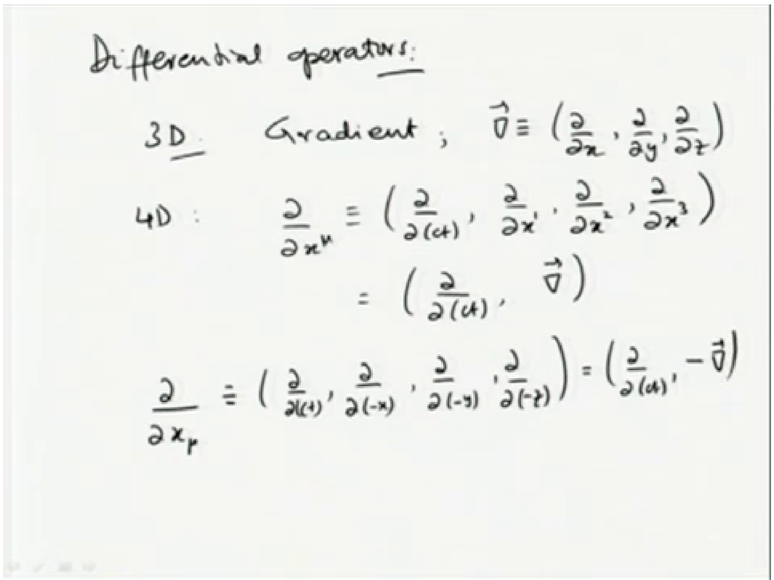
Nuclear and Particle Physics
Prof. P Poulose
Department of Physics
Indian Institute of Technology, Guwahati

Module – 09
Electroweak Interactions
Lecture – 02
Relativistic Quantum Mechanics- continued

We will continue our discussion on the relativistic quantum mechanics. We were talking about the relativistic notations and some of the how to combine the time coordinate, time parameter and the special coordinates in a single entity called a 4 vector, and how they transform under coordinate transformations.

(Refer Slide Time: 00:57)

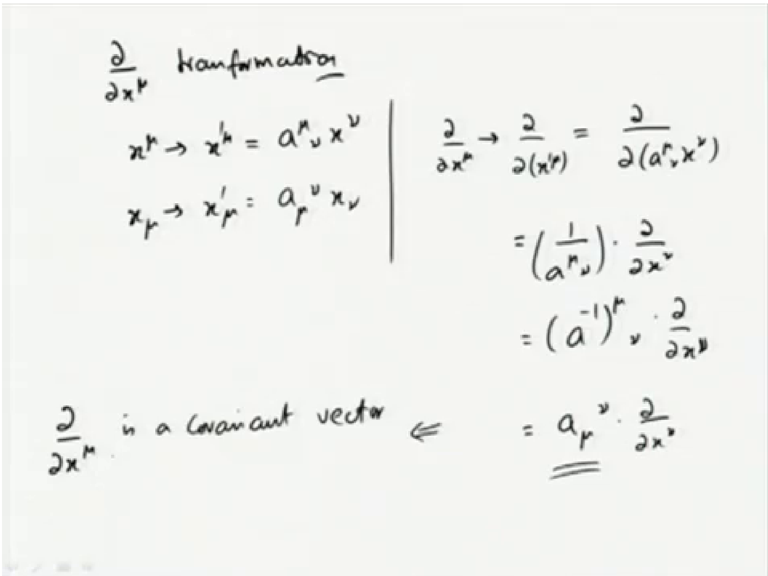


Let us start with differential operators. So, let us consider differential operators in relativistic notation see in 3 dimensional relative 3 dimensional case. We have gradient defined as dou by dou x along the x axis dou by dou y along the y axis and dou by dou z along the z axis.

And the 4 dimensional generalization of this can be written as dou by dou x mu, which is dou by dou ct which is our 0 th component and dou by dou x 1 dou by dou x 2 dou by dou x, which is dou by dou x dou by dou x y and dou by dou y dou by dou z etcetera or I can write it in a way which will separate the time component and the special component returned as the gradient.

This is one way of doing it and the other is with mu as a subscript on the x which is basically a x mu covariant. So, if you take the derivative with respect to the covariant x then you will have dou by dou c t dou by minus dou by dou minus x dou by dou minus y dou by dou minus z, because the special components of x mu covariant is minus x minus y minus z and this is essentially dou by dou ct minus gradient.

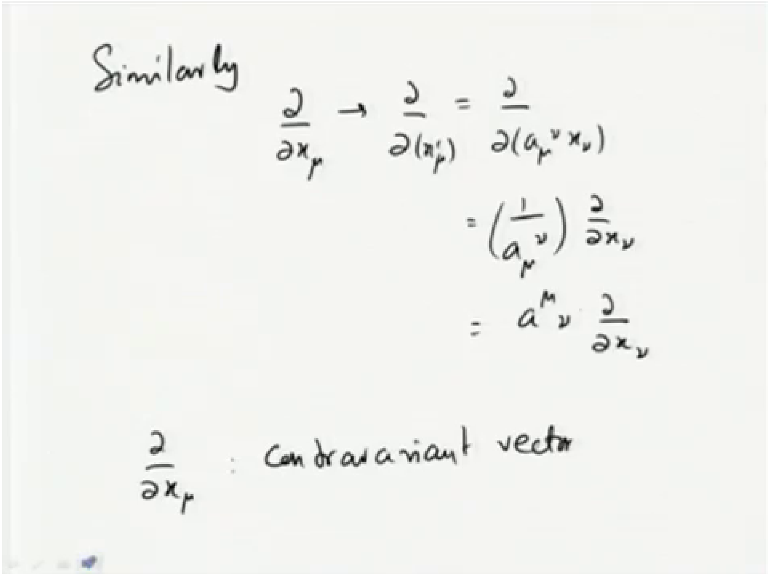
(Refer Slide Time: 03:08)



Let us look at how the dou mu dou x mu transform and the coordinate reservation. We know x mu goes to x prime mu, which is a mu nu x nu and x covariant goes to x prime covariant mu a mu nu x nu dou by dou y x mu contravariant, will go to dou by dou x prime mu, which is equal to dou by dou a mu nu x nu, which is equal to I can write it as 1 over a mu nu dou by dou x and this one over a nu mu is essentially mu nu th component of the a inverse matrix times dou by dou x nu, but we saw in earlier lectures that a inverse mu nu is essentially equal to a mu nu dou x mu. This is basically the way a covariant vector transform from our previous lecture. So, previous lecture says that this is the way a covariant vector transform.

So, this says that dou by dou x mu is a covariant vector it transform like a covariant vector. So, we could consider that as a covariant.

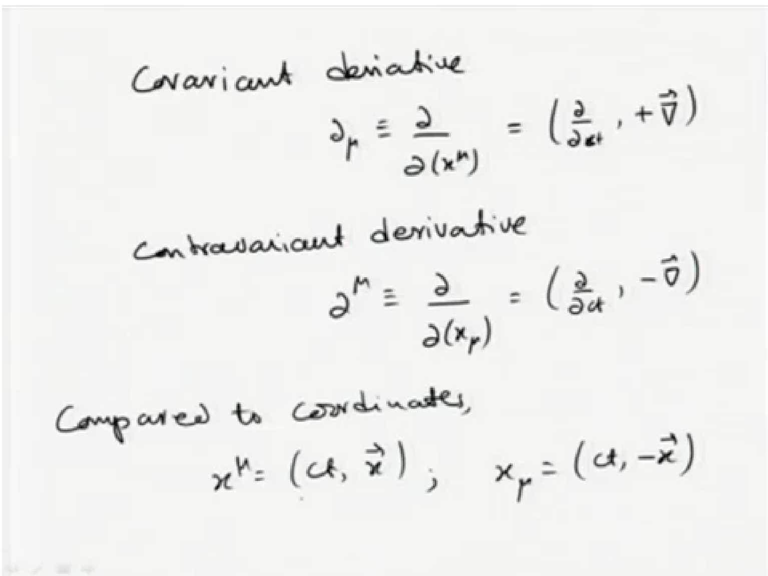
(Refer Slide Time: 05:23)



Similarly, if you consider dou by dou by dou x covariant goes to dou x prime covalent equal to dou by dou y a mu nu x nu is essentially equal to a mu nu dou by let me write it here, taking same argument as earlier this is one over a mu nu dou by dou x nu, which is essentially a mu nu dou by dou x nu nu now the subscript. And this is how contravariant vector transforms. So, dou by dou x mu is a contravariant vector.

So, we will actually denote these 2 in a slightly simpler notation.

(Refer Slide Time: 06:45)

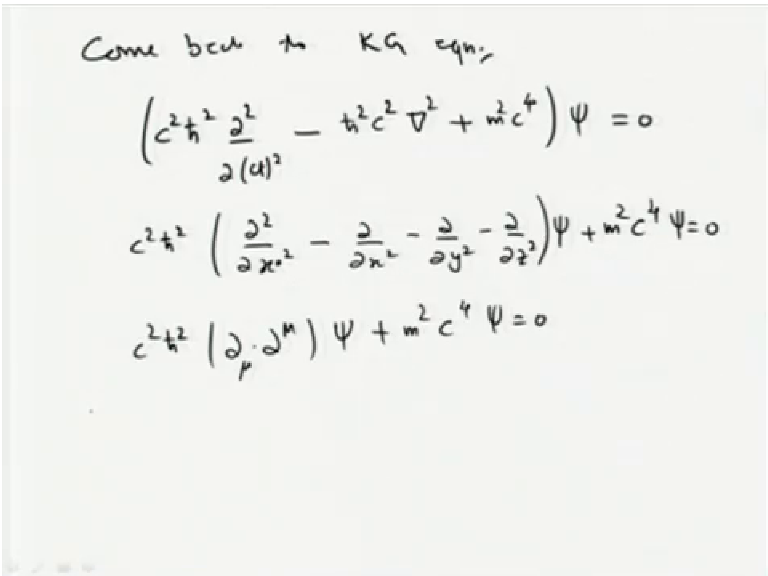


Covariant derivative operator dou mu is defined as the derivative with respect to the contravariant vector. And contravariant derivative operator is dou mu dou x nu all right. These are the notations that we will use when you have the subscript on dou mu this is a covariant vector like the earlier notations, but then this is this corresponds to derivative with respect to a contravariant x mu.

And similarly when we say contravariant derivative it is a superscript on though the partial derivative operator, which is equal to the partial derivative with respect to the covariant x. And this is essentially equal to dou by dou c t and minus here you know how actually a c that it is plus gradient operator dou by dou c t minus the gradient operator.

So, here one thing to notice is that compared to the coordinate x mu is equal to ct x and x contravariant covariant is ct minus x whereas, in the case of derivative operators the covariant vector has a plus sign in front of the gradient operator the special derivative part, and the contravariant derivative has minus sign with respect to the time component and this is exactly the opposite in the case of coordinates x mu and x nu covariant and contravariant.

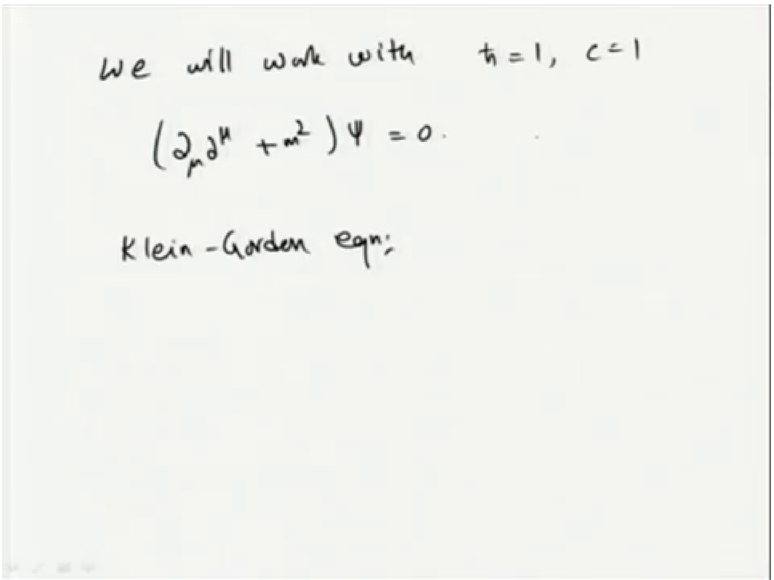
(Refer Slide Time: 09:21)



Now let us come back to Maxwells sorry come back to the Klein Gordon equation let me call this for short KG equation. So, we have minus ah. So, let me write it in terms of c square h cross square, second derivative with respect to time or c t minus h cross square c square grad square plus m square c 4 c size equal to 0 please refer to the previous lecture for this we have we had written down this.

So, now I can take c square h square as a common factor and then that will give you the square over though ct square, which is essentially let me write it as x 0 square the 0 th current minus dou by dou x square minus dou by dou, y square minus dou by dou z square ok, psi plus m square C 4 psi equal to 0 this can be written as c square h cross square dou mu dou mu the usual dot product between 2 vectors these are all actor operators, but dot product will give you similar expression as in the case of vector, psi plus m square c power 4 psi equal to 0. Now we will work in units of c square is equal to h cross square is equal to 1 ok.

(Refer Slide Time: 11:20)



We will work in a unit system h cross equal to one c equal to sorry h cross equal to one C equal to 1. This is all right we can do that this is a nu unit system that we are considering not the SI unit Lorenz heavy said au energy it is called. And in this unit system we will have some simple this is basically to simplify our notation, because in relativistic quantum mechanics both h cross and c will come appear in different terms. And then we if we put it equal to 1 we do not know how to write it, but then the question is whether information is lost or whether we will get confused at any point with the dimensions or with the physical quantities etcetera.

We will see that consistently at any point whenever we want to actually recover the h cross and c were to put that across and c dimensional analysis will let us put these in a unique, way maybe we will take it take up some examples to illustrate this later towards the end of this lecture also, but at the moment we will take it as that we can without creating any confusion we can take h cross to be equal to one and c equal to one and then work in this unit ah.

So, in that case you have dou mu dou mu plus m square is equal to acting on psi is equal to 0 is our Klein Gordon equation ah.

(Refer Slide Time: 13:35)

$$\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi + m^2 \phi = 0 \longrightarrow (1)$$

$$(1) \times (-i\phi^*) \Rightarrow \quad -i\phi^* \frac{\partial^2 \phi}{\partial t^2} + i\phi^* \vec{\nabla}^2 \phi + m^2(-i\phi^*\phi) = 0 \rightarrow (2)$$

$$(1)^* \times (-i\phi) \Rightarrow \quad -i\phi \frac{\partial^2 \phi^*}{\partial t^2} + i\phi \vec{\nabla}^2 \phi^* + m^2(-i\phi^*\phi) = 0 \rightarrow (3)$$

$$(2) - (3) \Rightarrow \left(-i\phi^* \frac{\partial^2 \phi}{\partial t^2} + i\phi \frac{\partial^2 \phi^*}{\partial t^2}\right) + \left(i\phi^* \vec{\nabla}^2 \phi - i\phi \vec{\nabla}^2 \phi^*\right) = 0 \longrightarrow (4)$$

Let us discuss this further we will consider dou square write it in many components dou square by dou t square. So, now, i work in units of c equal to one square phi minus grad square phi plus m square phi equal to 0, change the wave function notation from phi psi to phi I mean you know you know particular need to do that, but let me denoted by phi, because that is usual the standard way of denoting the fields who satisfy the Klein Gordon equation for the wave functions who satisfy the Klein Gordon equation.

Now let me multiply this by minus I phi. So, let me called this equation 1. So, 1 times minus i phi star will give me minus i phi star, second derivative with respect to time of phi minus i. So, plus i phi star grad square phi minus i phi star i can take m in the beginning m outside.

So, m square minus i phi star y equal to 0, let me called this equation 2. So, let me take the equation one star complex conjugate of this and multiply this by minus i phi. So, that will give me minus i phi dou square over dou t square phi star I have taken the complex conjugate of vacation 1 minus again that becomes plus i phi del square phi star minus m square only 1 minus sign there phi star phi equal to 0. So, this is equation 3. Now this equation 2 minus 3 will give me all right let me write it here 2 minus 3 m square terms are identical. So, they will cancel out.

Now, there is a minus i phi star dou square phi by dou t square plus i phi dou square phi star over dou t square ok. Now plus i phi star grad square phi minus i phi grad square phi star equal to 0.

(Refer Slide Time: 17:01)

$$\frac{\partial}{\partial t}\left(\phi^* \frac{\partial \phi}{\partial t}\right) = \phi^* \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi^*}{\partial t} \cdot \frac{\partial \phi}{\partial t}.$$

$$\phi^* \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left(\phi^* \frac{\partial \phi}{\partial t}\right) - \frac{\partial \phi^*}{\partial t} \cdot \frac{\partial \phi}{\partial t}.$$

First term in

$$(4) \Rightarrow \quad -i\left\{\frac{\partial}{\partial t}\left(\phi^* \frac{\partial \phi}{\partial t}\right) - \frac{\partial \phi^*}{\partial t} \cdot \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial t}\left(\phi \frac{\partial \phi^*}{\partial t}\right) + \frac{\partial \phi^*}{\partial t} \cdot \frac{\partial \phi}{\partial t}\right\} =$$

$$-i \frac{\partial}{\partial t}\left\{\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right\} \rightarrow (5)$$

Now if I take time derivative acting on phi star dou phi by dou t, that will give me 5 star dou square phi by dou t square minus i dou phi star over dou t there is only my sorry plus dou phi by dou t. Similarly for the other term time phi dou phi by dou t also we can write in the same fashion.

We use this to rewrite equation number 4 as ah. So, not that here we have a term phi star dou square phi by dou t square phi star dou square phi by dou t square is the first term on the right hand side here, but then that is equal to total time derivative or right derivative acting on both phi star and dou phi by dou t minus.

So, phi star dou square phi dou t square is what we have in this equation here, the first term on the left hand side of equation 4 apart from the minus i that is equal to time derivative acting on phi star dou phi by dou t together minus this term extra time on the right hand side of the first equation here, which is dou phi star by dou t dou phi by dou t.

So, that when we put back in equation 4 here will give us minus i phi star dou square phi by dou t square is replaced by dou by dou t acting on phi star dou phi by dou t, minus dou phi star over dou t dou phi over dou t plus we have taken a minus i from the first wound therefore, it will be a plus i phi will become a minus i know. So, there is a minus sign there because minus i is taken out and we have a gain this is phi dou phi by dou t phi star by dou t minus minus of minus plus dou phi star over dou t dou phi over dou t.

So, this is the first term in equation 4. So, this first term in equation 4 is equal to. So, let me only focus on the first term that will give me minus I these 2 cans the second term on the fourth term cancel.

So, that will give me minus i total derivative acting on phi star dou phi by dou t minus phi dou phi star over dou t. Similarly so, let me call this equation 5 ok. So, now, go back to equation 4 the second time is i phi star grad square phi minus i phi grad square phi star.

(Refer Slide Time: 20:45)

Similarly,

$$\vec{\nabla}\cdot(-i\phi^*\vec{\nabla}\phi) = -i\vec{\nabla}\phi^*\cdot\vec{\nabla}\phi - i\phi^*\vec{\nabla}^2\phi$$

$$+i\left[\phi\nabla^2\phi^* - i\phi^*\nabla^2\phi\right] = \Big\{\vec{\nabla}\cdot(-i\phi^*\vec{\nabla}\phi) - i\vec{\nabla}\phi^*\cdot\vec{\nabla}\phi$$

$$\vec{\nabla}\cdot(i\phi\vec{\nabla}\phi) + i\vec{\nabla}\phi\cdot\vec{\nabla}\phi^*\Big\}$$

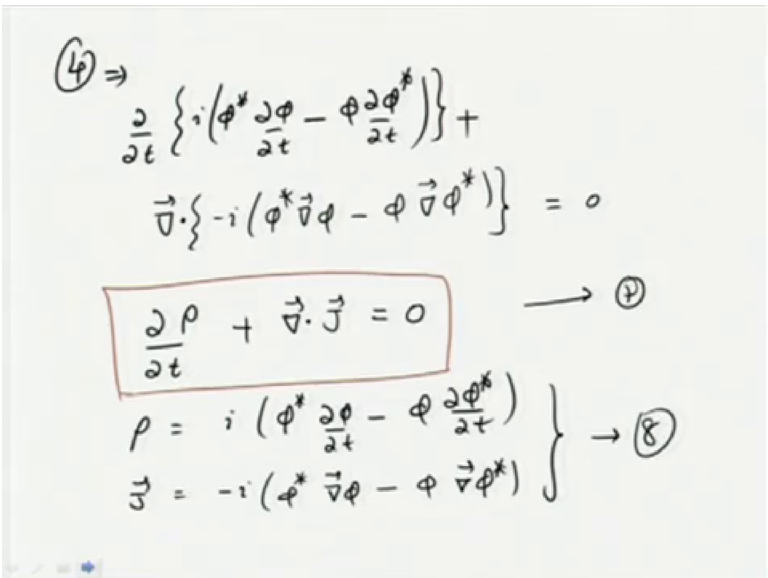
$$= -i\,\vec{\nabla}\cdot\left\{\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right\} \longrightarrow (6)$$

And in exactly similar way as we did in the case of the time derivative we can write grad gradient off minus i phi star grad phi dot here is equal to minus i grad phi star dot grad phi minus i phi star grad square phi.

So, the second term in equation 4 which is minus, which is plus I phi grad square phi star minus i phi star grad square phi can be written as del dot minus i phi star grad phi ok, minus minus i grad phi star dot grad phi. This is for the second time here a minus i phi star dot square phi and similarly for the other one, we have gain del dot i phi del phi minus minus of minus plus i phi sorry i del phi dot del phi star again the second and the fourth terms cancel.

And then we have this lhs or the rhs here is equal to. So, this is essentially equal to del dot minus i del dot phi star del phi minus phi del phi star ok. So, essentially what we have done is we have taken this equation 4 and written each of these term as time derivative for the first term acting on both phi star and dou phi by dou t time derivative is acting on both the terms both the factors and all there. And similarly for the gradient we have we written that in a more in a different fashion.

(Refer Slide Time: 23:47)



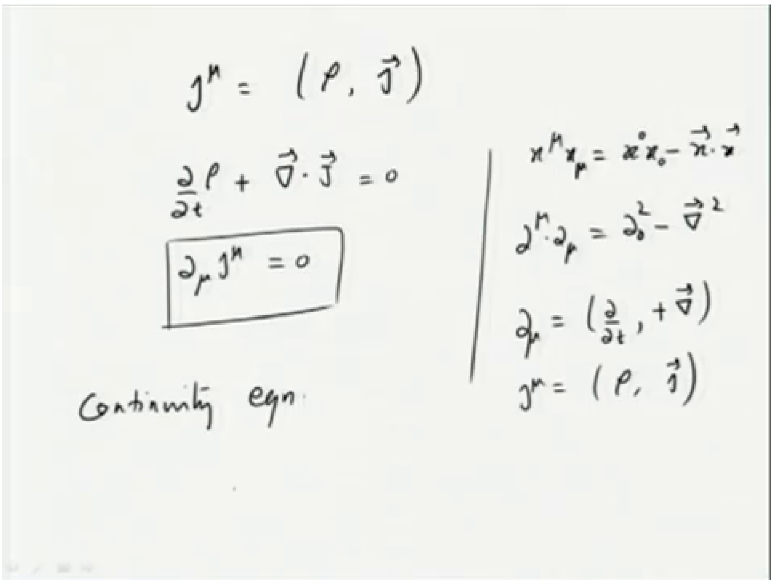
Why we are doing it because when we write it so, we have essentially equation 4 will give you time and derivative i star phi star dou phi by dou t minus phi phi star over dou t ok.

So, this is basically what we have from equation 5 here and similarly from let me call this equation 6 from 6 we have plus del dot minus i phi star grad phi minus phi grad phi star.

So, equation 4 can be written as time derivative of some term plus divergence of something else is equal to 0 I will write this in a compact way as dou by dou t of some rho plus divergence of some J equal to 0. So, this is 1 thing and now where rho is essentially as is rho is equal to i phi star dou phi by dou t minus phi dou phi star over dou t and j equal to minus i phi star grad phi minus phi grad phi star.

So, this thing here is a very compact notation compact equation you must have seen such equation in the earlier discussions at different under different topics, this is what is called the continuity equation ok.

(Refer Slide Time: 26:10)



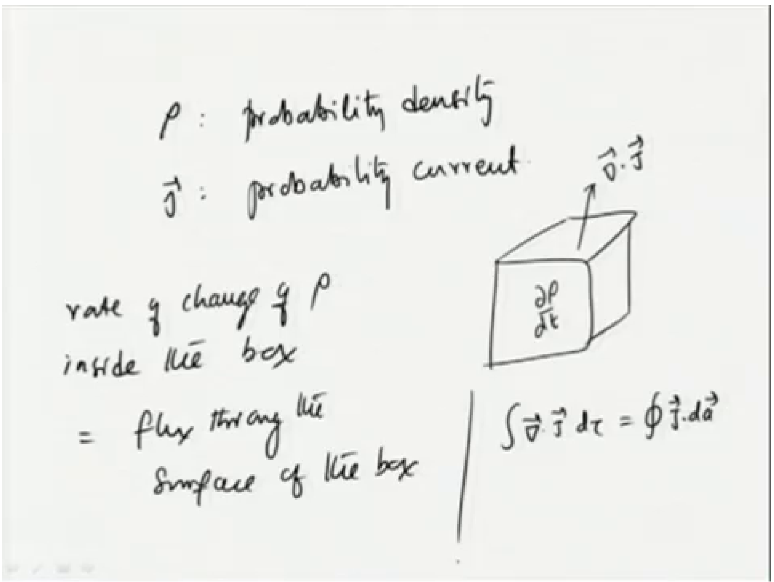
This before we come to that interpretation of this i can actually write more compact current j mu which component rho and j vector and the continuity equation dou by dou t of rho plus divergence of j then becomes equal to 0, then becomes dou mu j mu equal to 0.

He notice that between the special and the temporal components here the 0th component and a 1 2 3 components the sine is not minus, but plus just as an aside. The x mu x mu we had x 0 x 0 minus x dot x this was also the case, where we had the operator dou mu dou mu equal to dou 0 square minus grad square relative sign.

But here we do not have that relative sign; because dou mu is dou by dou t plus grads the covariant vector has a plus sign in this case. And j mu is a contravariant vector which also has a plus sign the way we have defined j.

So, this is the continuity equation e written in a compact way the physical interpretation is that all right. So, this is the continuity equation. If you look at the expression here in 8 equation 8, j is something which we have already familiar with in the case of non-relativistic quantum mechanics ok. So, there j is basically considered as the probability current.

(Refer Slide Time: 28:27)



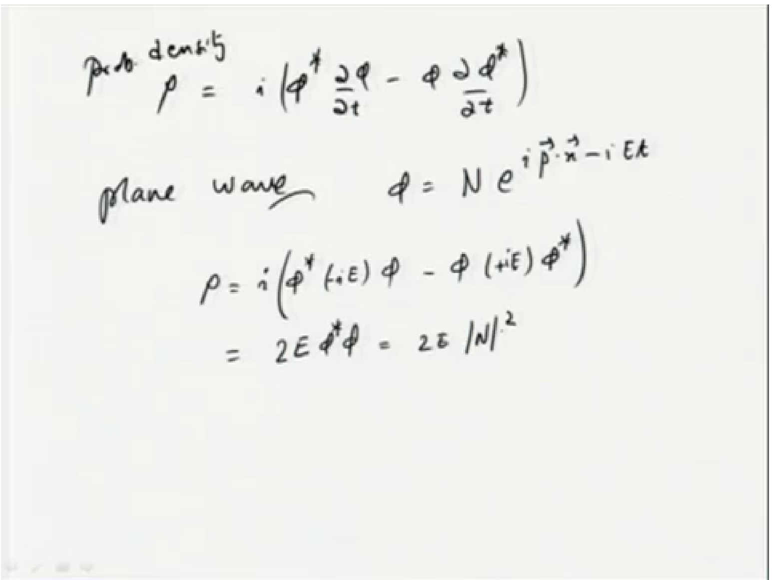
So, this here again we can interpret Rho as the probability density or the rate of rho is the probability density and j is the probability current. The interpretation is this that if you consider a small volume say a small cube rate of change of probability density inside the volume is the same as the rate of change or the divergence of the J, which essentially is because of the divergence theorem or the gausses theorem this corresponds to the flux going out of the surface from the volume you see. So, minus of the scene is actually the flux which is going out of this volume.

So, it essentially changes the says inverse that rate of change of the continuity equation will tell you the rate of change of probability density rho inside the box is equal to flux flux is j dot the area.

So, you have to use the gausses theorem divergence of a vector over integrate integrated over a volume is the same as J dot da a surface integral of the same vector over the closed surface which encloses the valley over the boundary of the volume. So, that is basically the flux the right hand side here is the flux of this thing of the field.

So, here it is the flux to the surface of the box and what flux is this the flux is basically the probability flux. So, it is it says for example, if you consider some number of particles inside this one, then it will actually tell you the rate of change of that minimum number of particles is equal to the number of particles that flows out or the probability for the number of particles 2 floors a kind of a statistical statement that is what we expect in quantum mechanics, but it essentially is very similar to the continuity equation that we have in any other physical situation like fluid dynamics water floor liquid flow all right.

(Refer Slide Time: 31:23)



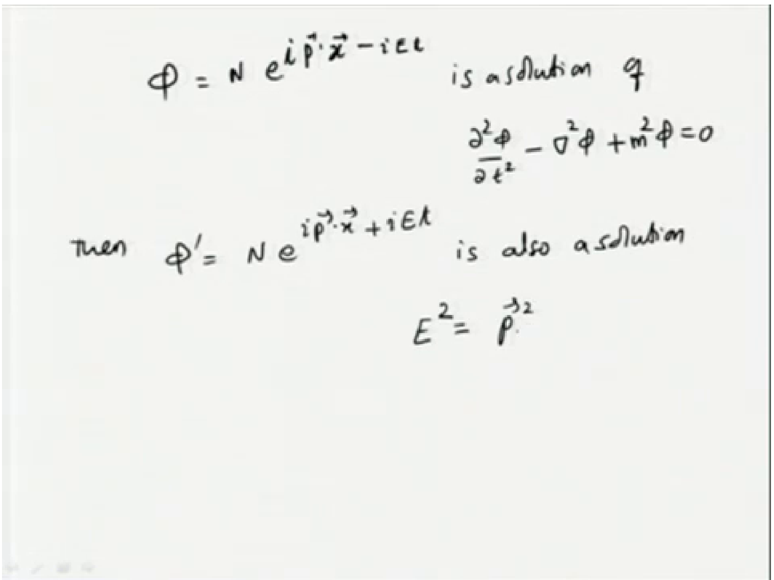
Now, so we have the expression for rho equal to i phi star time derivative of phi minus phi time derivative of phi star this is the probability density. So, the probability density, if i take a plane wave solution or plane wave for the phi, then I can write it as e pi e power i P dot x minus i energy times time this is a plane wave.

Then rho is nothing but i phi star remains as 5 star dou phi by dou t will take out minus i and E, then phi itself minus phi as it is phi star now will have n star exponential minus i P dot x plus i E t. So, when I take the time derivative it is plus ie that comes out and then phi star as it is this is.

So, there is an I here overall factor and then there is an minus i E in the first term together that will give me a plus E. So, plus e and the second term again I have a minus sign. So, minus i E inside the bracket and an i outside. So, that is again going to give you a plus E. So, total 2 E phi star phi, which is 2 E en square mod N square.

So, rho in the case of plane wave solution is twice the energy of the solution wave function and or energy of the particular corresponding to the wave function, and the normalization a square all right.

(Refer Slide Time: 33:57)

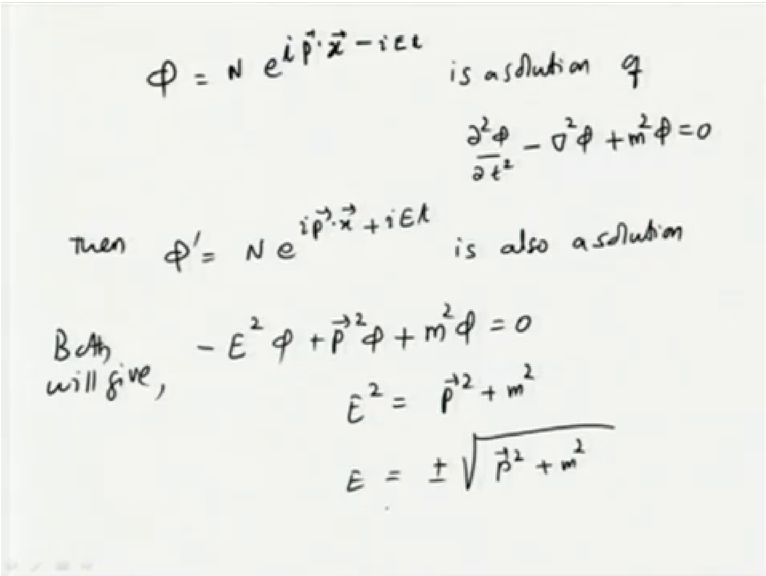


So, now, let us consider the 5 izzard is a phi equal to N E power i P dot x i P dot x minus i E t is a solution of Klein Gordon equation dou square phi by dou t square minus del square phi plus m square phi equal to 0, if this is all then phi equal to some other phi or phi prime.

So, let me take a yeah phi prime N e power i P dot x plus i E t this is also a solution. That is because the equation Klein Gordon equation is quadratic equation it actually second order derivatives are coming in.

So, for i plus i square and minus i square both will give you the same minus E square. So, essentially this is coming down to the situation where,we have to start with we have p square minus let me write on both of this actually give you, when you put it back in kg equation Klein Gordon equation both of this will give you, E square phi minus P square phi sorry there is already a minus sign there plus.

(Refer Slide Time: 35:46)



So, this is minus this is plus this is plus m square phi equal to 0 both of this will give you this.

So, this is essentially E square is equal to P square plus m square, which is basically the energy momentum relation relativistic energy momentum relation. And if I write down the expression for E I can actually write it as P square plus m square under root plus or minus.

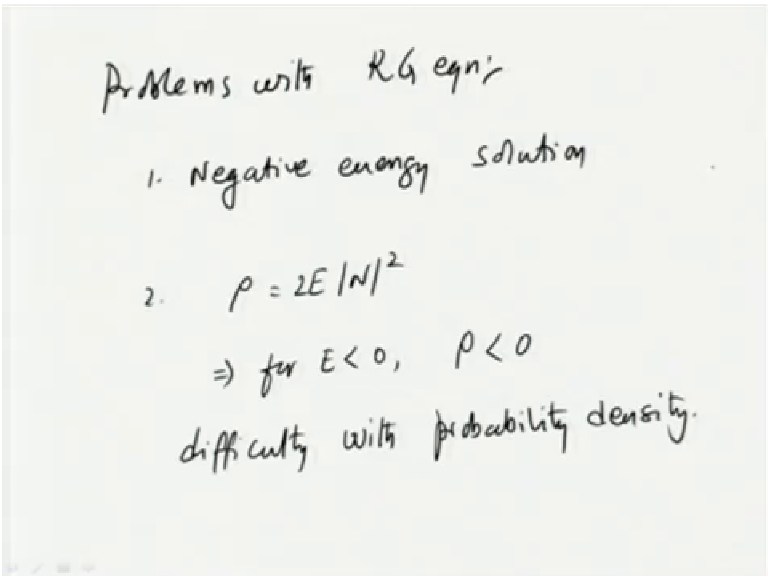
Now you can see that more clearly if whether I take plus E or minus E the equation should give the same result or the other way of saying it is that, E E is a solution to this thing minus E is also a solution I mean expression width plus e is a solution to the kg equation expression with minus E is also a solution to the kg equation.

When we interpret E as energy of the particle corresponding to the wave function y this then put us in a difficult situation, it tells you that you have a solution with energy E for this particle.

And you can order in that case-avoid a solution another solution with minus E. So, a negative energy solution emerges comes along with the positive energy solution. This is a problem, because we are talking still about free particles they are not put any potential here these are all free particle solutions.

We do not have any extra potential that is why E square is equal to P square plus m square only the kinetic energy term and the mass term are there in the energy. So, negative energy is not a physically viable case we do not want to have negative energy solutions, but the Klein Gordon equation actually give you solutions with negative energy. So, this is one problem.

(Refer Slide Time: 38:32)



So, problems with kg equation as it stands now are 1 negative energy solution. 2 for that let us look at the probability density in our case the probability density is equal to twice the energy times N square N square is not positive negative 2 is not negative. So, if E is negative then rho is negative.

So, that will give you for E less than 0 rho less than 0 how do we interpret rho then as probability density. So, difficulty with interpreting or let me simply write it as probability density you cannot have negative probabilities which is physically not valid.

So, these 2 problems are to be addressed to understand the relativistic equation a relativistic dynamics of particles through Klein Gordon equation we will come to how we actually address these questions in the next lecture.