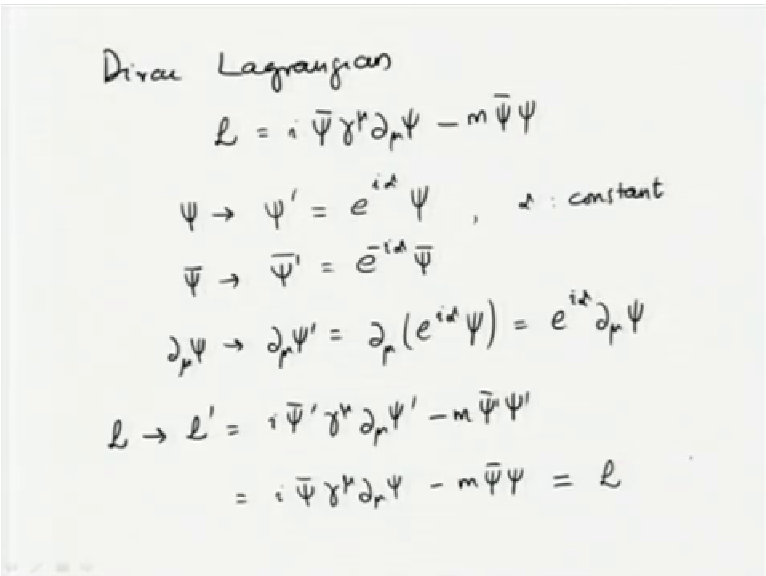

Let us consider the Dirac Lagrangian, by which I mean the Lagrangian that will give us the Dirac equation i psi bar gamma mu dou mu psi minus m psi bar psi. We sawed yesterday that this indeed leads to the Dirac equation for the wave function or phi psi or for the field psi. Now consider a transformation of psi which goes to some psi prime which is equal to some exponential i alpha times psi where alpha is some constant some kind of a phase factor which we add to psi, psi bar will go then e power minus i alpha psi bar and the other element then the Lagrangian is derivative of psi derivative of psi will go to derivative of psi bar psi prime.

We are not changing the transforming the space time coordinates at all, we are in the same space time coordinate we could even be in the same reference frame and changing only the form of the wave the field psi. So, this will now be derivative acting on the new field transformed field with all field with the exponential i alpha pre multiplying it.

The first factor exponential i alpha is independent of space time coordinates because alpha we have taken as a constant. So, that will not be affected by this and then we have essentially e power i alpha dou mu psi. Let us put all these back in the Lagrangian so, Lagrangian will go to some l prime say is i psi bar prime gamma mu, there is no change psi dou mu psi prime minus m psi bar psi when I multiply psi bar and psi together exponential terms cancel exponential factors cancelled.

Similarly the since psi bar prime has e power minus i alpha and dou mu psi prime also has a is similar to psi prime with the same exponential factor exponential i alpha, it actually cancels with each other. So, all together you will get back the original equation m psi bar was registered an in psi prime in the earlier equation here it is which is equal to L.

So, we see that the Lagrangian that we have written down in the first line is invariant under the transformation of psi go into psi prime equal to exponential i alpha psi with alpha a constant this transformation is called a gauge transformation a global gauge transformation.

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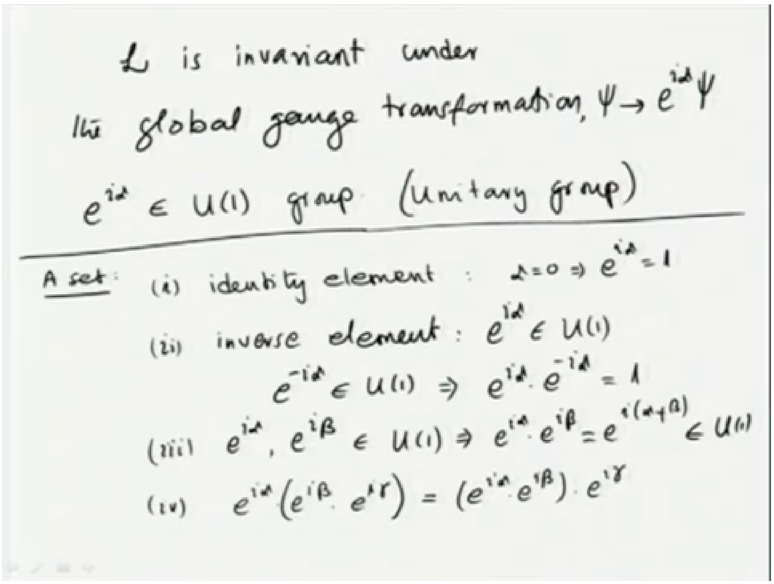

So, what we see is that L is, invariant under the let me call it global gauge transformation which actually says that psi goes to exponential i alpha psy. So, this is the standard terminology global gauge transformation.

Now, take e power i alpha if you know group theory you will recognize that this is an element of u one group if you do not know what is the meaning of a group, let us just briefly tell you what it is the a set you take any set that satisfy the following properties you take there should be an identity element in that.

In our case if you put alpha equal to 0 that will give you e power i alpha equal to 1 and you multiply anything by one will give you the same thing for example, it is an identity transformation when you take psi to psi itself. So, for alpha equal to 0 this psi will go to psi 2 it should have an inverse transformation. So, for group theory there is an inverse element, which means that if e power i alpha is an element of u, then e power minus i alpha is also an also an element of u 1. So, that e power i alpha e power minus i alpha is equal to unity the identity.

So, for every element there is another element when you multiply them together and that when you take them together you will get unity or identity, then other thing is that if e i alpha and e power i beta are 2 elements of u 1, in this case we are actually illustrating the group property by u 1, but you can this is applicable to any group. If these were 2 elements of the group which are considering you take any 2 elements which are which are I mean 2 elements of the concerned group.

Then a combination of that product of that e power i alpha e power i beta is equal to alpha plus beta is also an element of u 1. So, the set is saw that if you have 2 elements I mean when you consider 2 elements and then multiply them, then you will get another element of the group the product of 2 elements will be an element of the group.

This is the closure property is called the closure property when I say here multiplication in this case in this particular case it is actually the multiplication, but in general for any group it need not always be multiplication it is either multiplication or any other group operation whatever is the group operation.

So, along with the set you had to also define, what is the group operation? So, then you will see that all these things are satisfied when you combine these 2 elements which means that you are doing the group operation between these 2 elements. Such combination is associative which means if I take 3 elements e power i alpha e power i beta e power i gamma, it does not matter whether I club 2 of them first any the last 2 of them first and then multiply combined that resultant element with the first one or take the first 2 and combine them to give a resultant element and then combine that with the third this is the group associative property of the group elements.

So, any set that satisfy these condition these conditions are called is called a group in this particular case alpha can take continuous values any value and then therefore, it is called it is a continuous group u 1 is a continuous group u actually means unitary. So, it is a unitary group and there is only one parameter to represent the group element. So, that is why you have 1 in the bracket u, u and then bracket 1.

So, we will not go into any further details of group theory, we will advise you to go through any elementary book on group theory to familiarize yourself with the properties of group theory and algebra group theory etcetera especially the one which is of our interest is called is the a continuous group or are continuous groups are what we are interested in and the these groups are called Li groups.

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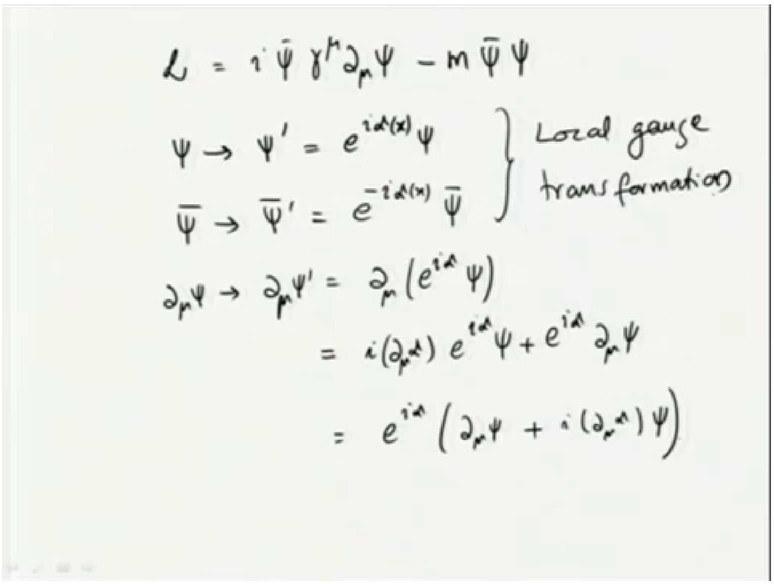

Now, let us come back to the transformation of the Lagrangian the Dirac Lagrangian, let me write that again i psi bar gamma mu gamma mu dou mu psi minus m psi bar psi, let us again consider a transformation psi goes to psi prime which is equal to e power I alpha psi. Difference is a difference with the earlier transformation is that we are now not considering alpha a constant, but a function of space time coordinate x, I will not write the index mu on the space time the x there because it becomes too clumsy for us..

So, when I say alpha s it is a function of space time coordinate, what about psi bar, psi bar will also transform similar way e power minus i alpha x psi bar, no problem. Let us look at derivative of psi it is derivative of psi prime now which is equal to derivative acting on e power I alpha psi I will not write the x dependence explicitly it is understood.

But now since alpha is not a constant, but a function of the space time coordinates the derivative will act on the exponential the factor. So, it is i times derivative with the derivative of alpha times same thing e power i alpha psi plus e power i alpha derivative of psi, where I can write this as e power i alpha dou mu psi plus i dou mu alpha alpha. So, there is an additional term in this one now coming up.

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$$\mathcal{L}' = i\bar{\psi}'\gamma^\mu(\partial_\mu\psi') - m\bar{\psi}'\psi'$$

$$= i\bar{\psi}e^{-i\alpha}\gamma^\mu e^{i\alpha}\left(\partial_\mu\psi + i(\partial_\mu\alpha)\psi\right) - m\bar{\psi}\psi$$

$$= \underbrace{i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi}_{\mathcal{L}} - \left[(\partial_\mu\alpha)\cdot\bar{\psi}\gamma^\mu\psi\right] = \mathcal{L}$$

$\mathcal{L}$ is not invariant under local gauge trn,

$$\psi \to \psi' = e^{i\alpha(x)}\psi.$$

So, what happens to that in the Lagrangian, you have now i psi bar prime gamma mu dou mu psi prime. So, this is your, i prime minus m psi bar prime psi prime, second term is invariant under this transformation because the phase factors cancel with each other the first term the phase factors cancel with each other, but there is an additional term. So, psi prime psi psi bar prime is e power minus i alpha psi bar then you have the gamma mu e power i alpha in the bracket you had dou mu psi plus i dou mu alpha acting on psi, we have multiplied by psi minus m psi bar psi I think in this expression I miss this psi here should be there in.

The exponential factors indeed cancel, but problem is that in addition to the original Lagrangian which is this these 2 terms you have a term which is basically you have a an i there and i here. So, it is minus dou mu alpha psi bar gamma mu psi. So, there is an additional term which is which means the Lagrangian different from the original one question is then also this particular this Lagrangian is not invariant under the transformation.

So, this transformation is basically called the this transformation is called local gauge transformation it is called local because alpha depends on the local points the point at which you are talking about it is not globally defined it is not defined for all the points it is a different it can be can have one value at x 1 and it can have another value at x 2. So, this is a local locally defined quantity different points will have alpha differing from each other. So, L is not invariant under local gauge transformation L as we have to written it is not invariant under the local gauge transformation psi goes to psi prime equal to e power i alpha x psi.

Now, supposing we say that where I want the say Lagrangians may not be this particular Lagrangians, but I want to have a modified Lagrangian, but I want the local gauge symmetry for that Lagrangian I insist on that what will happen in that case. So, what we are to do is. So, find a way to cancel this additional term here. So, this term which I have underlined by this red color.

Is addition in is an additional term in the new Lagrangian compared to the old one. So, somehow we have to arrange things. So, that this is canceled when you go from L to L prime. So, the Lagrangian has to be modified we also recognize that this is a dou mu alpha is actually a vector quantity. So, there is a vector quantity that has to be that is going to these pre multiplying a current like quantity here. The whole issue was with the derivative term you see that the derivative is not of the same form as same the transformations of derivative.

The transformations of psi is this way psi bar transform in a similar way, but look at dou mu psi, dou mu psi does not transform in a similar way the object that you have in the on the left hand side and the form of the object that you have on the right hand side are completely different that is because of that additional term the second term in this bracket here.

Otherwise it should have been fine like in the global case that is what we had in the global case we had this and this transforming in a very similar way. So, this is the same as what you have the same natures what you have on the left hand side and left hand side right hand side and the left hand side. Now how do you make this derivative in a in the same fashion is what we want to understand.

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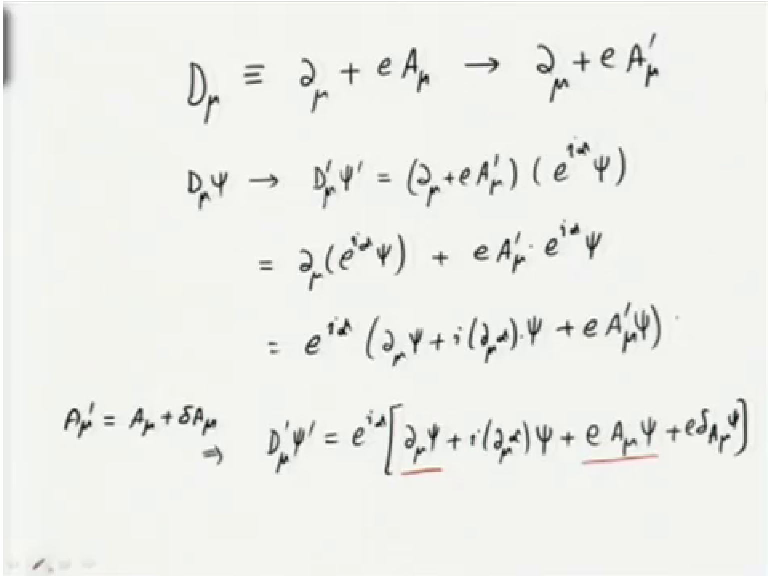

For that we define a new kind of derivative which is equal to or defined as the ordinary derivative plus e A vector quantity m, this has to be a vector because the first term is A vector. So, what we have in the second term should be A vector of course, we denote that by a mu whatever that vector is some vector field is handed multiplying this adding to the dou mu you to give a new kind of derivative and now what happens to the mu psi.

So, D mu psi will go to D mu psi prime which is equal to D mu or let me expand the D mu dou mu plus e A mu and psi not psi and e power i alpha psi, let me expand this first is the derivative acting on e power i alpha psi which we know are true what is the result. Second is e A mu e power i alpha psi so, altogether I have e power i alpha from the expression from the expression here that if or i alpha dou mu psi plus i dou mu alpha psi.

So, that is what I will write down first dou mu psi plus i dou mu alpha psi then you have another term plus e A mu, but now let us say along with the transformation of psi I also want psi prime to be changed. So, it is not just the D, but a D prime dou mu the first factor in D is not going to be changed. So, this will go to dou mu plus e a mu prime so, it is e may A prime that you have here. So, A prime is what you have here and that is it psi of course.

Now, if I take A mu prime as the original one plus some delta A mu, then that will give you D prime psi prime is equal to e power i alpha first term as it is, second term as it is, the other term is e A mu psi plus e delta A mu psi. So, in this so, let me take that the combine the terms first let me take the first term and third, first term and third term when you combine what you get is following.

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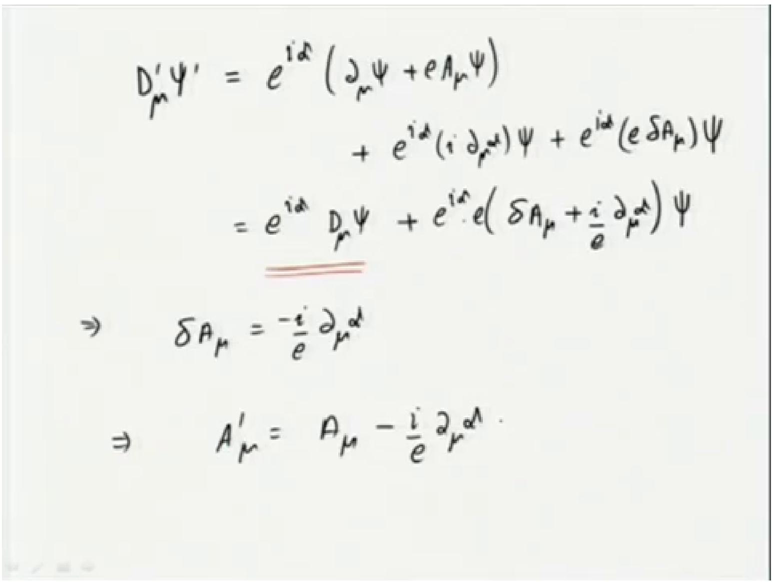

So, D mu prime psi prime is equal to e power i alpha dou mu psi plus e a mu psi, then you have e power i alpha and i dou mu alpha psi plus e power i alpha e delta A mu psi.

First align the first term 2 terms is essentially D mu psi then you have additional 2 terms. So, what you have to do is to make these 2 terms cancel each other so, that is all you had to do. So, this is e power i alpha let me take e also outside then that will give you delta a mu plus i by e dou mu alpha psi.

So, if for this to cancel you have to have delta a mu equal to i over minus i over e dou mu alpha. So, that will tell you that A prime mu is equal to a mu minus i over e dou mu alpha. So, essentially what we are saying so, this is clear what we are trying to say is that for the Lagrangian to be invariant under the (Refer Time: 28:08) transformation what we want is only the first term here the other the rest of it should vanish so, that is what we wanted.

And for that we can actually make delta A mu equal to opposite of the second term in that bracket then it will vanish. So, that is what we have here that is, what is leading to A prime mu equal to A mu processing very good.

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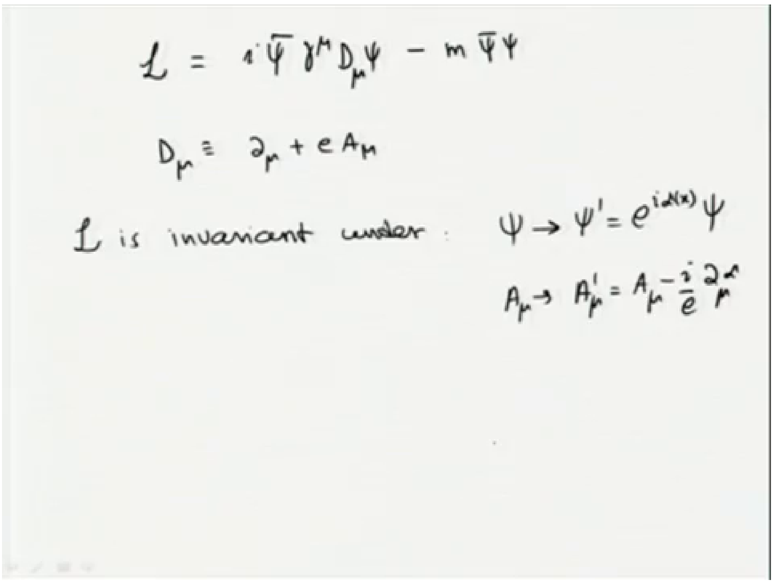

So, what we have is the following Lagrangian i psi bar gamma mu gamma mu D mu psi minus m psi bar psi with D mu defined as dou mu plus e A mu is invariant under the transformation psi going to psi prime equal to e power i alpha x psi and psi A mu going to A mu prime equal to A mu minus i over e derivative of the same alpha.

Together they make sure that the Lagrangian that is written down is invariant under these transform 2 transformations. So, we have a way out of this we have a Lagrangian new Lagrangian now. So, we have a new Lagrangian now, which has an additional term compared to the old one be considered, but; that means, the new Lagrangian invariant and along with the transformation of psi we had to also consideration consider a transformation of A mu at the new fields. So, together they are called the gauge local gauge transformation of A and psi all right so, this is what we have achieved that is very good.

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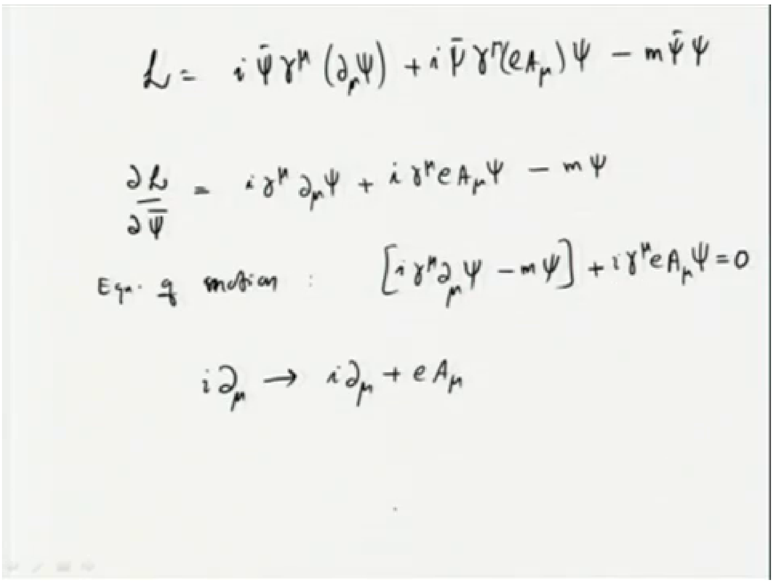

And now let us look at the equations of motion of psi bar with this. So, L is equal to i psi bar gamma mu dou mu psi plus i psi bar gamma mu e A mu psi minus m psi bar psi. So, this now will give you dou L by dou psi bar equal to i gamma mu dou mu psi plus i gamma mu e A mu psi minus m psi.

This is equal to 0 is your equation of motion because there is no derivative term of psi bar. So, this gives us equation of motion of psi as i gamma mu dou mu psi minus m psi recognized this as the free a particle free Dirac equation corresponding to a free particle determined I mean described by psi then there is an additional term plus i gamma mu e A mu psi equal to 0.

So, to interpret everything correctly now this looks like an interaction term, that we had written down earlier where we said that if I change the momentum from a free particle to any new free particle to interacting particle what I had to do is to change the momentum from i dou mu to i dou mu plus e A mu this is what I had to do to introduce the electromagnetic interactions, to accommodate these we had to actually main the gauge transformation change there is an extra i factor here the way we had done the transformation, but this can be taken care by considering dou mu is equal to dou mu i e A mu.

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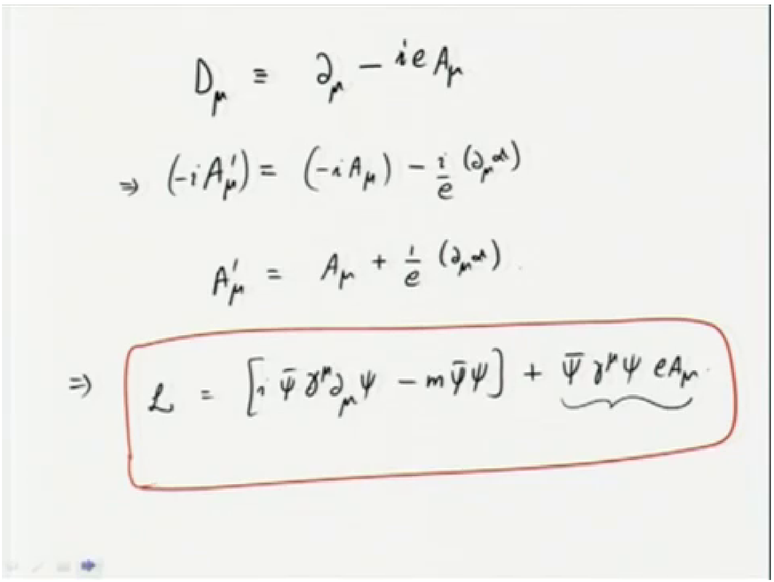

So, if I consider that then what we have. So, this is what the transformation that we will consider which means that I have multiplied e A by A minus i. So, that will give me everything should be alright, excepting that the transformation of a now is A prime is equal to. So, basically minus i A prime will transform as minus i A mu minus i over e dou mu alpha or I can say that A mu transforms us A mu prime equal to A mu plus 1 over e dou mu alpha.

So, this is the new A and with this the Lagrangian is equal to i psi bar gamma mu dou mu psi minus m psi bar psi plus psi bar gamma mu psi e A mu and indeed this is of the form that we had earlier for an electron or an charged particle represented by psi interacting with the electromagnetic field represented by A mu.

So, essentially what we have achieved by insisting on the Lagrangian to be invariant under the gauge transformation is that we have today we are then forced to introduce electromagnetic interaction of the particles in this thing, turning around we can say that interaction of the particle can be understood through the gauge transformations or the gauge symmetry of the Lagrangian.

This A mu that we have introduced A appears in the Lagrangian only in the interaction term only in the interaction term here see, but if you want to interpret this as a particle like a photon then of course, you have to also have to tell about what happens to the free part of the photon and how does it propagate, what is the equation of motion corresponding to the photon or even if you consider it as electromagnetic field you have to also get the Maxwell's equation out of this thing.

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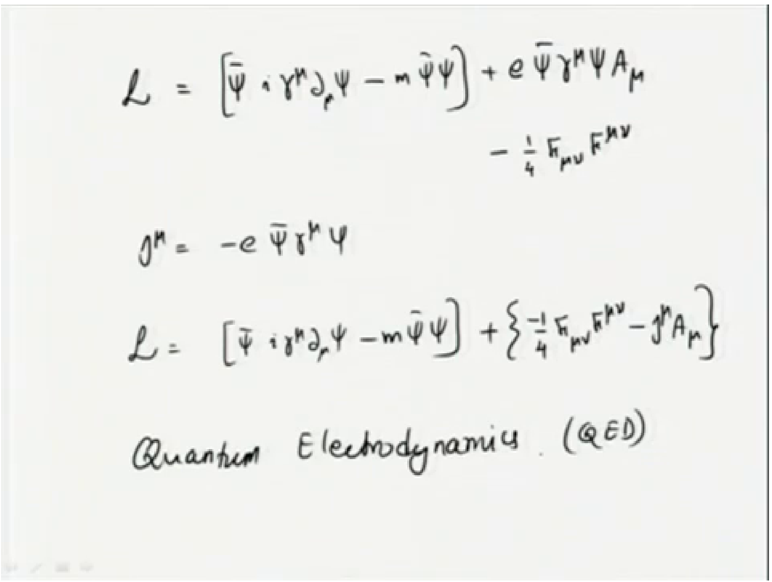

So, to do that what then we have to do is, to introduce the another term which will give you the rest of the Maxwell's equation with this. So, psi bar gamma mu dou mu psi minus m psi bar psi plus e psi bar gamma mu psi A mu is what I have I add 1 over 4 F mu u F mu u with this.

Now, look at the last 2 terms the last 2 terms if I consider J mu to be minus e psi bar gamma mu psi then L is psi bar I gamma mu dou mu psi minus m psi bar psi the free part of the Dirac Lagrangian plus I have one part minus 1 over 4 F mu u F mu u minus J mu A mu.

This is precisely that the, what is there in the curly bracket is precisely the Lagrangian that we used to get the Maxwell's equation. So, we have a free Dirac equation and a Maxwell's equation with the photon interacting with the electromagnetic field that term added. So, there is a free part for the psi there is a free part for A mu (Refer Time: 39:38) mu and there is an interaction term. So, these together will make sure that you have an equation of motion corresponding to the photon as well as the equation of motion corresponding to the charge the particle that you are considering through psi and these essentially is what is called the Quantum Electrodynamics QED.

And if you look at the Lagrangian it is clear the way we have been deriving the equation of motion from this that it will give you the correct equations of motion for the interacting psi as well as the Maxwell's equation and then one thing we can do with this Lagrangian is to take consider the quantum field theory framework and then get the Feynman rules from this, that we had listed down that we had listed down earlier.

So, we have actually free part of this free propagator version of the psi or the electrons or the Fermion particle that is represented by psi that we can get from this and there is also an interaction term from which we can get the interaction vertices and these are the things that we need to discuss any interaction of psi with electromagnetic field. Similarly we can also get the propagator for the photon from here and it; obviously, the interaction of the photon with the other one similar to these is exactly the same as the one which have we have just now set for the case of psi.

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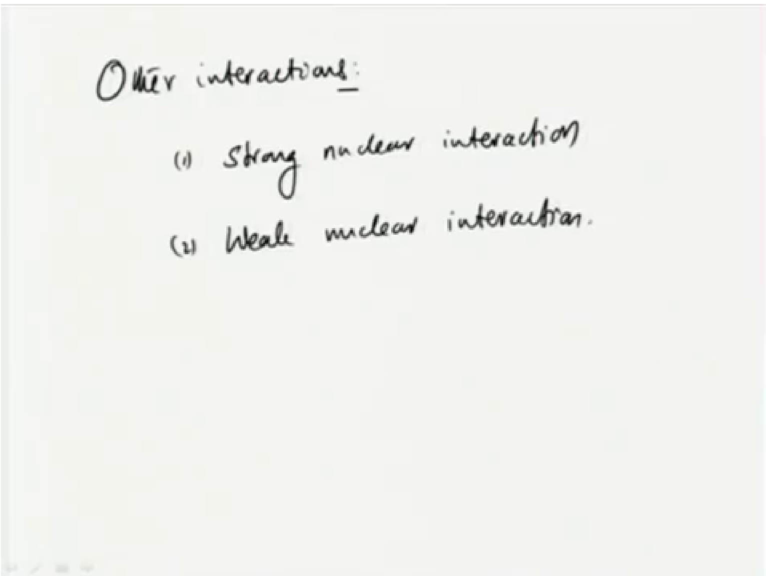

We will look at other interactions so, there is electrodynamics that we talked about, but then there is a strong nuclear interaction and weak nuclear interaction can we describe these interactions also through the gauge symmetry, similar to what we discussed for the case of electrodynamics is one thing which we will see we will come to that in the next lecture.