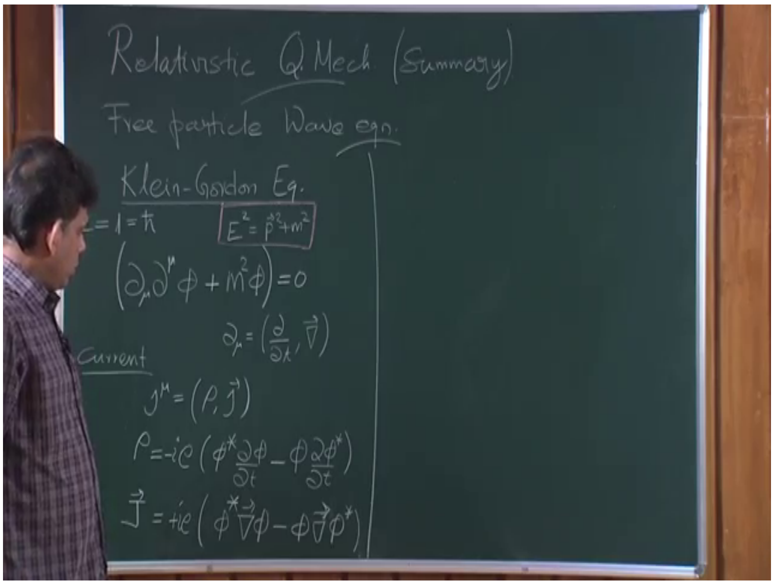
Nuclear and Particle Physics
Prof. P. Poulose
Department of Physics
Indian Institute of Technology, Guwahati

Lecture - 34
Interacting charged fermions – 2

We shall today, first summarize the three particles wave functions in quantum mechanics in a relativistic framework.

(Refer Slide Time: 00:41)



So, this we had first considered the Klein Gordon equation which coming from considering the relativistic energy momentum relation, when looked at from the quantum mechanical operators corresponding to these physical quantities. I will give you an equation that the wave function phi that is at representing in the particle should satisfy, which is called the Klein Gordon equation.

So, here the basis of this equation is essentially the energy momentum relation in theorem. And we have used the notation where dou mu, the derivative operator in the 4-dimension case along with special derivative including the time derivative ; in components, it has a time component time derivative, then three special derivative components combined in the gradient operator. We have we are considering a framework where in a units where c is taken to be 1. And so, is the h cross which is the Planck constants reduce Planck constant, which is also taken to be 1 so, there is the unit will be used.

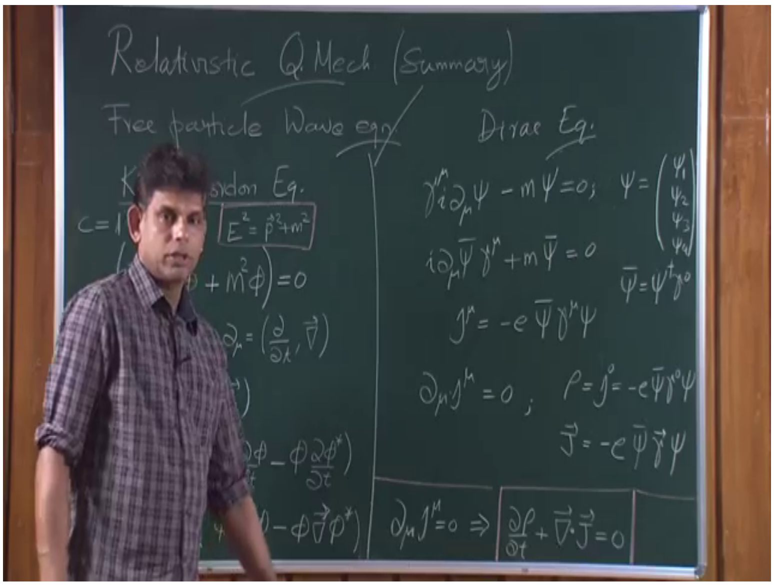
Now, when we talk about this equation and a free particle we can talk about also motion of the free particle so or the current associated with this. So, currents j mu, again we will use the for-vector notation, 0th component is represented by rho and 3 components clubbing the 3-vector j.

So, for this psi, the particle that is represented or that obeys the Klein Gordon equation, relativistic equation that is rho can be written as minus i e minus i e phi star the complex conjugate of phi time derivative of phi minus phi times time derivative of the complex conjugate of phi.

And the 3 vector j is plus i e phi star gradient of phi minus phi gradient of phi star. This is something which we had already seen earlier. So, I am not going to explain this any further, but essentially, we had looked at the Klein Gordon equation in detail at some point, and then these are the things some of the things that we had discussed there.

This j for vector current is divergence less, which means you take the 4 vector divergence of this that vanishes that is given in components this time derivative of rho plus gradient of j equal to 0.

(Refer Slide Time: 05:04)



Which is essentially I did not we recognize as the continuity equation, in some sense we had interpreted this as well in a sense, that this is going to tell as say for example, if you have some amount of charge in a volume considered, then if we look at the charge flow out of this volume.

Let us say confined in a sphere, then if you consider the flow of charge out of this sphere across the surface, that flux or the current the divergence of that current plus the rate of change of the current density, charge density should satisfy this equation, or if you take this to the other side will essentially get the decrease in the charge density is the same as the outgoing current or flux.

In the case of I mean we had another, another way of looking at the relativistic state of particles that is through Dirac equation. And in Dirac equation, we had considered a linearized form of this, which essentially give as i dou mu usually we give a which will which is the moment of operator in sense, gamma mu a new thing that we have introduced here which now act on psi together, minus mass of this m psi is equal to 0.

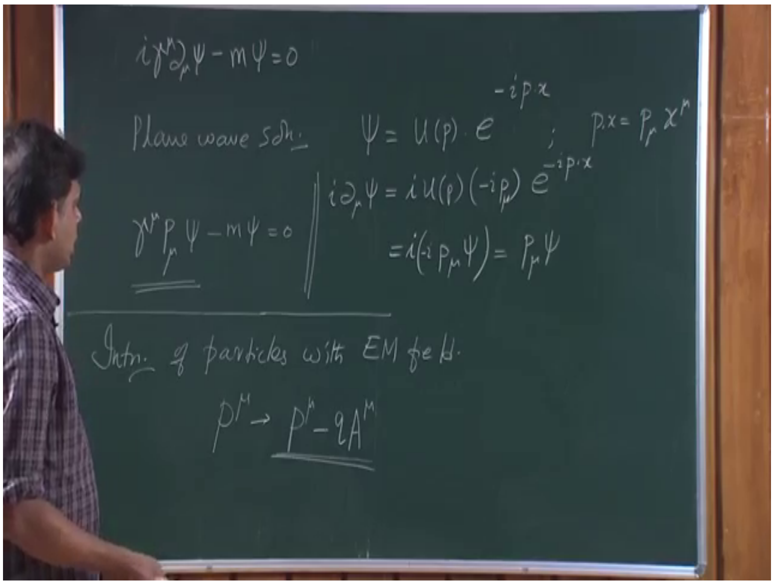
So, this now is called the Dirac equation, and psi itself is a 4 component objects, which in matrix form can be represented as a column matrix psi one psi 2 psi 3 psi 4, that is in way. Unlike, a single complex object like phi. So, one more thing that we have to understand is the conjugate Hermitian conjugate form of this equation which is written in this form i dou mu complex Hermitian conjugate, along with Hermitian conjugate one has to actually consider a little bit algebra more than just taking the Hermitian conjugate.

But essentially the right form of the co Hermitian conjugate equation is given in terms of this object psi bar which is defined as psi dagger Hermitian conjugate of psi along with a gamma 0.

With this, we can define the j mu the currents which is equal to minus e psi bar gamma mu psi. And this again will satisfy the continuity equation (Refer Time: 09:32) in the 4 vector 4. And so, we can in fact, again think about rho as the 0th component of this j is j 0 which is equal to minus e psi bar gamma 0 psi. And j V vector j is equal to minus e psi bar gamma vector psi.

Now, this vector means only that with the three components taken together in the form of vector. And they will satisfy the continuity equation as it is written there ok.

(Refer Slide Time: 10:34)



So, let us consider the Dirac equation, an write it in a slightly different way. So, the Dirac equation the way we had written as this operator i gamma mu dou mu acting one side minus m psi is equal to 0, and let us say consider a plane wave solution of this of the form takes psi to be some u depending on whatever the momentum only, and all the space dependence is through some kind of a oscillatoring oscillatory function exponential minus i p x this is what is called the plane wave solution, where p dot x is the dot product for vector dot product between p the momentum and x. So, this sorry, I am in the notation this should be p momentum p.

Now, with this we consider derivative of this psi that will give you nothing happens to you with it is a function only of p, from here you will get a minus i p index is mu. So, that has to be mu here. So, this mu will come out I mean so, then you have the same thing exponential minus i p dot x.

So, this is nothing but minus i p mu psi. So, that is not surprising because this is the momentum operator when it acts on p it will give you the momentum. Of course, we are not put the entire thing momentum operator is i times p mu. So, that would actually give you derive this thing. So, if I put an i here another i here then that would be this is minus i the whole thing. But then this additional I and that will give you simply p mu psi.

So, this is the momentum operator that is so, when you actually look at it this way, then this equation Dirac equation can be written as gamma mu p mu psi minus m psi.

Interaction of particles with electromagnetic field, for that what we had considered? We saw that we can simply change this momentum p to p mu plus whatever is the charge of this particle minus actually minus q of the charge of the particle with a mu. Where A mu represents the electromagnetic field, q is the charge of that particle, p is the momentum of that particle meaning for momentum, which includes energy as well as 3 momentum ok, particles motion must in the and the energy total energy of that.

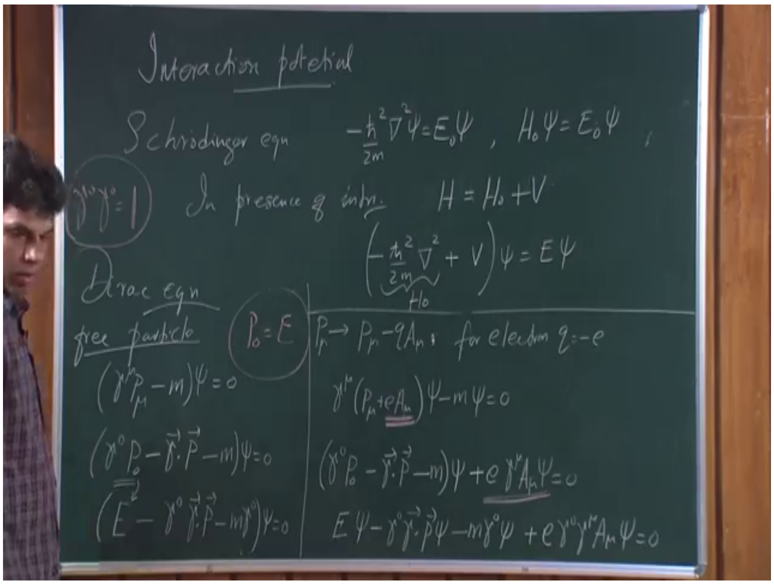
So, if we make change in the equation by changing the p to p mu by changing p mu replacing it by p mu minus q A mu, that adequately describes the particles interacting for the equation of motion, for the particle including it is interaction with the electromagnetic field. Of course, the electromagnetic field itself is not described otherwise that will have to be added through the Maxwell's equation which can be incorporated there.

But this is concerning the charged particle motion. The charged particle moving along with that it encounters electromagnetic field represented by m that is the radius. And essentially our idea is to consider how we study the interactions and the transition amplitude which will give information about the scattering of such particles by electromagnetic potentials.

Say for example, alpha particle when encountered the electromagnetic field of the nucleus. They got scattered in through the force experiment. How do we actually understand it? And then that was alpha particle, but here we are talking about an electron if an electron is sent to such a this one how do we actually understand it theoretically.

So, for that what are the equations for what are the what is the equation of motion that we should consider? So, basically the interaction potential has to be understood.

(Refer Slide Time: 16:24)



What do I mean by that? That is Schrodinger equation, let us say take Schrodinger equation Schrodinger equation for a free particle is minus h cross pi r del square over 2 m is equal to e psi is equal to e psi where psi is the wave function of the particle, density this is the free particle there is no potential (Refer Time: 16:58). So, this I can actually consider as H naught psi is equal to e naught psi or e psi energy psi (Refer Time: 17:08).

Now, this in the case of interaction potential in presence of interaction with a potential v, we have now total Hamiltonian H is equal to H naught plus V, and where now the equation is h cross square over 2 m grad square plus the potential, which is essentially H naught and is equal to e psi. This is different from the psi, maybe I can write it here as e naught, that is how we understand.

Let us see what happens in the case of Dirac equation. Dirac equation free particle let me write that free particle here for free particle, I have p mu gamma mu equal minus m psi equal to 0. Let me split this into the energy part of and the momentum term. So, that is gamma 0 e plus or the rather here it will be minus, this is p 0, p 0, p 0 is essentially the energy 0th component. Then you have p dot product of trip vector gamma and p minus m psi equal to 0. This is essentially energy, and I have the property of gamma matrix, that gamma square is so, gamma square is equal to 1.

So, let me use that property, I will use that property and then multiplied this by gamma 0, and then I have p naught equal to energy, this is essentially energy, minus gamma 0 I have multiplied this by gamma 0, and gamma dot p minus m psi, no m gamma 0. So, multiplied have written by a gamma 0. This is essentially what we have in the case of free particle.

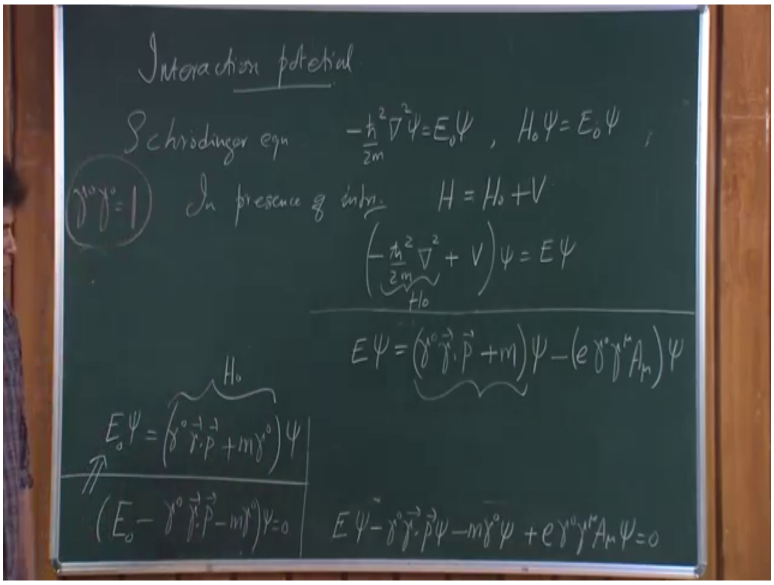
So now when we consider the interaction, we set p mu will go to p mu minus q m. In particular, consider for electron q is equal to minus e, and that will give you p mu plus a. So, let us write that, gamma mu p mu plus e A mu electron interacting with the mu A mu minus psi minus m psi equal to 0 is the Dirac equation for an electron interacting with electromagnetic field represented by A mu ok.

So, that is what it is, it is the electron (Refer Time: 21:20) additional term now. This again when we write it in this fashion we have gamma 0 p 0 minus gamma dot p ok. And let me add this term here and then take this term there. So, m psi this is a free part plus e times gamma mu A mu psi equal to 0. So, this is basically the additional term due to the interaction.

So, this can be written as again after multiplying by gamma 0 and I mean writing p 0 as e. So, maybe for clarity I will write here p 0 is equal to energy e. So, e psi equal to or rather e psi minus gamma 0 gamma dot p psi minus m psi m gamma 0 psi minus (Refer Time: 23:09) plus e gamma 0 gamma mu A mu psi equal to 0, ok. I will make a slight change in this one and write it as, perhaps I will rub this side off keep this there. So, that we can refer to it, then in the interaction case we have this equation.

So, let me write that again here, this is e, psi equal to gamma 0 gamma dot p plus m psi, minus this goes to the other side.

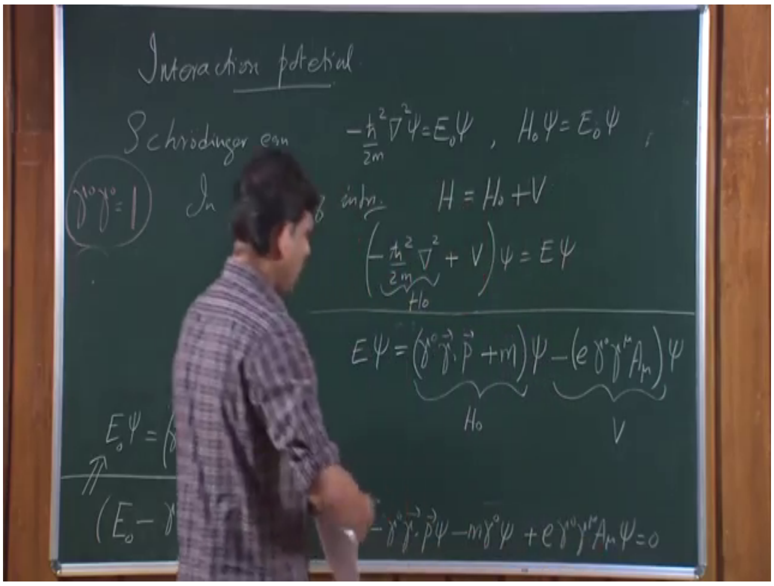
(Refer Slide Time: 23:59)



So, that will become minus e gamma 0 gamma mu A mu psi. So, this is essentially the H 0 part of this. From here we can write it as e psi is equal to gamma 0 gamma dot p V 0 actually, this is the free particle s, minus sorry, plus m gamma 0 psi. So, this is what this one will give.

Now, let us compare this and this, you will see that this is nothing but the H naught are the one which corresponds to H naught and this is basically the V ok.

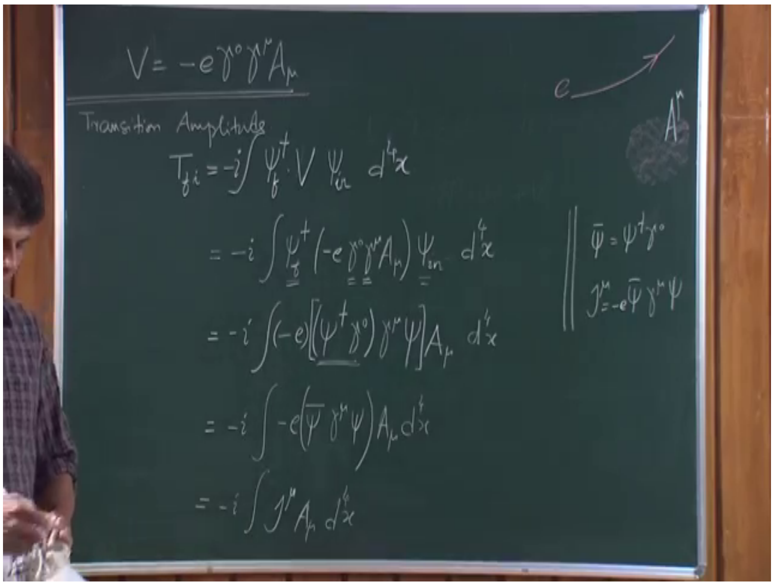
(Refer Slide Time: 25:19)



So, let me rub off everything else so that we can clearly see what is there here, and also the Schrodinger equation. So, the Schrodinger equation we had e psi is equal to H 0 plus

V psi here also we have e psi equal to H 0 plus V psi. So, the potential for an electron interacting with electromagnetic field is V equal to V equal to minus e gamma 0 gamma mu A mu.

(Refer Slide Time: 25:59)



So, this is the potential corresponding to electron interacting with electromagnetic field represented by A mu. Now, in this case, ultimately, we want to understand, how? Say for example, electrons scatter of some potential electromagnetic potentials. So, this is something which electromagnetic potential and there is an electron.

Now, how do we actually see their interaction? So, again we had defined what is called the transition amplitude sometime back. This is the transition amplitude which is related to the probability to see an electron interacting with A mu and scattering of to with some financier. So, electron having with represented by an incoming wave function psi, interacting with some potential v. So, to start with we considered to some V, and that V is this represented by this and V. And finally, going into a final electron; So, that outgoing electron was given as a psi dagger in fact.

Now, we have to consider all possible momentum of this. So, we have to consider the where is what possible combinations of this that we have to actually integrated over the spatial volume all over, wherever it is catering into not any particular directions, but you have to consider the probability for that scattering to happen, for electron going into any directions with all in all spatial directions in that sense.

We had an i minus i rather in along with this i. So, this is something which we had discussed earlier, when we discussed this catering amplitude. We are discussed in the nuclear physics conducts, how the Rutherford scattering happened etcetera, we had discussed in detail what is the meaning of transition amplitude. And so, I do see you to actually take a look at it and then, understand this.

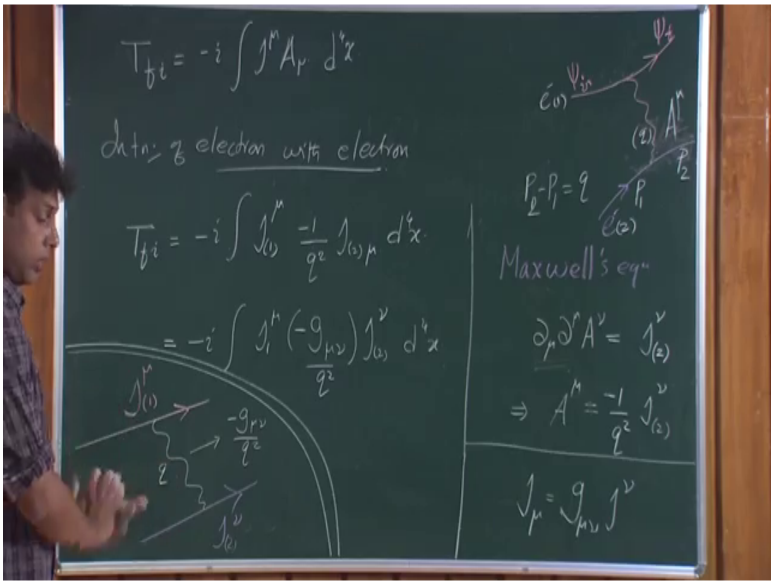
So now this minus i integral psi f dagger representing the electron which is going out and then we have a minus the potential is minus e gamma 0 gamma mu A mu, ok. This is summed over gamma mu dot A mu that is how it is; psi in integrated over the 4-dimensional spatial volume, space time volume. This can be written now as minus i integral psi f dagger. I will take this gamma 0 along with this and rest of it is minus e ok, gamma mu even minus e, I can take out. So, let me do it this way.

So, I have an integral oh minus e psi dagger then there is a gamma 0 and there is a gamma mu, and A mu a mu I can take out. These are matrices in the Dirac space, but A is not so, A I can take out, e I can take out without any, for I will actually be able to write it in this fashion ok. So, this is what I have, but I know psi bar is equal to gamma dagger gamma 0, and j mu is equal to psi bar gamma mu psi.

So, here indeed I can actually write this, we get see this integral minus e e psi bar gamma mu psi interacting with A mu or taking the dot product with A mu d 4 x. For that is essentially equal to minus I integral minus e write j had a minus e with this minus e psi bar the, remain thing is nothing but j mu and A mu ok, right.

So, we have the transition amplitude written in terms of the vector current, and the potential for vector potential representing the electromagnetic field. And the electron is represented by the current electron current, and this is the electromagnetic field and transition amplitude is given by this thing. So, this is one thing it will keep in mind, let me write that again. So, we have T f i is equal to minus i integral j mu A mu.

(Refer Slide Time: 33:06)



If you look at this here, we can think about this as, say electron initially represented by psi i finally, represented by psi f psi in and psi f, and then the photon which is represented by some (Refer Time: 33:30) it is interacting with this, and this is essentially represented by current j mu, and interacting with the new mu.

So, that is the picture we have. Of course, we have integrated over the spatial (Refer Time: 33:50) how do we represent the interaction of two charged particles? Interaction of electron with electron, this current interacting with this electromagnetic field generated by another electron; So, this A mu is essentially coming from another electron ok, another electron, how?

This is connected with this, we know that Maxwell's equation written in the in the form of or in terms of the potential is a nu, dou mu, dou mu a nu this equal to j nu; where j nu right hand side is the source. So, this is now given by the source electron second electron let me represent it by 2 and here let me represent this by one, electron 1 and electron 2. So, j 2 this is the source of A mu and we said we want to consider the electric field due to another electron represented by e 2.

So, that e 2 current is the source, and the corresponding electromagnetic field. A mu will satisfy this equation, and a solution of this will give you this again we had considered earlier, otherwise a let us just assume that this is so 1 over q square j nu.

But this q square is essentially coming from the fact, that if you take plane wave solution for A e power i q x kind of a thing. And then this dou mu dou mu will bring down a factor of q square. So, this is essentially needed to balance that that is how it is. You do not worry if you do not know the details. So, let us just assume this at the moment. So, this is a good representation of electromagnetic field generated from a current j where q is essentially given by ok, the difference in the momentum. So, p q minus p 1 is the q so, we essentially consider some a with momentum q greater. So, p 1 is split into q and p 2 so, p 1 is equal to p 2 plus p 2 or p 1 minus p 2 is q. So, that is what you have there.

Now, let us see how do we write this same transition amplitude T initial to final representing that electron interacting with this that we will see is psi j. Now we will write it as j 1 A mu mu and mu mu is 1. Strictly speaking you need here minus 1, because again a plane wave solution will have exponential i p dot x, there is an I factor which will come along with q, factor of I into I because of 2 derivatives. So, I square which will give you minus i. So, it is minus 1 over q square j 2 mu equal to in d for x.

Now, this is what we have or I can write it in a slightly rearranged form, j 1 mu minus g mu nu over q square, and jth nu differ nothing subtle that I have done here. I have just written this j mu as g mu nu A mu that I could always write, j mu is equal to j nu g mu nu j nu.

Now, let us pictorially represent is we have one electron represented by current j mu. And we have another electron represented by the current j 2 nu j 2 nu. And we have the photon field like that we have something connecting them, essentially, we know that that is the electromagnetic field A mu electromagnetic field that is connecting this. And now, this is given by g mu nu over q square; where q is the momentum that this photon carries.

So, in is this is exact this is the way we will understand the particle interactions. So, let us focus that for a minute, and then we will conclude today. So, essentially, we have now one electron, another electron, but since they have the electromagnetic field with them they interact with each other through their that, and then their interaction and such is given by this. And the transition amplitude which is finally, related to the scattering cross section or the probability for such interaction, probability for such an initial set of electron going into a final set of electrons, two electrons through this form.

This is the way we will understand the interaction of electron and electron. It is not the complete story, but essentially that captures all most all the features of whatever the electron interaction interacting with another electron, yes.

So, in addition to the electromagnetic field we will see that there is also the weak interactions that one has to consider when we talk about electron electron interaction. But that is going to be later.