Nuclear and Particle Physics
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Module - 10
Guage Symmetry
Lecture - 40
Lagrangian Framework

Today, we will discuss the Gauge Symmetry which is the way of understanding the interactions of particles.

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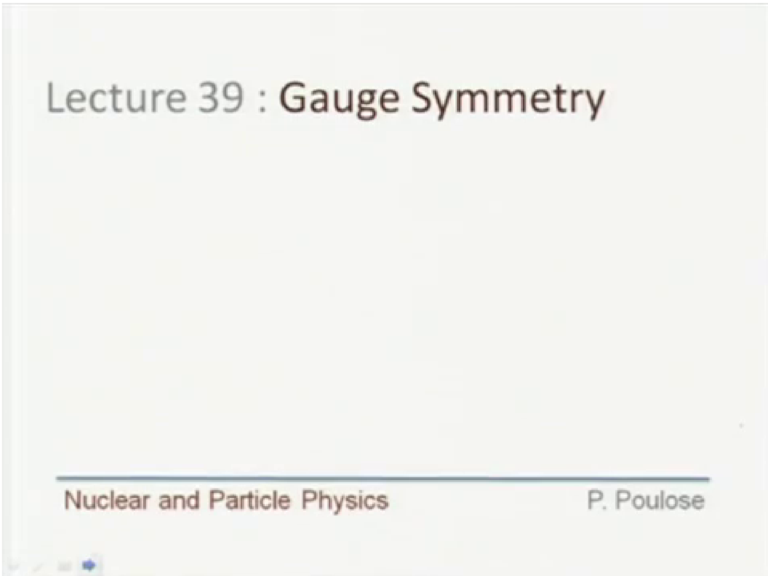

As we go on to be clear; what we mean by that so, far we have been discussing, the interactions of particles through their equations of motion and we did motivate the Dirac equation how it arises and also the Klein Gordon equation and then basically use the perturbative technique to compute the cross section we went on to describe the weak interaction also along the line of the electromagnetic interactions.

Now, we will try to see how everything can be understood from a symmetry point of view or which then can be taken as the basic principle guiding principle and things can be built from these guiding principles for that we will start with what is called the Lagrangian of the system.

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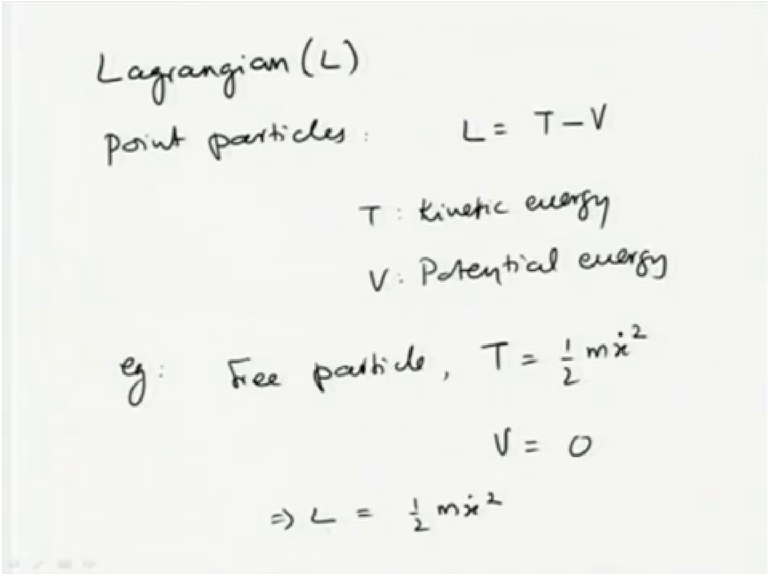

So, the basic thing here is the fundamental thing is the Lagrangian of the system and the equations of motion can be derived from the Lagrangian, for a 4 point particles a discrete system of point particles we can disc talk about the Lagrangian as in for many cases it is the kinetic energy minus the potential energy. So, T here is the kinetic energy and V is the potential energy.

For example free particle kinetic energy is half mass of the particle times the velocity square if I consider for simplicity one dimensional system particle moving in one dimension moving then it is velocity is the position coordinate time derivative of the position coordinate and kinetic energy is half m x dot square and potential energy for every part of free particle is 0 that gives us the Lagrangian equal to half m x dot square.

Now, without deriving we will describe, what is the equation of motion?

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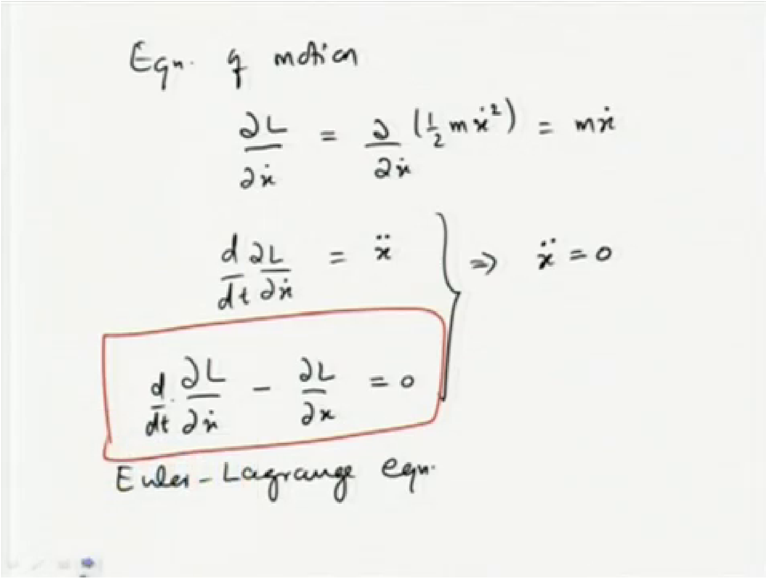

In this case, you take the derivative of the Lagrangian with respect to x dot and that will give you derivative of derivative with respect to x dot of half m x dot square which is equal to m x dot. Now you take the time derivative of this that is x double dot and equation of motion for any Lagrangian with one such coordinates is dou time derivative of derivative of the Lagrangian with respect to velocity minus derivative of the Lagrangian with respect to the coordinate itself is equal to 0 that is the equation of motion which is called Euler Lagrange equation of motion equations of motion.

So, this gives us in our present case x double dot equal to 0 which is the statement that acceleration of the particle which is free in the moving is equal to 0 that it will move with constant speed. So, this particular Euler Lagrange equation can be generalized to more than one coordinates and in particular we are interested in continuous system of particles for continuous system of particles continuous system say block of matter solid block or things like that we can we have infinitely many degrees of freedom.

So, there are infinitely many particles very large number of particles so, infinitely many means loosely speaking and this can be then considered as for the position of the particular point. So, we can assign some kind of a function which describes the describes the configuration of the system at a particular time.

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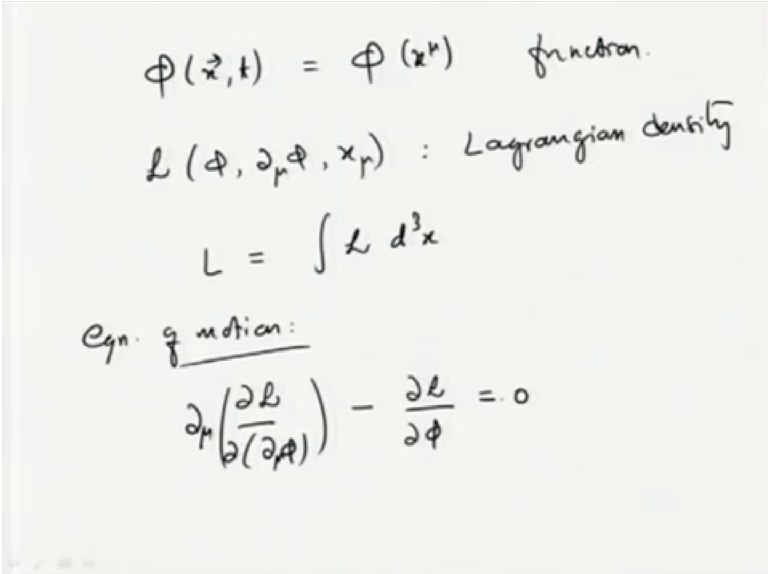

So, for that actually we rely on what is will define some function which is a function of the spatial coordinates and the time coordinates or in general we can write it as a function of the 4 vector and we can consider the Lagrangian now as a function of instead of the coordinates and the velocities the guy is like in the case of the particle mechanics.

We can now consider it as function as a function of these phis the which are the functions of the coordinates and the derivatives spatial and time derivative of this and it can also depend explicitly on the spatial coordinate and this is basically what we will consider to start with is the Lagrangian density in this case and actually Lagrangian is obtained by integrating this Lagrangian density over a volume over the volume of the system and the equations of motion generalized to derivative of the Lagrangian density with respect to the dou phi dou mu phi and take the derivative of this whole thing minus derivative of the Lagrangian density with respect to phi equal to 0.

Again we are not going to prove this we are not going to derive this, we are not going to discuss anything more about how this equation is arrived at, but one can actually look at any standard classical mechanics book including gold stein classical mechanics by gold stein or any other field theory classical field theory book then or quantum field theory book where introduction to classical field theory is given to understand what is the Lagrangian mechanics and how do you get the equations of motion in Lagrangian mechanics.

Now, we will, but illustrate this in our situations.

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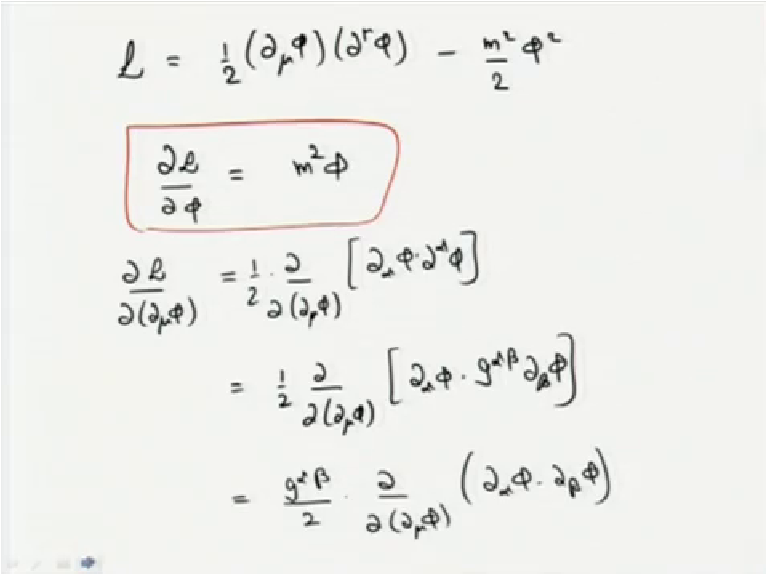

So, let us consider a particular Lagrangian density I will write it as 1 over 2 dou mu phi dou mu phi minus m square over 2 phi square for some field phi, some function of x mu phi is some function of x mu. So, in that case I can write dou L by dou phi I will get it as m square phi and dou L over dou derivative with respect to dou mu phi, how do I write it second term there is no dou mu for the first term has. So, what I will do is, to write it in a more convenient way.

It is there is a factor of 1 over 2 there and I will write it as dou alpha phi dou alpha phi . So, that there is no confusion with the mu that we are taking alpha is a dummy index and mu written in the Lagrangian density is a dummy index. So, I can write it in this fashion so, half derivative with respect dou mu phi or now since I am taking derivative with respect to a covariant derivative of phi I will write everything in the square bracket here in terms of the covariant derivative I can do that by using g mu nu or g alpha beta or the metric tensor g..

So, I will write it as g alpha phi as it is and this one is g alpha beta dou beta phi, I hope everybody understands understand the operation that we are doing g alpha beta is a metric which is independent of space time coordinates for special theory of relativity. So, we will take that out in our formalism we do not have any space time dependence for the metric.

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$$\frac{\partial}{\partial(\partial_\mu \phi)} \left(\partial_\alpha \phi \cdot \partial_\beta \phi \right)$$

$$= \left[\frac{\partial}{\partial(\partial_\mu \phi)} (\partial_\alpha \phi) \right] \cdot (\partial_\beta \phi) + (\partial_\alpha \phi) \cdot \left[\frac{\partial}{\partial(\partial_\mu \phi)} (\partial_\beta \phi) \right]$$

$$= \delta^\mu_\alpha \cdot \partial_\beta \phi + \partial_\alpha \phi \cdot \delta^\mu_\beta$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{g^{\alpha\beta}}{2} \left\{ \delta^\mu_\alpha \partial_\beta \phi + \partial_\alpha \phi \cdot \delta^\mu_\beta \right\} =$$

So, then we have dou alpha phi dou beta phi what is this, derivative of derivative with respect to mu dou mu phi of dou alpha phi and dou beta phi I have just rewritten anything here is equal to first take the derivative I will go slow here. So, that we understand the steps and then we will go a little faster later on.

First let me take the derivative with respect of the first factor here dou alpha phi and the other one is a spectator plus do the other way around dou alpha phi is a spectator and the derivative acts on dou beta phi, which is equal to this 2 are the same as long as you have same index on both the one which you are differentiating and the one which you are differentiating with respect to so, this essentially gives you a delta mu alpha.

The kronecker delta which says that this is equal to one if mu is equal to alpha is equal to 0 otherwise dou beta phi plus similarly dou alpha phi delta mu beta. In case you get lost with the indices what you have to check at every stage is that the number of free indices on either side and in all the terms are the same and other in a wherever you have a repeated index that is summed over and that you want the that to be summed over.

So, here for example, each of this term on the right hand side has a mu as a contra variant upper index and an alpha and a beta as the covariant indices for both the terms and on the left hand side we have a covariant alpha and beta indeed and a covariant mu in the denominator which is equivalent to a contra variant mu in the numerator. So, that is fine this is what you have look back at what you have in dual derivative of L with respect to dou mu phi g alpha beta. So, let me write that down derivative of L with respect to dou mu phi is then equal to g alpha beta by 2 g alpha beta contra variant by 2 into whatever else that we have here in the this thing. So, that is delta mu alpha dou beta phi plus dou alpha phi delta mu beta taking all these contractions.

Now, again there is only one free index alpha beta are dummy indices contracted and on the left hand side we have only one free index. So, it all agrees now let us put this g alpha beta and multiply the first term with that, what is happening is it will. Firstly, the delta mu alpha will change the alpha 2 alpha n g alpha beta 2 mu maybe let us do that slowly in the next slide.

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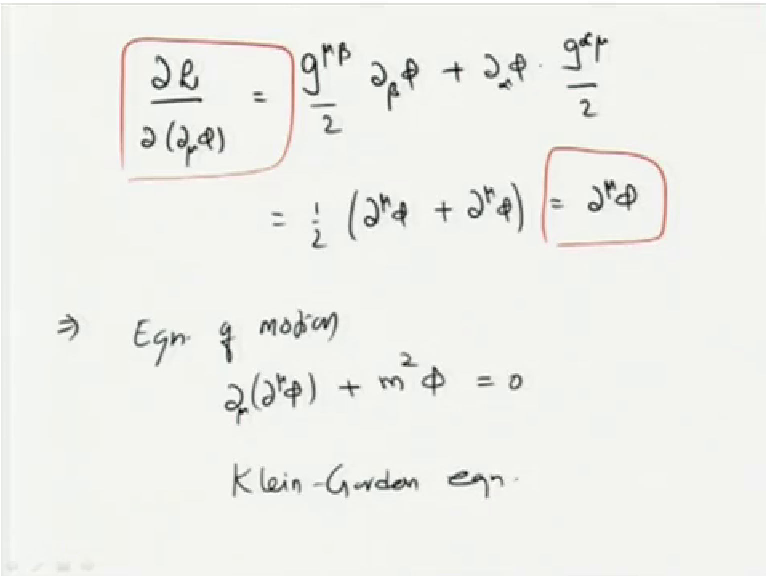

So, this therefore equal to g alpha beta I will use the delta function first to change mu 2 alpha sorry alpha 2 mu. So, I have a g mu beta over 2 this is only the first term into dou beta phi and for the second one we have a delta beta mu. So, mu will be sorry beta will be changed to mu. So, we have dou alpha phi as it is g alpha beta alpha mu beta is changed to mu again by 2.

So, this is equal to 1 over 2 the role of g mu beta acting on dou mu dou beta phi is to raise the index beta to mu make it covariant contra variant. So, we have first term dou mu phi, second term instead of beta you have an alpha no other difference. So, it is dou mu phi which is equal to dou mu phi without a half factor that is it all right. So, that is what you have here and you have one part this one which will go into the equation of motion and the other part here. So, that will give you equation of motion as derivative of dou mu phi minus m square of phi sorry actually there is a minus sign here already. So, it should be plus m square phi this is equal to 0, what is this, this is the Klein Gordon equation.

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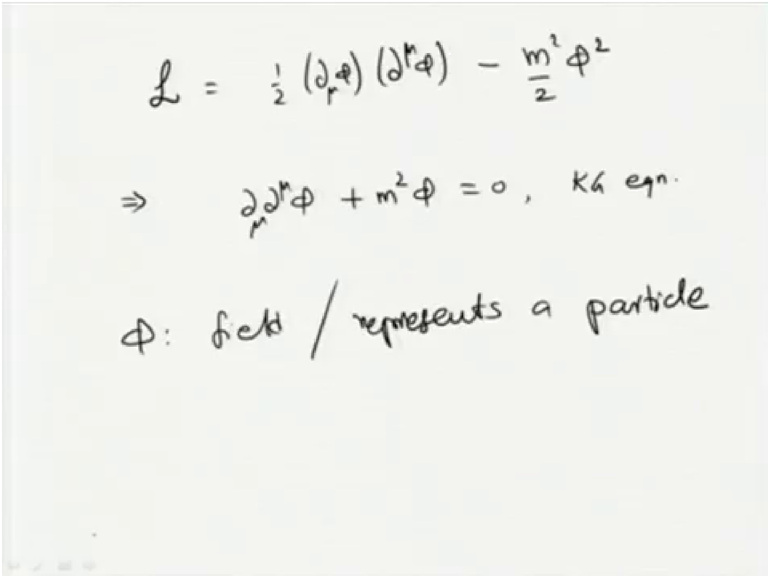

So, summary is that if I use the Lagrangian density half dou mu phi dou mu phi for mu summed over minus m square by 2 phi square for a function phi that will give us dou mu dou mu phi plus m square phi equal to 0 the Klein Gordon equation. So, we can say that the Lagrangian that we have written down is the Klein Gordon Lagrangian or the Lagrangian corresponding to a Klein Gordon field or a field that satisfied for a function that satisfies the Klein Gordon equation.

Now, these functions are called fields and we can actually say that this represents or correspond to a particle. So, this is a field is the one which will represent a particle. So, this is what is the quantum field theory does in relativistic quantum mechanics when we discussed the Klein Gordon equation we considered the phi as wave function and there is a subtle difference here when we actually consider this phi now as a field, again we will not go into the details of what is what it can be taken as a kind of a technical difference here, but the bottom line for us today here is that we represent particles now by fields or rather the fields acting on some vector space or the state vector and corresponding to each different particles will have different fields representing them.

So, one such field is this phi there is a way to actually relate the spins of the particles to understand the spins of the particles also and we will see that usually the scalar particles are the ones which are described by for the spin 0 particles are the ones which are described by Klein Gordon kind of Lagrangian that we have just now written down which we are staring at. Now another type of particle that we had encountered earlier is this Dirac the ones which satisfy the Dirac equation and in passing we had mentioned that this is basically the type of particles which are described by Dirac equation are spin half or fermionic particles for example, electron.

So, let us consider that kind of a Lagrangian.

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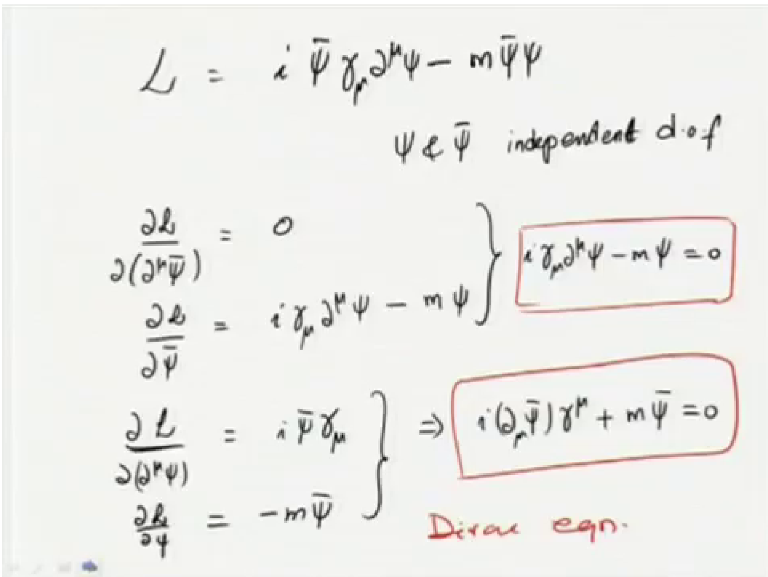

Consider the Lagrangian i times psi bar gamma mu dou mu psi minus m psi bar psi should take this Lagrangian and then consider psi and psi bar are independent degrees of freedom independent their functions. So, a Lagrangian is a function of psi psi bar and dou mu psi in this case.

Now, let us look at the equation of motion with respect to there are 2 such (Refer Time: 24:12) degrees of freedom. So, with respect to each of these we had to consider the equation of motion when we consider it with respect to psi bar we have no derivative term for psi bar. So, this is equal to 0 and when we consider dou mu dou L by dou psi bar we have I gamma mu dou mu psi minus m psi and equation of motion will tell you derivative or dou mu of the first line here plus or minus dou L by dou psi bar second line is equal to 0 since the first one is 0 we have i gamma mu dou mu psi minus m psi equal to 0 indeed we recognize this as the Dirac equation.

So, this Lagrangian gives us the Dirac equation, what about the other degree of freedom psi. So, when we take derivative with respect to dou mu psi we have i psi bar gamma gamma mu and dou L by dou psi is equal to minus m psi bar together this will give you i dou mu psi bar gamma mu plus m psi bar equal to 0 indeed this is the conjugate equation corresponding to the above Dirac equation. So, we have the Dirac equation and the conjugate equation from this Lagrangian given here.

Now, this it can therefore, be considered as the Dirac Lagrangian if you want to name it. So, psi here is the field corresponding to the fermionic particle and it is a function of the space time coordinates and momentum as we had written down earlier, all right.

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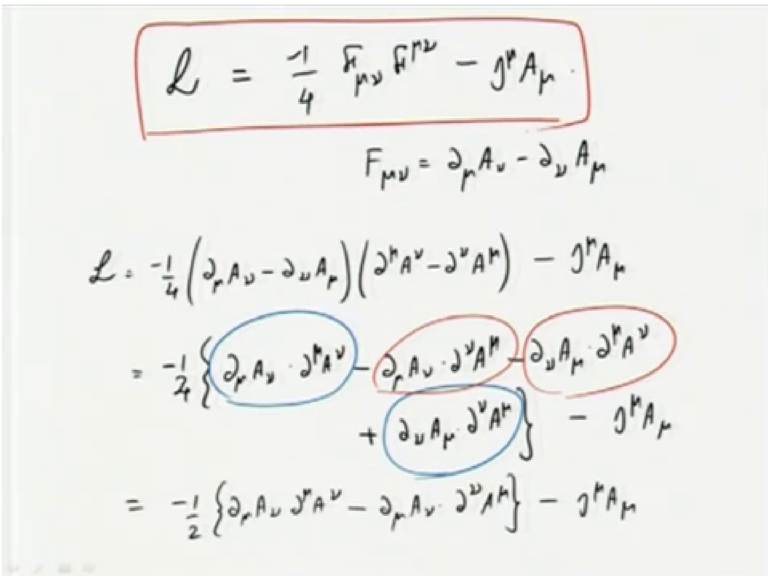

So, then let us go on to another Lagrangian which I will write as minus 1 over 4 F mu nu F mu nu F mu nu the field tensor corresponding to the electromagnetic potential minus some current J mu A mu potential A mu, F mu nu for us is equal to dou mu dou dou mu A mu A nu minus dou nu A mu. So, if I expand the Lagrangian F mu nu I have minus 1 over 4 dou mu A nu minus dou nu A mu contracted with dou mu A nu minus dou nu A mu minus J mu A mu remember also the Lagrangian density should not have any free index. So, all the indexes are summed over. So, then only you will have an invariant quantity Lagrangian density is a Lorentz invariant called Lorentz scalar.

So, this is when I open the bracket dou mu A nu dou mu A nu minus dou mu A nu dou nu A mu minus dou nu A mu second term in the first bracket times the first term in the second bracket plus dou nu A mu dou nu A mu. So, that crosses the bracket there minus J mu A nu. So, this is equal to first term and the last term in the clay curly bracket if you look at they are the same excepting that mu and nu are interchange since they are dummy indices if you interchange it nothing happens. So, I mean they are dummy indices.

So, then they are the same terms similarly you will recognize that the second term and the third term dou mu A nu dou nu A mu is minus A dou nu A mu dou mu A nu are the same for example, when you have dummy indices like this you could change the upper and lower indices without changing anything you will see that they are the same. So, I will leave that as a small exercise for you to raise everything and I recognize that they are the same.

Essentially this term that I will circle with a blue line so, the one which I circled with the blue line are the same and adds up similarly the other 2 the left the other 2 which are left, so, the which I circle with red are also the same. So, they also add up together you have minus 1 over 2 dou mu A nu dou mu A nu minus dou mu A nu invert the switch A nu dou nu A nu minus J mu A nu. So, that is your Lagrangian density in terms of the potential derivative of the potential (Refer Time: 31:47).

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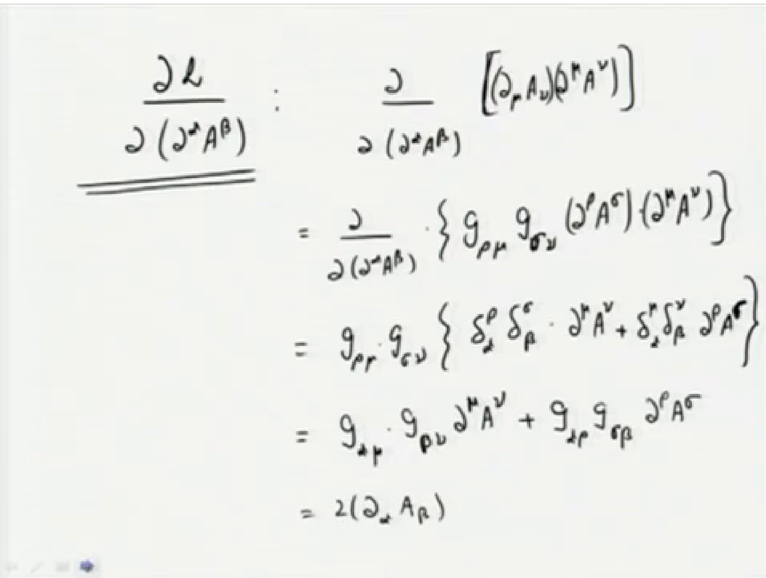

Now let me take the derivative of the Lagrangian with respect to dou alpha A beta. So, I have taken alpha beta in order not to confuse with this mus there in the Lagrangian let me take the first term here. So, the first term is going to give you so, this is what we want to look at. So, let me take derivative with respect to dou alpha A beta or first term of the Lagrangian apart from minus 1 over 2 is dou mu A nu dou mu A nu. So, dou mu A nu dou mu A nu. So, this acts only on that in the (Refer Time: 32:45), this is equal to derivative with respect to dou alpha A beta again, let me write it as g rho mu g nu sigma or sigma nu dou rho A sigma. So, idea is to raise all the indices of dou mu and A mu A nu and you have dou mu a nu.

Now, you can apply this take the derivative here. So, we have rho rho mu rho sigma nu was a common factor which is independent of space time or I a coordine A the vector potential. So, we do not have this affected by the differentiation, but the other one first it will give you a delta of rho alpha and delta of sigma beta and dou mu A nu as it is plus you have delta of mu alpha delta of nu beta and dou rho A sigma which is equal to.

Now delta rho alpha when you will change the rho on the g rho mu to alpha g alpha mu g beta nu dou mu A nu plus similarly g alpha rho g sigma beta dou rho A sigma which is essentially equal to dou alpha A beta and the other one also give you rho alpha rho beta. So, this is twice this so, first term here will give you twice dou alpha A beta and do a similar analysis.

And you will see that the second term will give you dou L over dou of dou alpha A beta as minus dou alpha dou beta A beta minus dou beta dou alpha L.

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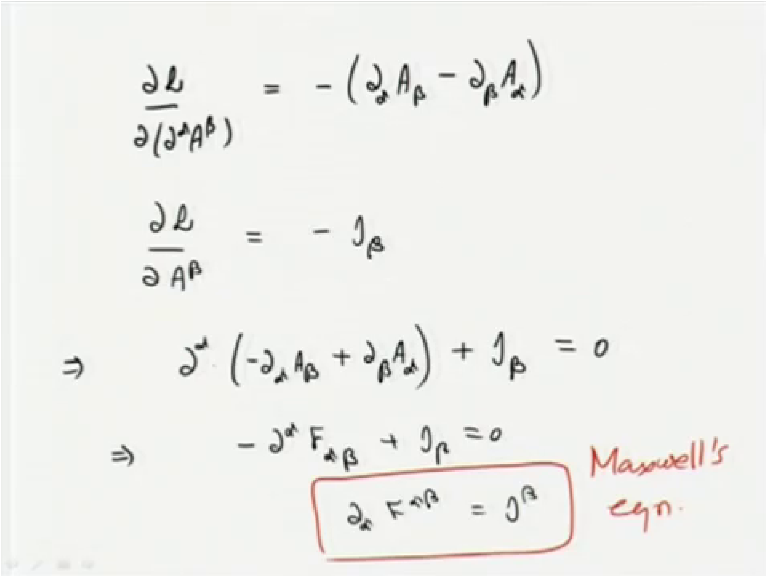

So, and this is the first part other term in the Lagrangian is J mu A mu with A minus sign. So, that will give you dou L over dou A alpha equal to minus J alpha, here let me take the same J beta and that will give you J beta. So, together that will give you derivative of the first line with respect to alpha in the numerator of minus dou alpha A beta plus dou beta A alpha plus J beta is equal to 0.

So, this is equal to essentially minus dou alpha F alpha beta plus J beta equal to 0 or dou alpha F alpha beta equal to J beta you recognize this as the Maxwell's equations. So, therefore, the Lagrangian that we considered it here leads to Maxwell's equation all right. So, we started we have described 3 different Lagrangian.

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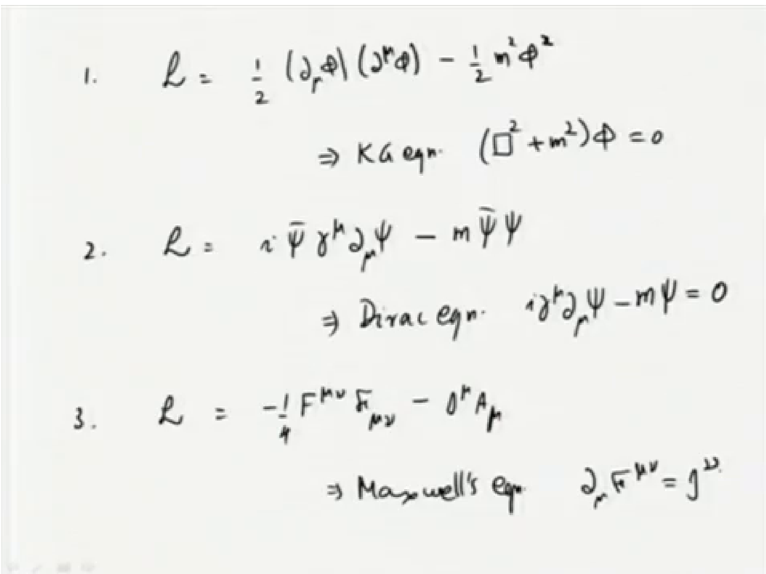

One L is equal to 1 over 2 dou mu phi dou mu phi minus 1 over 2 m square phi square and. So, that this gave us the Klein Gordon equation which is essentially box square plus m square acting on phi equal to 0 then we looked at i psi bar gamma mu dou mu psi minus m psi bar psi and that gave us the Dirac equation i gamma mu dou mu psi minus m. So, i equal to 0 and then we considered the Maxwell's Lagrangian F mu nu F mu nu minus J mu A mu and that gave Maxwell's equation dou mu F mu nu equal to J nu.

So, we have someone somewhat familiarize ourselves with the Lagrangian and equations and then we understand how we can write down the Lagrangian corresponding to the Dirac equation, Klein Gordon equation, Maxwell's equation which are the 3 familiar equations that we had discussed in the earlier this ones. So, the task that we have now next is how to write down the Lagrangian with interactions of the particles included.

So, not just the free particle is not a very interesting thing that will not tell us what the dynamics is it tells us only how the free particle is propagates that is fine, but if you want to understand the dynamics how they behave under various different situations especially under the basic forces and we need to include the interaction in that theory. So, we will just now see how this interaction can be taken into account this time.