Nuclear and Particle Physics
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Module – 09
Electroweak Interactions
Lecture – 08
Dirac Equation – continued

We were discussing the Dirac equation which is another way of describing the relativistic quantum particle.

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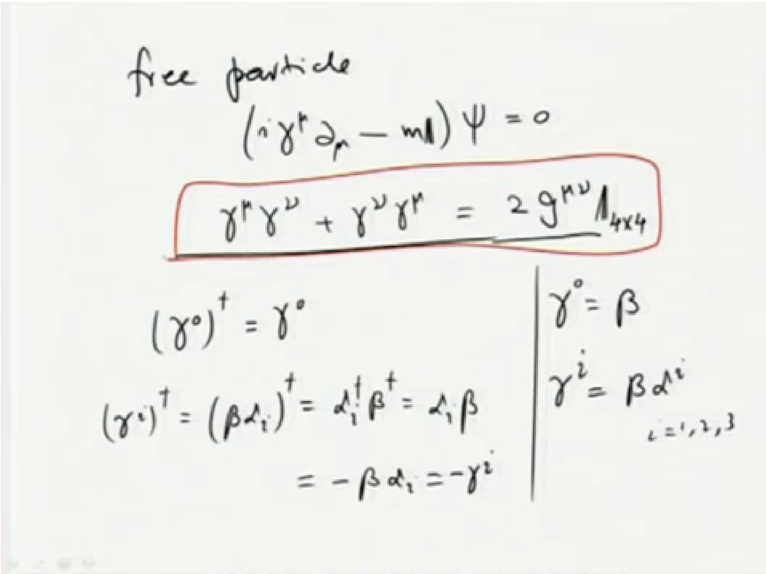

So, we said for a free particle, we could consider the equation i gamma mu dou mu minus m where m is the mass of the particle and acting on psi equal to 0 where psi represents the wave function in represents the particle. And it is the wave function of the particle gamma mu are objects which anti commute which means if we take gamma mu gamma nu with 2 different indices gamma nu gamma mu added to that will be equal to 0.

If mu is not equal to nu and equal to 1, if when mu is equal to 1 that if gamma mu square is equal to 1, gamma mu square plus gamma nu square will then be equal to 2. So, together for any mu and any nu, we can write this and gamma 0 dagger the Hermitian conjugate is equal to gamma 0. We have seen that actually gamma can be written in terms of gamma 0 can be is equal to beta, remember, our original form of the Dirac equation in terms of beta and gamma and gamma i is beta alpha i where i now runs from 1 2 3.

So, gamma i square sorry gamma i dagger the Hermitian conjugate is beta alpha i dagger which is equal to alpha i dagger beta dagger which is equal to alpha i beta because alpha and beta are Hermitian coefficients are Hermitian then or the Hermitian conjugate of each other and then this because of the anti commuting property of alpha and beta it can be written as minus beta alpha alpha i which is nothing, but minus gamma i. So, then this says that gamma 0 is Hermitian and gamma i where when you take the special code components i equal to 1 2 3, then that is anti Hermitian. So, this is what we discussed yesterday.

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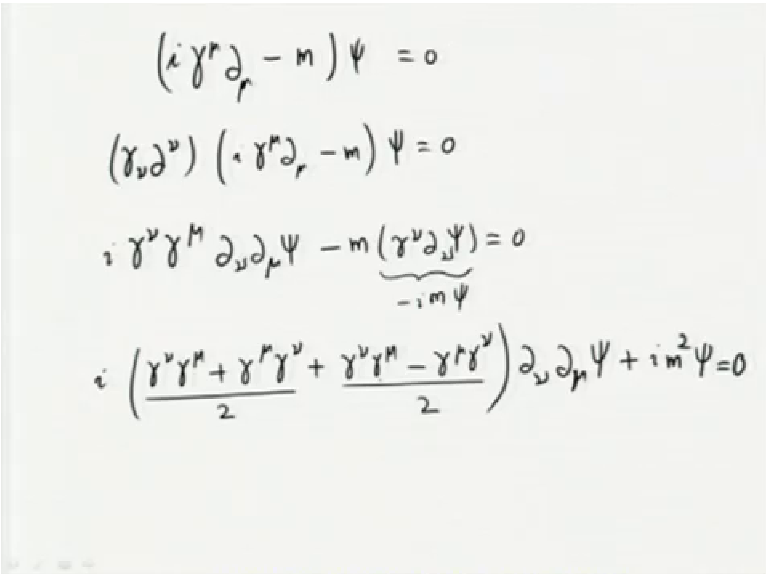

And now let us go ahead and then see further what else we want to understand here if you take gamma the Dirac equation i gamma mu dou mu m psi equal to 0, multiply it with gamma nu dou nu gamma nu d nu of the operator where nu is summed over and in Dirac equation mu is summed over gamma mu dou mu minus m psi equal to 0, what do we get? First it is gamma, well, I could write it as gamma nu gamma and the other way around also. So, I have gamma nu i gamma nu gamma mu dou nu dou mu by the way, gamma nu gamma mu are the matrices or one way of representing them in matrices or they do not commute with either.

So, when you take one across the other you will have to be careful whereas, the operator differential operator has nothing to do with these. So, they are this gammas are basically some constant as far as the derivative operator is concerned and then they will be can be easily taken across them without worrying about any other issues. So, that is what we have here. This is the first term and then the second term is m gamma nu dou nu psi in is equal to 0.

First let us look at the second term gamma nu dou nu psi because of the Dirac equation is now minus i m psi, right. So, this whole thing is minus i m psi because of the Dirac equation, the first line on this slide and therefore, what we have. So, and the second one what I will do is to write it as gamma nu gamma mu plus gamma nu as gamma mu gamma nu, I added gamma mu gamma nu it..

So, I had to subtract the same thing, but then what I will do is I will take half of this and take half of gamma mu gamma nu along with minus gamma mu gamma nu and half of this together they sum over 2 gamma nu gamma mu faster the third term will give you gamma mu gamma sorry gamma nu gamma mu and second term and forth term cancel each other and then we have a dou nu dou mu this thing acting on side and you have plus i m square psi equal to 0.

So, the first term, let us look at the first term it has 2 parts one is symmetric under the exchange of mu nu if you take the first and gamma mu gamma nu gamma mu plus gamma mu gamma nu that is symmetric under the exchange of mu and nu. So, I change mu to nu the first term becomes second term and second term becomes first term whereas, the second part in the bracket here gamma nu gamma mu minus gamma mu gamma nu is anti symmetric under the exchange of mu and nu if I change mu to nu.

The first term becomes a second term, but there is a minus sign there. So, when second term becomes the first term and there is without the minus sign. So, this all will pick up an overall minus sign when I change mu to nu and nu to mu. So, that is why we call it an anti symmetric under the x it is anti symmetric under exchange of mu and exchange of the indices..

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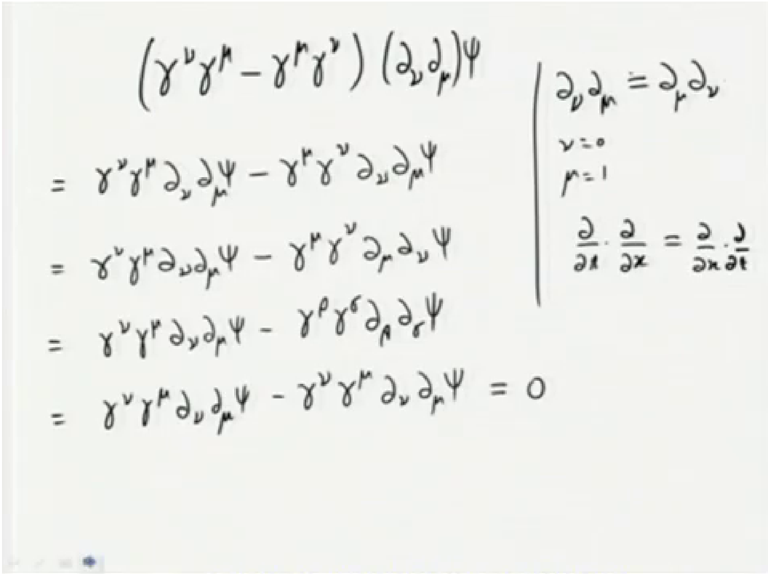

So, let us look at that anti symmetric part gamma nu gamma mu gamma mu gamma nu and there is a 2 nu dou mu this thing dou mu dou nu is actually symmetric.

This is basically differential operation and it does not matter whether if you take say nu equal to 0 and mu equal to 1, you will get differential operator dou by dou t and dou by dou x it is the same as writing dou by dou x first and then dou by dou t after that. So, this is like interchanging mu and nu. So, therefore, dou mu dou nu dou mu is symmetric or let me actually write it as equal to dou mu dou nu it is the same object.

So, let us look at this acting on psi. So, this first one is gamma nu gamma mu dou nu dou mu psi minus second part is gamma mu gamma nu dou nu dou mu psi and not done anything just expanded it. Now let me leave the first term as it is and, but interchange mu and nu this interchange is actually changing this to the dummy indices or first let me change the leave mu and nu as it is.

But since we have dou mu nu equal to dou nu mu I can write this as dou mu dou nu psi now this is exactly the same in the exactly the same form as the first term if it is confusing to you because of mu nu let me actually do the following let me change it to some rho and sigma rho and sigma because it is a dummy index summed over independent of the other terms, I can just change it into anything else alike. Now I can instead of writing the first term is as it is instead of writing changing the rho to mu I can change again rho to nu which I can do and sigma to mu.

So, consistently I have to do all the rhos to nu and all the mus to sigmas to mu. Now this is exactly like the first term well you could have done it just by observing the second line here and then come to the last conclusion, but we are done it in steps. So, that it is not confusing to yes. So, this is now first and second terms are exactly the same therefore, equal to 0..

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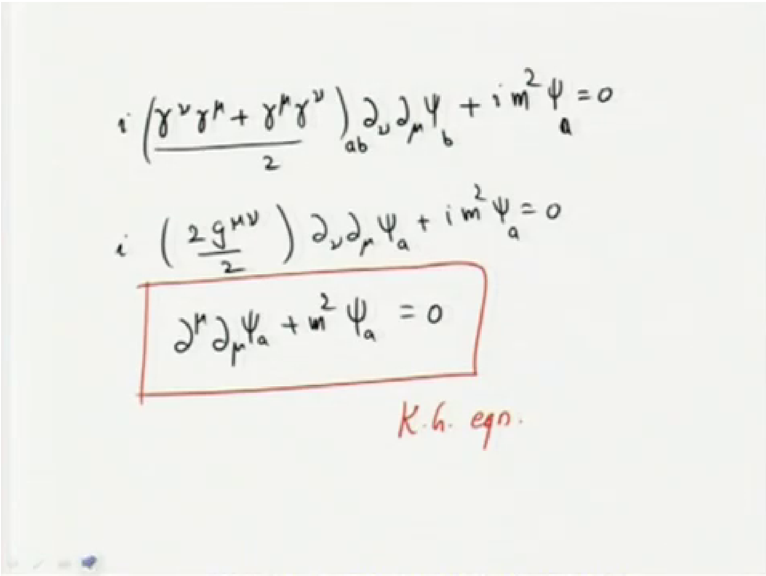

So, in this expression here the Dirac equation might have operated on by gamma nu dou nu or d we have i gamma mu gamma mu plus gamma mu gamma nu over 2 dou nu dou mu psi plus i m square psi equal to 0 is what here other term is identically equal to 0.

But now we had an anti commutation relation between gamma matrices and that tells us that it is equal to 2 g mu nu over 2 dou nu dou mu psi e plus i m square psi equal to 0 i is common, I will take that away 2 from the first term cancels with 2 in the denominator of that term g mu nu will raise the index of dou nu to mu. So, I will have a dou mu dou mu psi plus m square psi is equal to 0 it is still a matrix equation. So, this actually says that each component of psi satisfies this equation right there is no other regime.

So, here basically let me also write down then the Dirac matrix symbols also each of this is a b psi b a this is what the component wise form of the Dirac equation multiplied by dou nu gamma mu and then it comes to the next line it is again a b, but g mu nu is these are constant as far as the Dirac matricing is concerned.

So, when you actually write let us go back to that when you actually write this expression here the anti commutation relation we have a 4 by 4 unit matrix on the right hand side which is understood since I mean, if you get confused you please write that each time there, it is similar to having an unit matrix of 4 by 4 unit matrix in m. So, that you can add the first 2 terms these 2 terms in the Dirac equation you cannot add some there are 2 different quantities.

So, gamma mu is a matrix for the 4 matrix therefore, what you can add to it is a 4 by 4 matrix. So, where is that 4 by 4 matrix in the second term it actually m times unit matrix that is what you have. Similarly here in the second line, in the anti commutation relation also, it is exactly the same thing that you have let me highlight that now let us go to our this equation what we have here. So, this essentially is a and a e a the wave function now has 4 components and each component satisfy this equation.

And I believe you know this equation this is the Klein Gordon equation which each component of the web psi satisfied all right. So, that is not surprising because we need a square e square equal to P square plus c is the m square which is a result of special theory of relativity the energy momentum relation that be satisfied by anything. So, this is another thing that we can see.

And now let us come back to the Dirac equation in the case of Klein Gordon equation. The wave functions that satisfy the Klein Gordon equation we had a continuity relation written in terms of a conserved current a current whose 4 divergence vanished that we could get looking it or we could construct that from the wave function phi that satisfied the Klein Gordon equation what we did in that case is.

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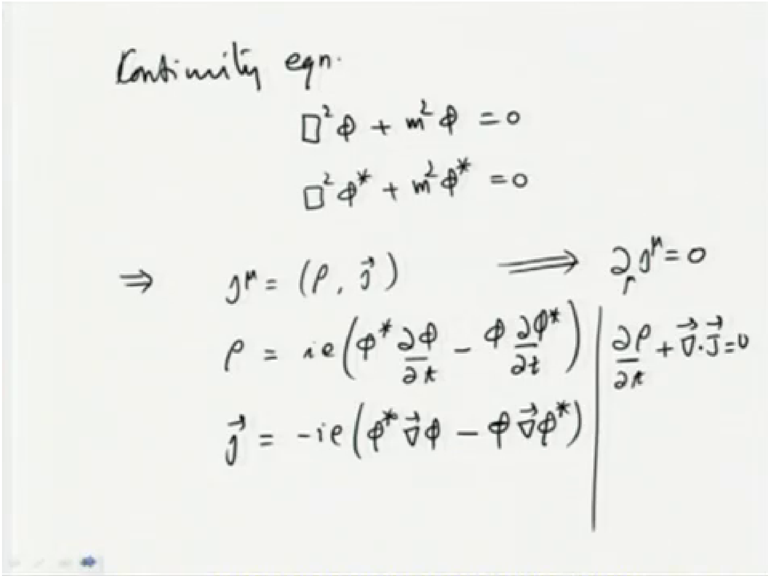

So, continuity equation; how we did that; this the following is the following way the box square phi plus m square phi equal to 0 is the Klein Gordon equation for the wave function phi and then we took the complex conjugate of that which essentially told us that the phi star also satisfies the same Klein Gordon equation.

And these led to J mu which we could write as 2 components rho and J where rho is equal to ie phi star dou y over dou t minus phi dou by dou t of phi star and J is nothing, but minus ie phi star grad phi minus phi grad phi star and with this we had dou mu J mu is equal to 0 and that in terms of the components is dou by dou by dou t of rho plus divergence of J equal to 0 which is basically the continuity equation.

Let us do a similar thing for the Dirac equation.

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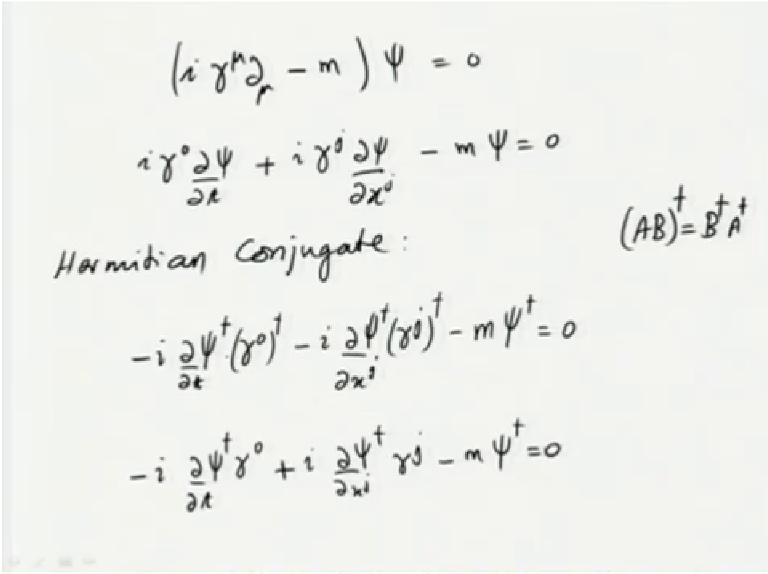

So, what we have is gamma mu dou mu minus m psi equal to 0 take the complex conjugate of that or let me first write it in a in terms of the time and the special component separately dou by dou t of psi gamma 0 plus i gamma j dou psi by dou x j minus m psi equal to 0 take the complex conjugate of that that will give you minus i.

So, actually here because we are dealing with a matrix equation we had to take the Hermitian conjugate. So, let me write that for you let us take the Hermitian conjugate of the equation I will be equal to minus i a b if 2 matrices are there the Hermitian conjugate is basically b dagger a dagger the Hermitian conjugate of b first and then the Hermitian conjugate of a this is because there is a transpose of the matrix involved apart from the complex conjugation in Hermitian conjugate minus. So, this Dirac equation will give you minus dou by dou t of t psi dagger.

The complex conjugation or the transpose will not affect dou by dou t. So, it is actually psi dagger that will be affected psi that will be affected and then the second one is gamma 0. So, gamma 0 dagger minus i. Similar way dou xj psi dagger gamma j dagger minus m psi dagger equal to 0 if we know what gamma 0 dagger is it is gamma 0 itself. So, we have minus i dou by dou t psi dagger gamma 0.

But gamma j dagger is minus gamma j. So, there is a change of sign there it is plus i dou by dou xj psi dagger gamma j minus m psi dagger which is equal to 0. So, this is now a little difficult situation here because the its not will not really be able to write it in a covariant fashion for that that is because this is now minus i dou by dou t of psi dagger gamma 0 minus dou by dou t dou xj of psi dagger gamma j instead of a plus; it is a minus related to the zeroth and the special components.

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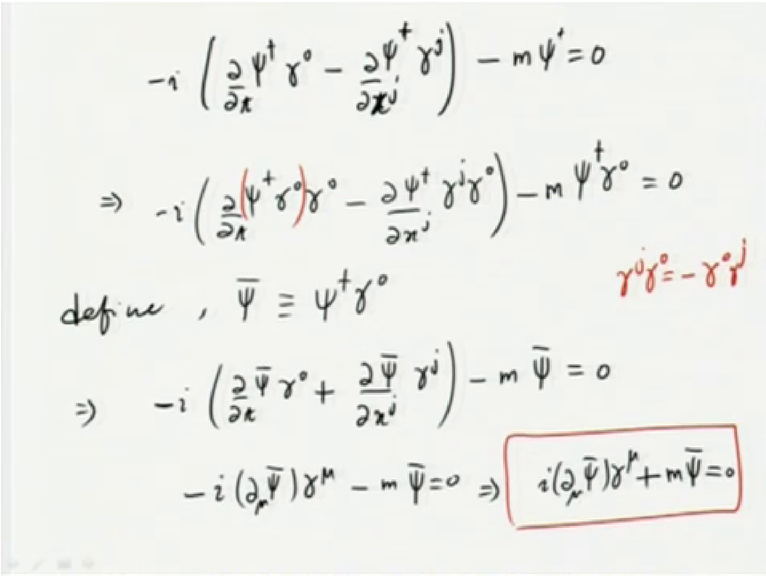

The way out is to multiply it by a gamma 0 from the right when I multiply by gamma 0, this will give me minus i dou by dou t psi dagger gamma 0 another gamma 0 minus dou psi dagger xj gamma j gamma 0 minus m psi dagger gamma 0. So, everywhere I multiply by a gamma 0 from the right side and define psi bar as gamma dagger psi 0, this will give me minus i first term I will club psi dagger and gamma 0.

In the in that that will give me psi bar, but the second term i had to switch gamma j and gamma 0 when i switch gamma j and gamma 0 and write it as gamma 0 gamma gamma j gamma 0 is gamma 0 gamma j with a minus sign. So, there is a switch of sign again. So, that is what we will make use of. So, that will give me dou by dou t of psi bar and there is a gamma 0 which is left out minus in this case, I had to switch this gamma 0 and gamma j club the gamma 0 with psi bar that will give me a change of sign.

And dou psi bar over dou x j what is left is gamma j minus m psi bar equal to 0. So, there is an overall minus sign and what is there in the bracket is basically a total derivative for the 4 derivative of psi bar dotted with gamma mu. So, this is nothing, but i dou mu psi bar gamma mu minus m psi bar equal to 0 or I can write it as taking away the minus sign i dou mu psi bar gamma mu plus m psi bar equal to 0. So, this is what we will consider as the conjugate equation corresponding to the Dirac equation.

So, now we have these 2 equations one is the Dirac equation.

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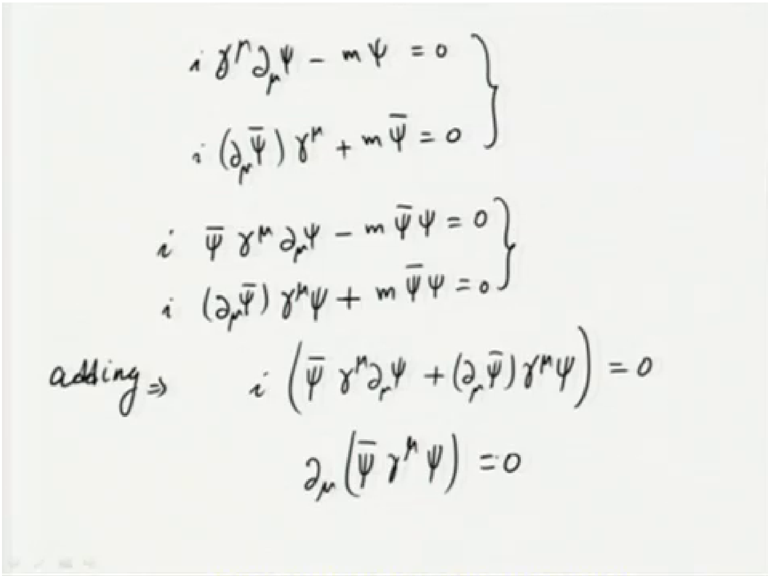

I gamma mu dou mu psi minus m psi equal to 0 and I have dou mu psi bar gamma mu minus m psi bar, sorry, plus m psi bar equal to 0 first multiply the first term with their gamma psi bar from the left. So, that will give me i psi bar gamma mu dou mu psi minus m psi bar psi equal to 0 and second term multiplied by psi from the left from the right i dou mu psi bar gamma mu psi minus m same term sorry plus m psi bar psi equal to 0..

So, these together and these together; add the second set what do we get? I is there, you have a psi bar gamma mu dou mu psi plus dou mu psi bar gamma mu psi equal to 0 m psi bar psi cancels with minus m psi bar psi since gamma mu is a constant as far as the special coordinator time coordinate is considered the derivative will not act on that. So, we can actually write this as total derivative acting on psi bar gamma mu psi.

The derivative acting on that so, this is the divergence of psi bar gamma mu psi which is equal to 0..

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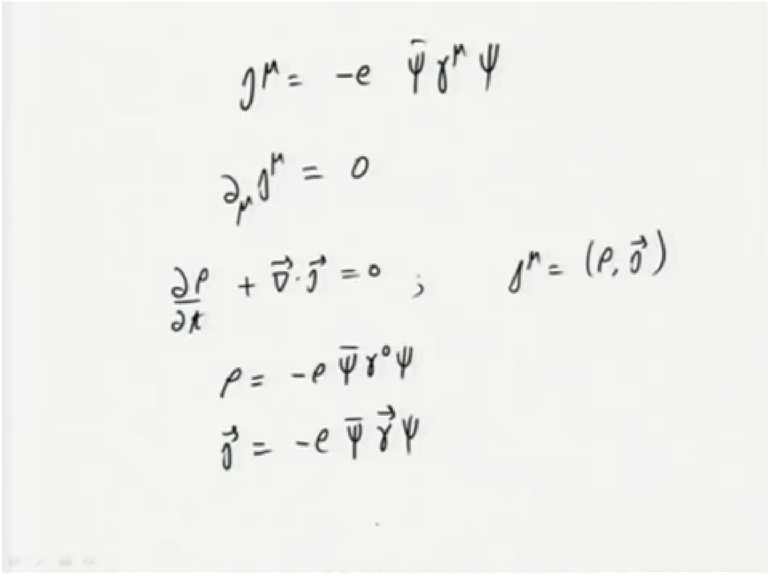

So, we will define j mu as minus e psi bar gamma mu psi and it is a divergence less or come the third derivative of divergence of j mu is equal to 0 this is basically the continuity equation dou by dou t rho plus divergence of j equal to 0 with j j mu written as rho the first zeroth component and j 3 vector as the 1 2 3 components.

And what is rho? Rho is nothing, but minus e psi bar gamma 0 psi and j is equal to minus e psi bar gamma vector psi when I say gamma vector psi meaning is that for each component of j we have a corresponding component of gamma there we added this minor e like in the case of Klien Gordon equation to interpret it physically as the charged density the rho and the current density j. So, that is what we have here in the case of the Dirac equation.

Let us now consider a plane wave solution for the wave function.

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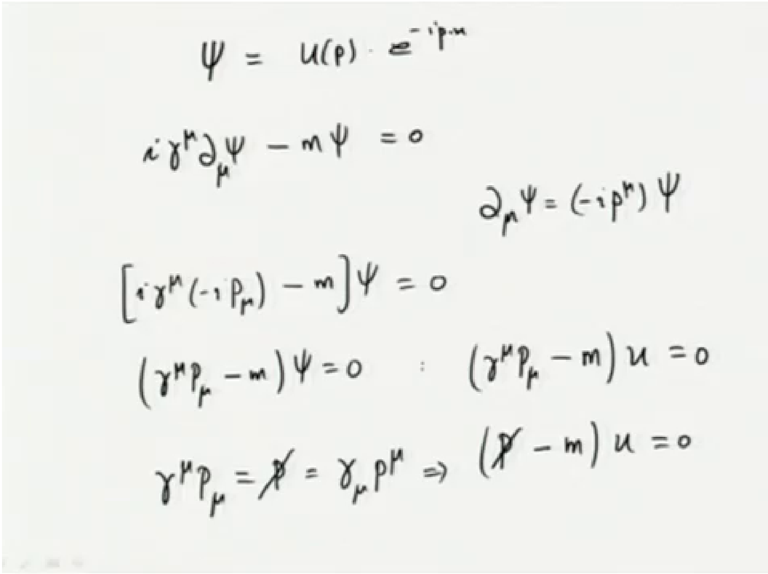

Psi can then be written as some u which is a function of momentum times e power i minus i Px which has all the spatial dependence and time dependence in h P denotes the momentum of the particle. Now remember that we have a column vector with 4 components written in compact way as represented in a compact simple psi and on the right hand side u should also therefore, be a column vector or which it has 4 components whereas, the exponential is just some normal function.

Now, let us look at the Dirac equation with this psi it when I take the derivative of psi with respect to x mu that will bring out minus i P mu and rest of it is psi. So, that will give you i gamma mu minus i P mu minus m psi equal to 0 or I can write this as gamma mu P mu minus m psi equal to 0 i factors go away or I could also write it as gamma mu P mu minus m acting on u equal to 0.

Because exponential i P dot x is anyway not equal to 0. So, that is easy for short like I will denote this as P slash m u a standard way of denoting gamma mu P mu is P slash this is just notation meaning is dot product of gamma mu with P mu. It is also equal to contra variant P and covariant gamma the same with this notation we have the Dirac equation for the plane wave solutions as P slash minus m psi equal to 0. So, for free particles, we can actually consider this as the Dirac equation.