Nuclear and Particle Physics
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Module – 09
Electroweak Interactions
Lecture – 04
Relativistic Electrodynamics

So, we are discussing the Maxwell's equations. In the relativistic case, Maxwell's equations written in the normal way in terms of the electric and magnetic fields that itself, it is a as a set Lorenz invariant, but we could write yesterday in a more compact and compact way the Maxwell's equation 2 of the Maxwell's equations. In fact, and the way we had written that it is clear the Lorenz invariance is clear in that this.

Now, today let us continue our discussion on this and see how to take care of the other 2 equations. So, let us consider the object that we had introduced yesterday now that we have now that we are familiar with this.

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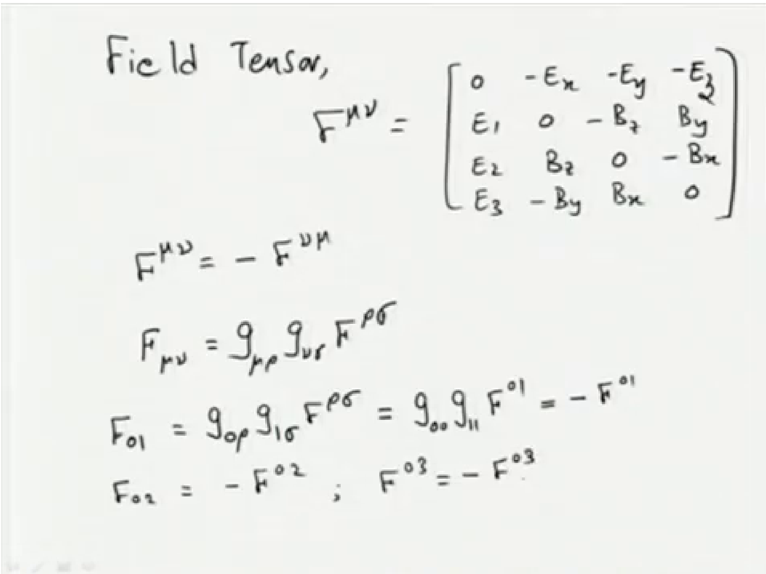

We will have I will call this as the field tensor F mu nu, we did not introduce or call it by any name as today, so, but it is usually called the field tensor, it is a second rank tensor and in the matrix form, we can write it as 0 minus E x minus E y minus E z, E 1 0 minus

B z B ym E 2 z B z 0 minus B x E 3 minus B by B x 0. So, this is what we had written down yesterday.

And properties are that if mu nu is equal to minus F nu mu anti symmetric under the interchange of these 2 indices mu and nu and that we said it is a second rank tensor. So, therefore, it is actually going to all right, transform in a like a second rank tensor, we will come to that maybe at some other point of time or it is not important here, let us see how do we write down the lower components.

Let us look at the covariant form of this tensor F mu nu. So, we have to lower 2 of these indices, if you want to write it in terms of the contra variant tensor. So, F rho sigma; so, we will connect with F mu nu covariant in this fashion g mu rho g nu sigma, this is very similar to the raising and lowering of the indices of vectors covariant to contra-variant and contra-variant to covariant.

But then this let us see; what are the components of F lower 1. So, 0 0 or the diagonal elements are all zeros. So, it is still anti symmetric with respect to the interchange of these 2 indices. So, the diagonal elements are 0s off diagonal elements F 0 1 is equal to g 0 mu is fixed to be 0, right and nu is fixed to be 1. So, I can write it as g 0 rho g ny is now 1 and sigma is anything F dou sigma dou sigma dou and sigma are summed over all.

But we know that g is a symmetric diagonal matrix. So, if one component is one index is 0 the other index is also 0 for it to be nonzero while. So, so g 0 0 is 1 and g 0 with any index. So, if rho is not equal to 0 the first g the component of the first g is going to be 0 then. So, therefore, the only non vanishing term that you will get here is g 0 0 and sigma equal to 1. So, rho equal to 1 sigma equal to 1. So, this is rho equal to 0 sigma equal to 1.

So, F 0 1 lower index indices is related to F 0 1 upper index with the same indices upper and lower cases and this is equal to g 0 0 is plus 1 g 1 1 is minus 1; therefore, altogether there is A minus sign F 0 2 is similarly minus F 0 2 F 0 3 is minus F 0 3.

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$$F_{10} = g_{11} g_{00} F^{01} = -F^{01}$$

$$F_{12} = g_{11} g_{22} F^{12} = F^{12}$$

$$F_{13} = F^{13} \;; \quad F_{23} = F^{23}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

We will now consider we will now consider mu equal to one and nu is either 0 or 1. So, if it is 0 1 2 3. So, F 1 0 is equal to g 1 1 g 0 0 F 0 1 which is equal to again minus F 0 1. In fact, since it is anti symmetric in with the interchange of these 2 elements, we do not really have to write this down at all we can get it from F 0 1. So, we will not write the rest of it at the 1 2 element is nu.

So, 1 2 is g 1 1, g 2 2, F 1 2 and both 1 1 g 1 1 and g 2 2 are minus 1s. Therefore, together they will give up and therefore, the sign is not changed in this case, similarly for F 1 3 which equal to F 1 3 and you can see that F 2 3 is also equal to F 2 3 contra-variant. So, altogether, we can write the matrix F mu nu covariant as 0 there is a sign change there. So, E x E y E z again, similarly for the first column minus E x minus E y minus E z diagonal elements are zeros.

In the case of off diagonal elements in the second or belonging to corresponding to the magnetic field there is no change in the sign compared to the contra-variant case. So, it is B x B is minus B z B y plus B z in the third row minus B y in the fourth row and 0 diagonal plus B x here minus B x here and A 0 in the diagonal term.

So, this is basically the contra-variant covariant F mu nu. So, between contra-variant and covariant F mu nu the electric field components change sign and magnetic field components remain the same that is easy to consider another object.

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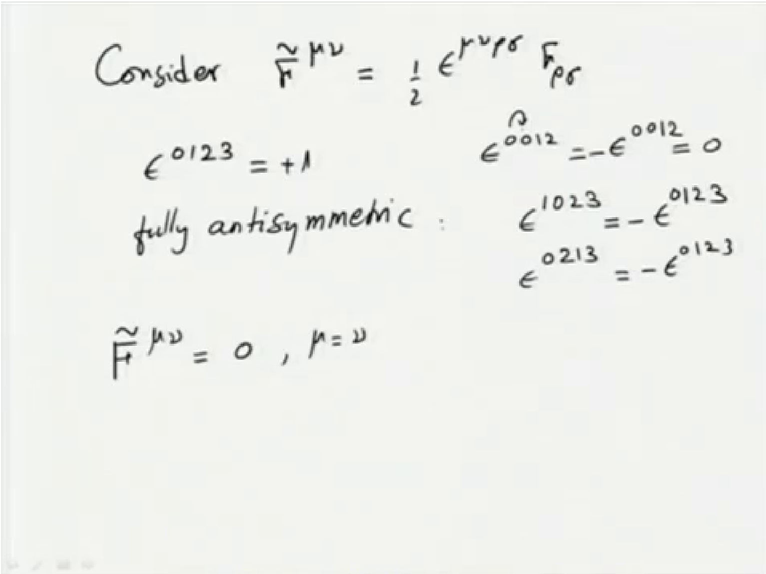

Which we will denote by F tilde mu nu equal to epsilon mu nu rho sigma F rho sigma covariant with a half here epsilon mu nu rho sigma is defined in this fashion F epsilon 0 1 2 3 is equal to plus 1 and it is fully anti symmetric.

What do I mean by that meaning interchange of any 2 indices keeping the others at their positions will give you A minus sign that is for example, epsilon if I interchange the first 2, I will get 1 0 keeping the third and fourth at the same position 2 3 is equal to minus 0 1 2 3 or it could be 0 2 1 3 where I have interchanged 2 1 1 in epsilon 0 1 2 3. So, I get a relative minus n, etcetera.

Let us look at F tilde mu nu clearly, if mu is equal to nu because of the anti symmetric property of epsilon mu nu rho sigma if both of them are the same, both if we need 2 indices are the same that will give you A 0 that is only way because one object will be say for example, if I take a epsilon 0 0 1 2 that is going to be equal to epsilon interchanging this to this and this to this.

Second first and second will give me the same thing at 0 0 1 no 3, but we already said that interchanging any 2 should give you A minus sign. So, that will give you consistent solution as a 0, there is no other way that you can get anything equal to minus one only way is to set it to equal to 0. Therefore, F mu nu is equal to 0 for mu is equal to nu and it is anti symmetric with the interchange of mu nu because epsilon mu nu rho sigma is anti symmetric with the interchange of mu nu.

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\begin{aligned}
\tilde{F}^{01} &= \frac{1}{2}\left(\epsilon^{01\rho\sigma} F_{\rho\sigma}\right) \\
&= \frac{1}{2}\left(\epsilon^{0123} F_{23} + \epsilon^{0132} F_{32}\right) \\
&= \frac{1}{2}\left(\epsilon^{0123} F_{23} - \epsilon^{0123} F_{32}\right) \\
&= \frac{1}{2}\left(\epsilon^{0123} F_{23} + \epsilon^{0123} F_{23}\right) \\
&= \epsilon^{0123} F_{23} = -B_x
\end{aligned}
$$

$$
\tilde{F}^{02} = -B_y, \quad \tilde{F}^{03} = -B_z
$$

Now, let us get the components F tilde 0 1 is equal to half epsilon 0 1 mu nu F all right. So, F tilde 0 1 is equal to half epsilon 0 1 rho sigma F rho sigma rho and sigma are summed over. Now we see that since epsilon already has fixed 2 of the indices 0 and 1 rho and sigma can take the rest of it rest of the indices which are 2 or 3. So, it can even I set rho equal to 2 sigma can take only one value which is 3 because we cannot repeat the indices.

So, here rho is equal to 2 sigma is equal to 3 add this to other possibilities. So, that is what you mean by take summation over rho and sigma what are the other possible rho values, we have taken one possibility which is 2 we cannot take 0, we cannot take one. So, the other possibility is 3 once you take rho equal to 3 sigma I speaks to 2 F 3 2, but that is equal to epsilon 0 1 2 3 F 2 3 minus epsilon 0 1 2 3.

Because epsilon 0 1 3 2, if I interchange 3 and 2, it will pick up A minus sign. So, that is equal to minus once F 3 2, but I will do the same thing with epsilon sorry same thing with F 3 2 as well 0 1 2 3 F 2 3 when I change F 3 2 to F 2 3 that will pick up another minus sign. So, making it plus epsilon 0 1 2 3, F 2 3.

Now, this were the identical terms and together that will give me epsilon 0 1 2 3, F 2 3 which is equal to 1 factor of one for epsilon 0 1 2 3 and F 2 3 is the covariant F 2 3 2 3 is minus bx. So, it is minus B x; now you understand why we had taken a half factor there it was in anticipation that we had put a half there.

So, that we can get A minus B x without any 2 factors there or any other factor in a similar fashion, you can actually do your homework and then get F tilde 0 2 as minus B y F tilde 0 3 as minus B z ok.

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$$\tilde{F}^{12} = \frac{1}{2}\left(\epsilon^{12\rho\sigma} F_{\rho\sigma}\right)$$

$$= \frac{1}{2}\left(\epsilon^{1203} F_{03} + \epsilon^{1230} F_{30}\right)$$

$$= \epsilon^{1203} F_{03} = -\epsilon^{1023} F_{03}$$

$$= \epsilon^{0123} F_{03} = E_z$$

$$\tilde{F}^{13} = -E_y \,, \quad \tilde{F}^{23} = E_x$$

And F 1 2 tilde is equal to half epsilon 1 2 rho sigma F rho sigma equal to half epsilon 1 2 are fixed what are available rho could take 0 or 3 let us take it to be 0 to start then sigma is 3. So, F 0 3 plus epsilon 1 2 3 0 or rho equal to 3.

The other possible value F 3 0 and that gives us epsilon 1 2 0 3 F 0 3 and epsilon 1 2 0 3 is minus epsilon 1 0 2 3 where I have interchanged 2 and 0 the second and third indices which again to get into a form epsilon 0 1 2 3, I have to interchange one and 0 first 2 indices. So, that will give me a plus 0 1 2 3 F 0 3 which is equal to factor of 1 for epsilon 0 1 2 3 and F 0 3 F 0 3 is F 0 3 is E to z. So, this is equal to E z.

Similarly, we can get F tilde one 3 as minus E y and F tilde 2 3 as E x the rest of the terms are obtained by taking their interchange of the indices from whatever we already have. So, put it putting it together.

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$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F^{\mu\nu} \xrightarrow[\vec{B} \to -\vec{E}]{\vec{E} \to \vec{B}} \tilde{F}^{\mu\nu}$$

I have F tilde mu nu equal to 0 minus B x minus B y minus B z B x 0 E z minus E y B y minus E z 0 E x B z E y minus E x 0.

Let me also write down F tilde F mu nu along sides F mu nu, we had 0 minus E x minus E y minus E z E x E y E z 0 minus B z B y B z 0 minus B x minus B y B x 0. So, you can see that we can get one from the other how do you get one from the other F tilde mu nu is obtained by taking.

Let us do the other way you start with F mu nu change E to B all the components. So, wherever you have an E x you replace it by B x E y goes to by E z goes to B z and wherever you have a B x you change to minus E x and E y to minus E y, E z to minus E z. So, that will give you F tilde mu nu or the other way around with this a.

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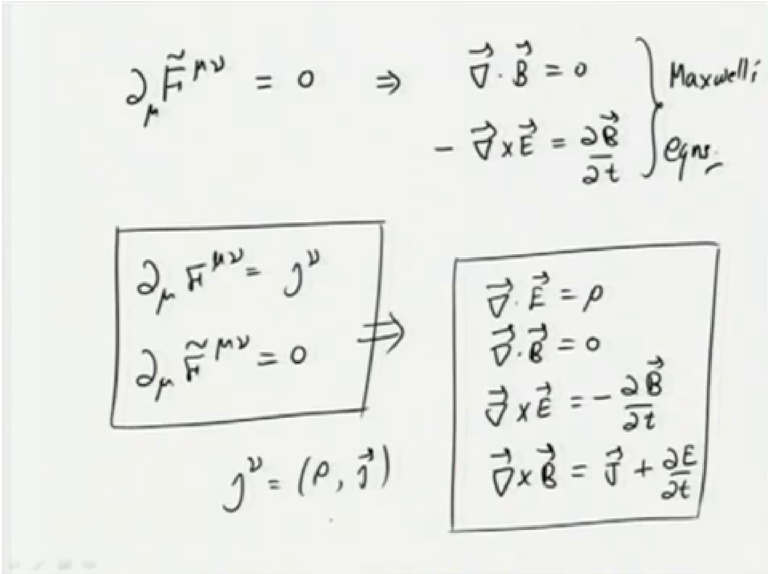

Let us take F tilde mu nu take the for derivative of this set it to equal to 0 to start with and then that will give you as an exercise this is an exercise divergence of B is equal to 0 and minus curl of E is equal to dou B by dou t, you will recognize this or these 2 as 2 of the Maxwell's equations.

So, together with the earlier dou mu F mu nu equal to j nu dou mu F tilde mu nu equal to 0 for any nu give you divergence of E is equal to 0 sorry divergence of E is equal to rho divergence of B equal to 0 curl of E equal to minus dou B by dou t curl of B which we got earlier is equal to j plus dou E over dou t this is a compact way of writing the 4 Maxwell's equation in a covariant and a explicitly or covariant fashion or Lorentz invariant fashion.

I mean Lorenz covariant fashion. So, this is what we have and going on I am just wanting to this here j nu is a for vector with first component or 0 th component equal to rho and the charge density and 1 2 3 component equal to the current density all right. So, let us come to expressing in the field now in terms of the potential.

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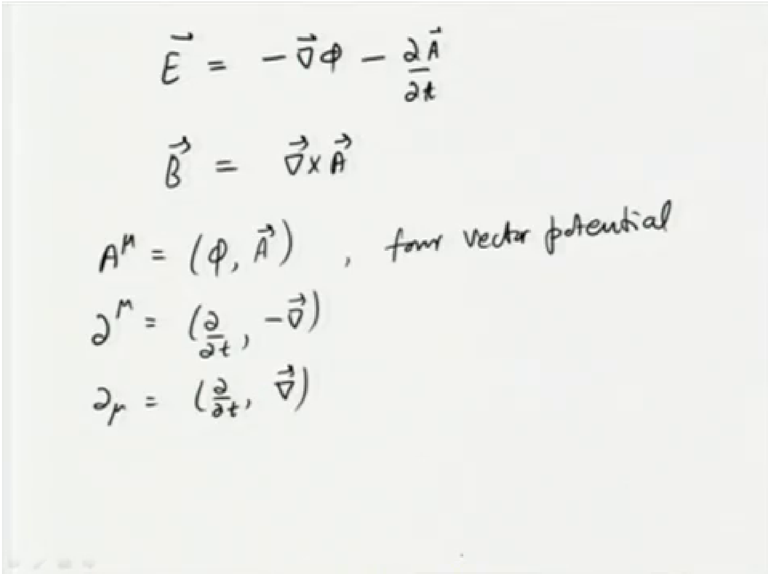

We know we can write electric field as gradient of scalar potential and time derivative of vector potential.

The first term is the electrostatic field and second m corresponds to the dynamic electro dynamic term or which is due to changing magnetic field first term as curl free and second term not necessarily curl free curl of the second term is going to give you the variation in the magnetic field curl of a is magnetically e. So, that is what we have for B; B is curl of A. So, if we have a time changing the time varying electric field magnetic field that will induce an electric field which is given by this.

Let us focus on in the electric field expression, let us denote the electrostatic field potential phi and the vector potential a together as a for vector we can call it the 4 vector potential and we already know dou mu is equal to dou by dou t minus gradient dou covariant is dou by dou t plus gradient.

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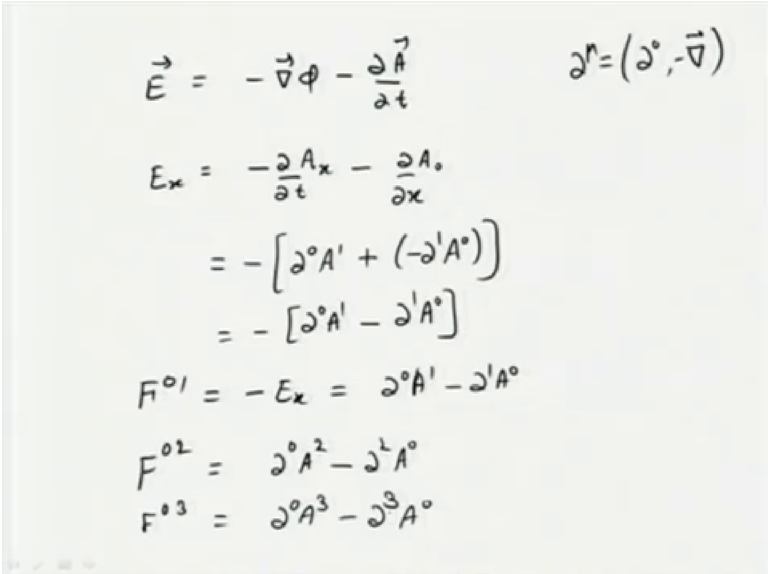

So, now let us look at the E x. So, let me write that again for the general this one is grad phi minus dou by dou t of E x is let me first write the derivative term minus dou by dou t A 1 or a x minus grad dou by dou x a zero. So, this is equal to minus of dou by dou t is dou 0 A 1 minus. So, now, it should be plus dou by dou x is actually minus dou a by dou x is dou 1.

So, we have dou mu equal to dou by dou t which is dou 0 and minus grad. So, there is A minus sign. So, there should be A minus sign here. So, we have dou 0 A 1 minus dou 1 and A 0 which is equal to minus dou 0 A 1 minus dou 1 A 0, but if you look at the field tensor F mu nu the zeroth mu equal to 0 nu equal to one will give you minus E x.

So, taking F 0 1 as minus E x gives us dou 0 A 1 minus dou 1 A 0 as F 0 1 in a similar fashion you can get 0 2 as dou 0 a 2 minus dou 2 A 0 F 0 3 as dou 0 A 3 minus dou 3 A 0 now.

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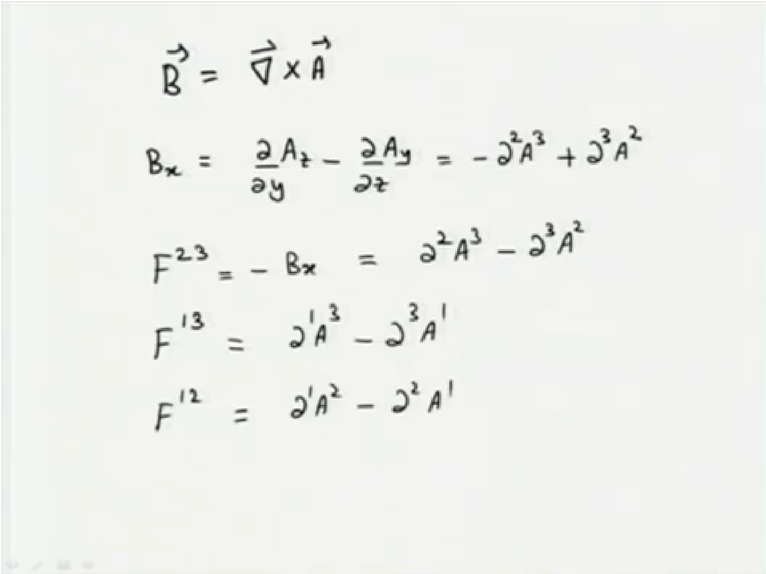

Let us switch over to B is equal to curl of a B equal to curl of a B x is equal to dou by dou y A z minus dou by dou z of a y which is equal to minus dou 1 dou 2 A 3 minus of minus plus dou 3 A 2. Since we know F 2 3 is equal to minus B x, we have that equal to dou 2 A 3 minus dou 3 A 2. Similarly, F 1 3 equal to dou 1 A 3 minus dou 3 A 1 and F 1 2 is dou 1 a 2 minus dou 2 A 1 together.

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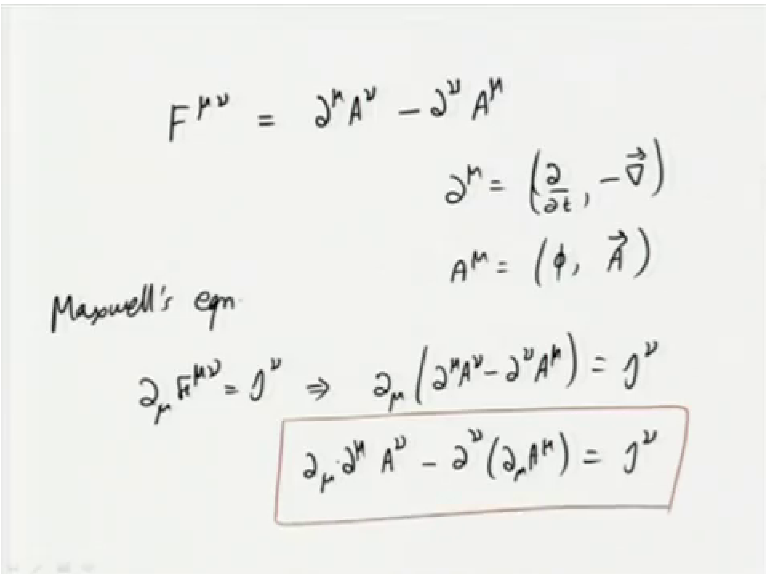

We can write F mu nu as dou mu a nu minus dou nu a mu well dou mu is dou by dou t minus grad gradient a mu is 0 and sorry phi and what about the Maxwell's equations dou mu F mu nu is equal to j nu. So, that will give taking the derivative of F mu nu written in terms of a nu and a mu, we have dou mu dou mu a nu minus dou nu a mu equal to j nu this is nothing, but dou mu dou mu acting on a nu minus dou mu acting on dou mu a mu is equal to j nu.

So, the Maxwell's equation written in terms of the potential in general looks like this, now let us come to something very interesting partly something which we already know earlier from our elementary electrodynamics discussions.

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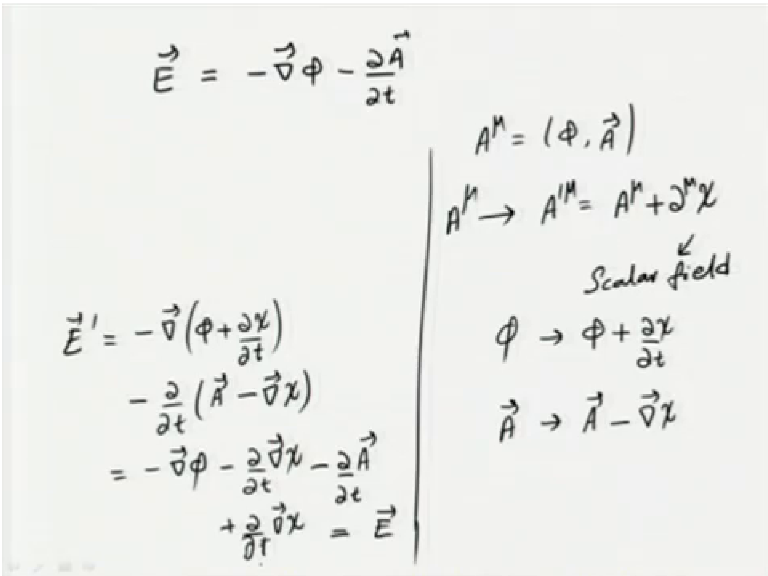

If you consider the expression for the electric field when we consider the expression for the electric field E equal to minus grad phi minus dou by dou t of A.

If I take a mu and make it change make a change in it to A prime mu which is actually equal to the original a mu plus for derivative of some scalar field. So, chi is some scalar field and our scalar function in this case its scalar sky is some scalar function in that case we have t prime equal to. So, essentially this change in a mu will sorry it should have been a mu phi will go to phi plus time derivative of chi and a goes to original a plus gradient of phi A minus grad chi because gradient has and relative minus sign compared to the normal contra-variant vectors.

So, this will tell me that I have a gradient of phi plus time derivative of a scalar function minus time derivative of original A 3 vector A minus gradient of chi and this is equal to minus grad phi minus derivative of grad chi minus dou by dou t of A minus of minus plus dou by dou t of grad chi which is equal to the original E.

Because the second and the fourth terms are exactly the same excepting that their signs are different. So, bottom line is that if we change a mu to a mu prime which will which picks up a derivative of a scalar field in addition to the original mu then the electric field remains the same.

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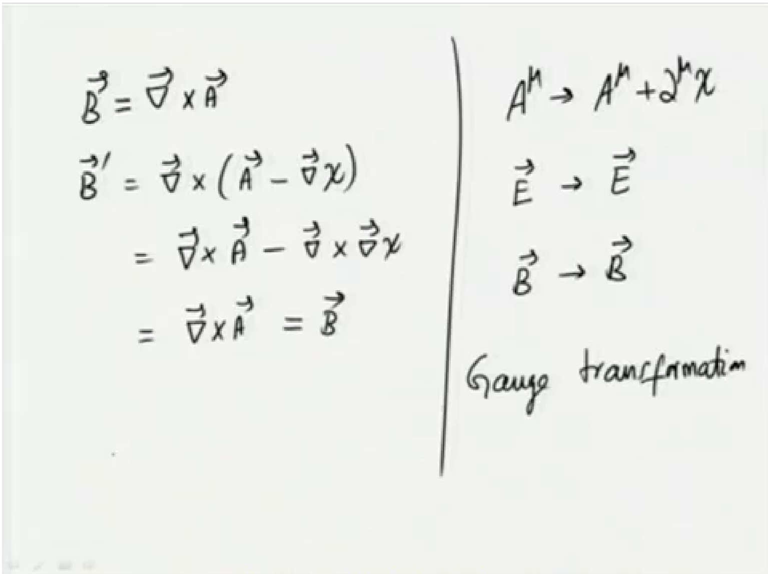

How about the magnetic field magnetic field B is equal to curl of A which is equal to curl of or B prime. Now is equal to curl of A minus grad chi which is equal to curl of A minus curl of gradient of chi and curl of a gradient of any function any scalar function is going to be equal to 0 is equal to 0. It is a property of the curl of a gradient. So, this is equal to curl of A which is nothing, but the original B. So, B and A are invariant as A goes to A plus 4 for the gradient for dimensional gradient of a scalar function chi electric field is not changed magnetic field is not changed.

So, there is a freedom in choosing the potential a mu you cannot change the zeroth different components of a mu independent of each other arbitrarily if the change in a mu is in this fashion that a mu goes to originally mu plus F or gradient or dou mu of a scalar function any scalar function denoted here by k chi, then the physical electric and magnetic fields are not changed. So, if we consider the case where electric and magnetic fields are measurable and therefore, physical quantities, then A is you know uniquely fixed for a given A and B or we cannot directly attribute therefore, any physical meaning to a mu actually that is not exactly the way we interpret.

We will say that there is some arbitrariness in A mu. So, to physically interpret it as anything we have to remove these unwanted degrees of freedom, but this particular freedom or arbitrariness in the potential is very important in particle dynamics we call this the gauge transformation this particular way of changing a mu to a nu a mu is called gauge transformation and we see that the electric and magnetic fields are invariant under such gauge transformation.

So, we can say physics is invariant under such gauge transformation and this is going to be a very important point note when we come to the dynamics of the particle in quantum field theory. So, what we will do is to see how to remove the unwanted degrees of freedom you name a mu and try to fix this. So, that later on when we give physical meaning to this mu we have physical quantities without any arbitrariness in it.

There are different ways to do that.

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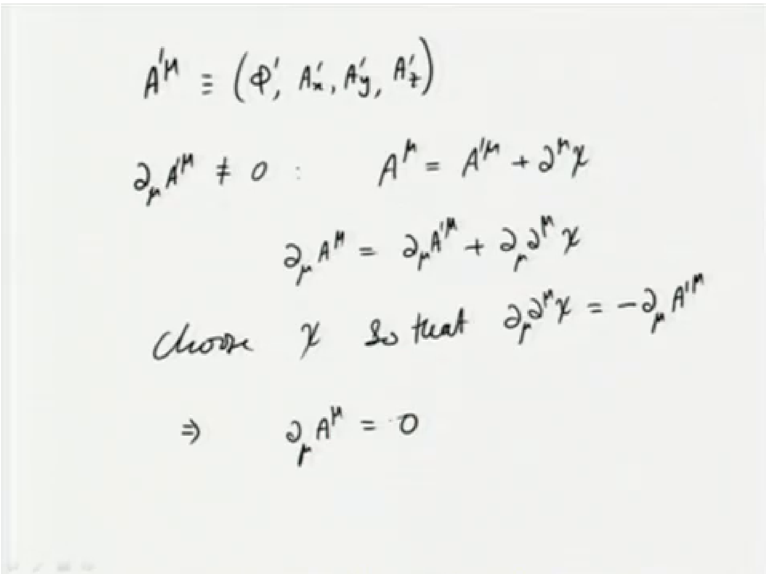

So, let us look at one way of doing it consider a mu. So, it has 4 components phi A x, A y, A z when we say these are not uniquely fixed we can actually think about some relation between these quantities that will try to fix this, let us consider the 4 divergence of A mu for any arbitrary choice of a mu it is not guaranteed that dou mu; A mu is equal to 0.

But let me start with some potential which I denote actually by a mu prime or a prime divergence of that is not equal to 0, but then I make a transformation to A mu which is equal to a prime mu plus dou mu chi where chi is some scalar field scalar potential scalar function take the derivative of a mu that is a derivative of a prime mu and dou mu dou mu of chi now we can choose chi. So, that dou mu dou mu chi is equal to minus dou mu a prime mu.

Since this transformation is allowed for any scalar field let us say we get to some field there is a scalar function kind let us choose a scalar function which satisfies the condition dou mu dou mu chi is equal to minus dou mu a mu prime that will then give us dou mu A mu is equal to 0. So, we can always choose some scalar function to set the potential with which is divergence less to start with if it is no divergence less then we can always consider the scalar field which will give us which will set change the mu. So, that the nu mu is divergence less.

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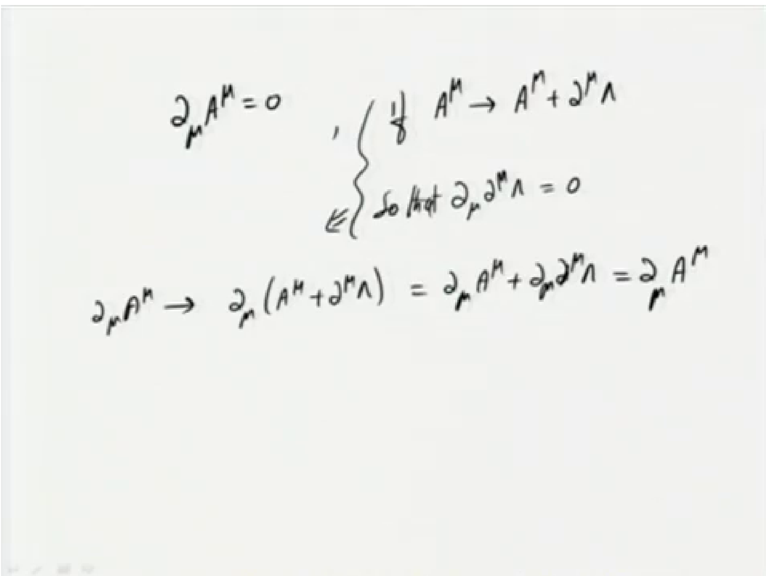

We have an a mu which satisfies this condition is that the only arbitrariness in that that is not if we further make any change to some A mu plus divergence A mu plus dou mu lambda where lambda is some other scalar fields this scalar field cannot be completely arbitrary because in that case the for divergence of A mu may not be equal to 0, but if we take dou mu a mu of lambda to be equal to 0 or if we choose lambda. So, that dou mu dou mu a mu lambda equal to 0 then. So, this is by choice; so, that this is equal to 0.

This will give us dou mu A mu going to dou mu A mu plus dou mu lambda which is equal to dou mu A mu plus dou mu dou mu lambda which is equal to dou mu A mu because the second term is equal to 0 that is the property of the chosen lambda. So, we have we can fix this in this fashion. So, after fixing A mu; so that therefore, divergence equal to 0 there is still some arbitrariness in that and that arbitrariness is that we can again and dou mu lambda. So, that dou mu dou mu of lambda is equal to 0 without changing the electric and magnetic field.

So, the electric and magnetic field or the physics remains the same still. So, we will come to this gauge freedom and how the Maxwell's equations are written with within this and tell a little more about the gauge conditions that we have just now mentioned or discussed in the next discussion.