Nuclear and Particle Physics
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Module – 09
Electroweak Interactions
Lecture - 01
Relative Quantum Mechanics

In the next few classes, we will describe how to handle the dynamics of the elementary particles. For this, we have to consider a relativistic quantum mechanics. What we usually study in the introductory quantum mechanics classes is a non-relativistic picture, which is suitable to study the properties of atoms etcetera. But when we consider scattering of high energy electrons or high energy protons on either protons or electrons or other particles, we need to actually consider the relativistic effect.

It is not enough to actually just consider the Schrodinger picture and then make corrections to the Schrodinger equation and other mathematical formalism by considering some aspects of relativistic effect, rather it is needed to find out a formalism suitable to the relativistic systems that is what we will first understand in the first few lectures from now on.

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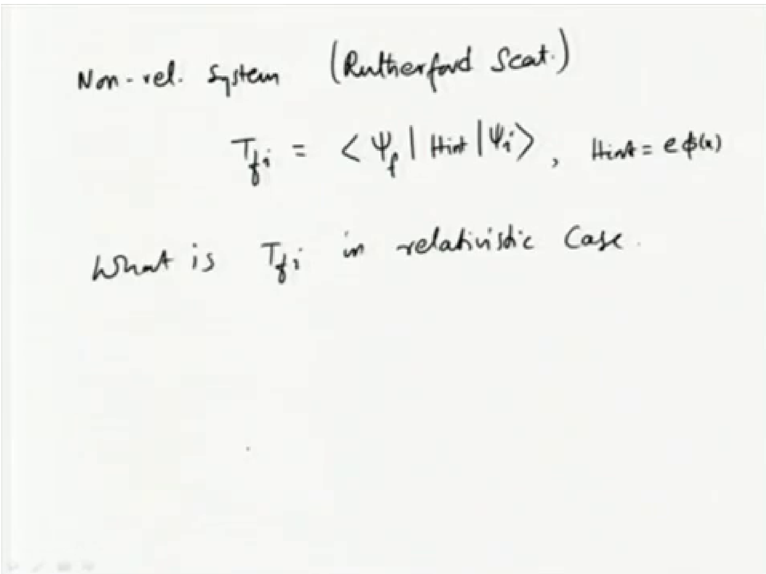

First question in the case of scattering as non-relativistic system like the Rutherford scattering, what we had considered in the previous one or two lectures is the transition amplitude from initial state to a final state is essentially equal to psi f H interaction psi i, where psi i and psi f are the initial and final wave functions of the electron or whichever particle is being affected by the interaction Hamiltonian given by H i. In particular, if an electron is scattered off a proton or any other nucleus etcetera, then we can in the non-relativistic picture consider H interaction the electrostatic potential experienced by the electron which is basically e phi x or minus e phi x depending on how you take that this thing phi. Now, question is what is the corresponding thing in the relativistic case? so that is what we have to deal with.

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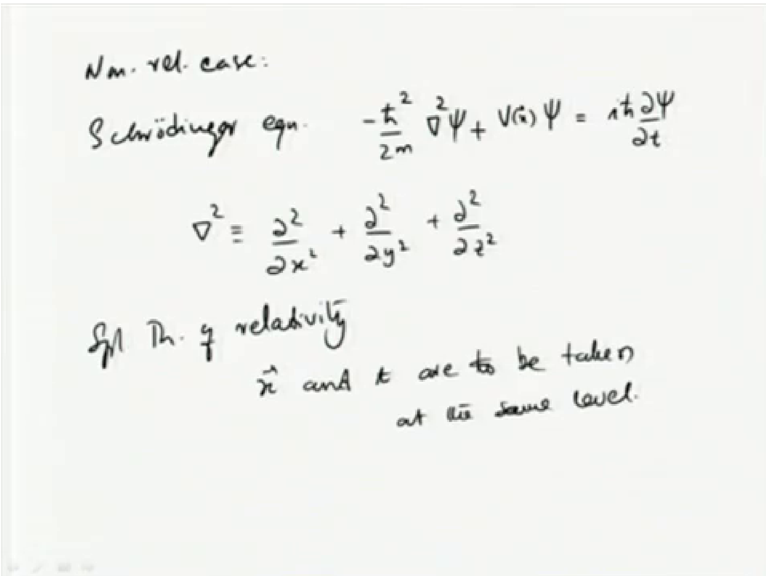

Now, let us start from the beginning in the non-relativistic case the equation of motion is basically the Schrodinger equation which is it can be written as h cross square over 2 m grad square psi for a wave function psi which essentially gives you the kinetic energy term plus the potential energy term v x psi x is now a three-dimensional vector equal to i h cross dou by dou t the energy operator acting on psi.

This is the non-relativistic Schrodinger equation why cannot we take this relativistic equation, because if you look at the kinetic energy term del square is essentially in Cartesian coordinate system second derivative with respect to x plus dou square by dou y square plus dou square by dou z square. So, the position is taken or the operator with the partial derivative with respect to the position coordinates is quadratic second order. While the right hand side of the Schrodinger equation tells us that the time derivative is first order, but in special theory of relativity, we say that we cannot actually have much distinction between x and t or the spatial coordinates and the time coordinate.

In fact, when you go from one frame to the other, they mix together; so their identity as time coordinate and spatial coordinate is a little vague there, we have to take them in the same footing, then only we will get the kinematics correct. Whereas, in Schrodinger equation, this is not the case Schrodinger equation is quadratic with respect to the spatial derivative while first order with respect to the time derivative, this is a problem. So, this how do you actually treat x and t at the same level and get a dynamical equation.

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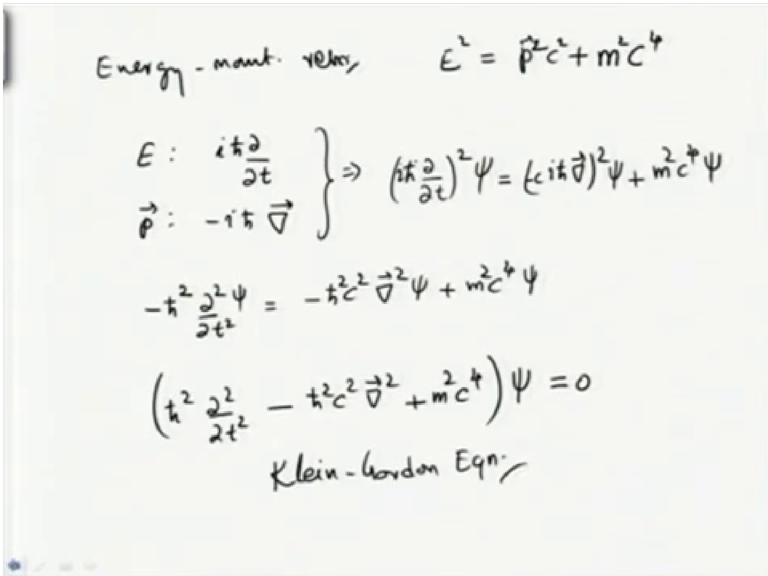

One approach is to rely on the relativistic energy momentum relation, which says that the energy of the particle is related to its momentum in a quadratic way E square is equal to p square c square plus m square C power 4. The operator corresponding to E is i h cross dou by dou t, and the operator corresponding to p is minus i h cross grad. So, this tells us that we can in the operator from write this as i h cross dou by dou t square energy operator acting on psi equal to minus i h cross grad square acting on of course, there is a C there. So, let me squeeze that in minus c i h cross grad square acting on psi plus m square c four acting on psi. Or other words, we can just square this and then you will get minus h cross square dou square by dou t square psi equal to minus h cross square of c square del square psi plus m square C 4 psi.

Or I can write it as h cross square dou square by dou t square plus minus h cross square c square grad square plus m square C 4 as the operator acting on psi equal to 0. And this is one of the ways to overcome the difficulty of the Schrodinger equation to get an equation which dictates the dynamics of the relativistic particle represented by a wave function psi with mass m. So, the equation is known as Klein-Gordon equation after applying a Klein Gordon. Let us look at some notations in relativity relativistic dynamics kinematics that we will consider which will simplify a lot of writing.

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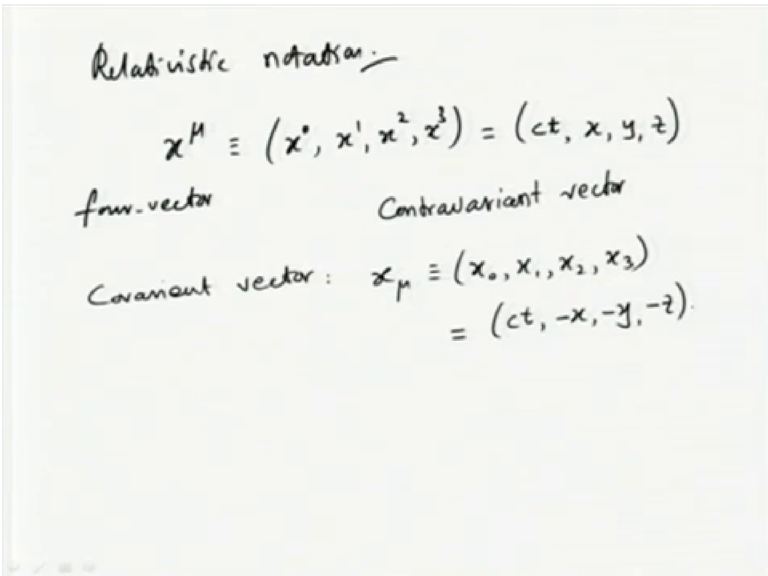

So, let us start with the relativistic notations and other things, let me call it just notation here. First thing is the position coordinate. As I said we will take the time coordinate and spatial coordinate together, and write it as a four vector instead of three spatial vector three vector which denotes the spatial coordinates and time coordinate separately. We will club them together as entity having four coordinates now.

Let us denote this by x with an upper Greek index mu. Mu can take values 0, 1, 2 and 3; and in components I will then write this x mu having four components as x 0 as the zeroth component, x 1 as the first component, x 2 as the second and x 3 as the third component. When we interpret this as the four-dimensional coordinate of a particle then we will consider x 0 as c t, where c is the speed of light and t is the time parameter or time coordinate, so that c t together has the dimensions of length. And then the coordinate x, y and z the normal position coordinates in Cartesian coordinate system x, y and z. So, x, y and z together with c t gives us a four vector. We call this a four vector, four vector representing position coordinate.

Now there are two ways to write this when we represent it with an index mu, I can write it either as a subscript or as a superscript. And what is given here we have written it as a superscript, and therefore, this is basically called a contravariant vector with upper index. Now, the vector when we write with the index as a subscript it is called a covariant vector x mu, and I can write it as x 0, x 1, x 2, x 3 conventionally we will take this as c t same as x with upper index 0 or superscript 0, but the spatial coordinates remain changes sign. So, x 1 with a subscript one is minus x. So, between the upper index and lower index, x mu and x mu lower superscript, there is a relative sign and difference between the time coordinate and the spatial coordinates, and this is an important convention that we had to keep in mind.

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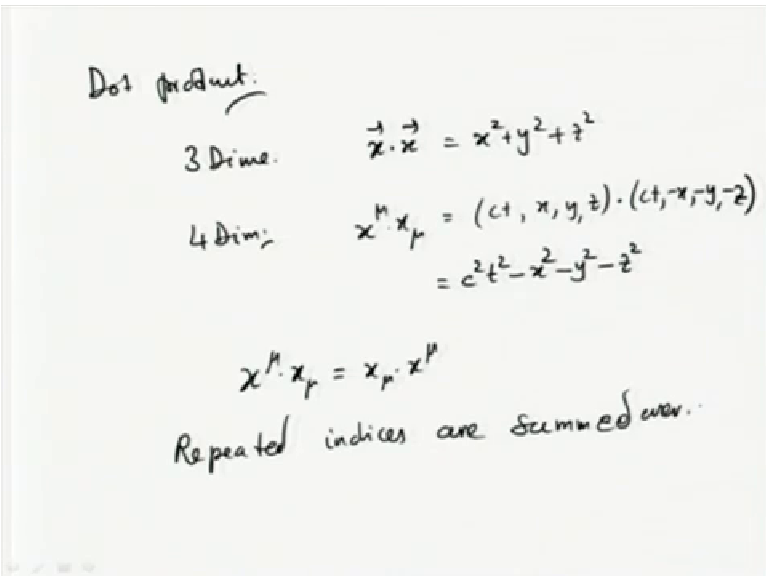

Now, this will become apparent why we are considering this when we consider say dot product bit of two such vectors. In three dimension case, we have x dot x equal to x square plus y square plus z square. And similarly in four dimension, we have x mu x mu c t x y z dot c t minus x minus y minus z equal to c square t square minus x square minus y square minus z square. There are two things that you should notice here one is that when I write this mu in this fashion, I sum over this index mu; the other thing is that when I take a dot product I have taken one vector as a covariant vector, the other one as contravariant. It does not matter whether I take first the covariant and then the contravariant or first the contravariant and then the covariant. They will both give you the same result c square t square minus x square minus y square minus z square in this case. So, convention is that repeated indices are summed over. This is convention that we will keep so that we do not have to write the summation sign symbol everywhere all the time.

Once we have an agreement that whenever there are two indices one upper index and one lower index, then we will sum over them. If so, this is what we will have in these lectures as well. And whenever we have two indices which we do not want to sum over we will explicitly make a statement regarding that this thing. So, until I say something like x mu is not summed over you can assume that if they are repeated in a term then they are summed over. This x mu x mu is equal to c square t square minus x square minus y square minus z square, this one main thing that we had to keep in special theory of relativity. This is basically an invariant quantity when we change from one frame to the to another thing.

I am assuming that you have some familiarity with special theory of relativity. And you know what is the meaning of going from one frame to the other, but what is the relation between two different moving frames. So, what is the difference, what are the relations of coordinates and between two different Lorentz frames as they are called. So, the Lorentz transformation as it is called gives you the relations. And then I assume that you are somewhat familiar with that.

And it says that there is an invariant quantity if you take x mu x mu this is similar to an invariant quantity called x square plus y square plus z square under rotations. We will come to this in a moment in a little illustration later. But otherwise the invariance is of the quantity with a relative sign between the spatial and the time coordinates not c square t square plus x square plus y square plus z square. This can be brought out by kinds considering an covariant vector and a contravariant vector in the manner we have just mentioned. There are other conventions that we can follow to get the same results; for example, it is also equally possible to consider the time coordinate changing sign keeping the other ones positive in the case of covariant and contravariant, but we will consider this convention throughout our discussion.

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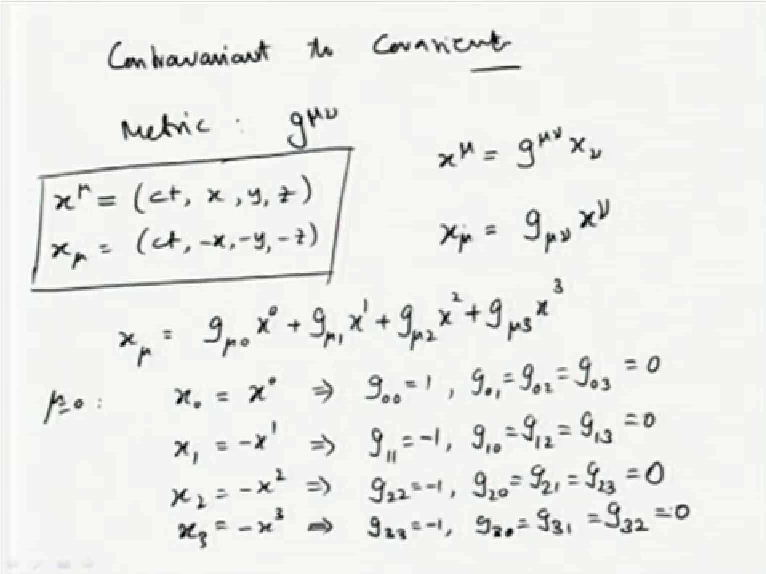

Now that we have said there is a covariant and contravariant vector, how do we convert from contravariant to covariant, is that possible or covariant to contravariant? For that, we consider what is called a metric g mu nu with two indices; x mu can be written then as g mu nu x nu covariant, and the other way around x mu covariant can be written as g mu nu lower indices x nu contravariant. Here again the repeated index nu in both expressions is summed over. So, let us take x mu or x nu the covariant thing that is x mu 0 x 0 plus g mu 1 x 0 upper index x 1 upper index g mu 2 x 2 plus g mu 3 x 3. So, this is the meaning of this x mu equal to g mu nu x nu.

Now, consider mu equal to 0 case that will give you x 0 equal to we know x 0 is equal to x 0, because let me write it here, x mu for us is equal to c t, x, y, z; x covariant is equal to same c t, minus x, minus y, minus z all right. So, let me keep that in mind. And therefore, x mu x 0 with 0 lower index is the same as x 0 upper index, so that will give you that g 0 0 is equal to 1; and g 0 1 equal to g 0 2 equal to g 0 3 equal to 0; otherwise x 0 would be related to x 1, x 2, x 3.

Similarly, x 1 is equal to minus x 1 and that gives us g 1 1 is equal to minus 1 and 1 0, 1 2, 13 equal to 0. And x 2 equal to minus x 2 gives again g 2 2 equal to minus 1, g 2 0 equal to g 2 1 equal to g 2 2, 2 3 equal to 0. Now, x 3 equal to minus x 3 will give g 3 3 equal to minus 1, 3 0 equal to g 3 1 equal to g 3 2 equal to 0 that fixes all the g the components of g mu nu.

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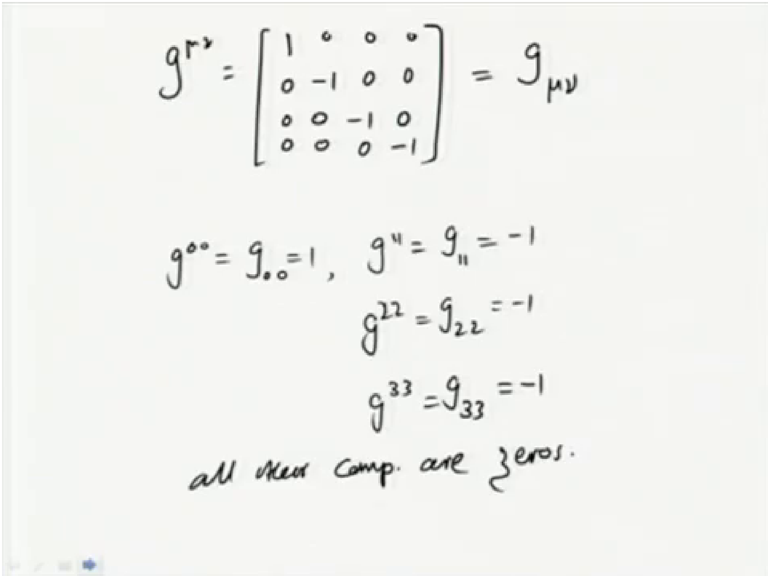

And thus we get g mu nu as let me write it as a matrix or the only the diagonal elements whenever these two indices are the same then that is nonzero; whenever the indices are different then that is 0. So, only diagonal elements survive and of this g 0 0 is plus 1, g 1 1, 2 2, 3 3 the special parts are minus ones. So, you will have a diagonal matrix 4 by 4 with 1 minus 1 minus 1 minus 1 as the diagonal entries, it is a fully symmetric matrix. Similarly, g mu nu with lower indices also can be written exactly the same way we can proceed exactly the same way as we did earlier. So, we can say that g 1 1 is equal to g 0 0 equal to g 0 0 lower index g 1 1, g 1 1 which is equal to 1, which is equal to minus 1. g 22 equal to g 22 equal to minus 1 3 3 equal to g 3 3 equal to minus 1, all other components are 0s.

So, this is our metric. And this metric again can be different we can choose different metric a different a metric which is different from this. Say for example, one case that we said earlier is that instead of changing the sign of the special coordinates, when you go from contravariant to covariant, we could change the sign of the 0th component and that will still give you a relative sign between c square t square and x square plus y square plus z square, and that is what is important in special theory of relativity. So, we could as well have other metric which will work, but here we will stick to this in our discussions, so that is about the four vector coordinates.

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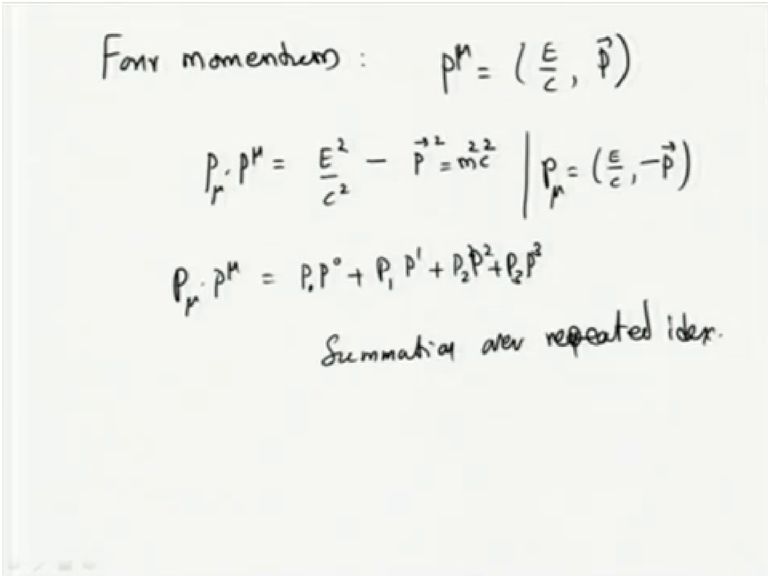

Then let us consider the four-momentum, four-vector momentum or for short four momentum. p mu is equal to here we will consider the energy as the 0th component; and for dimension to be correct you divide it by c, and you have a the normal three-momentum as the 1, 2, 3 components. Again p mu p mu is equal to E square over c square minus p square that is because p mu covariant is E over c minus p. This is equal to m square c square that is what we mentioned in couple of lectures before that if you take the square of the four-momentum then that is basically the mass square times c square. Square of the four momentum meaning dot product of the four momenta with itself which means you have to take a covariant and contravariant this thing and this is what we have.

And this is equal to m square c square because of the energy momentum relation E square is equal to p square c square plus m square c 4. So, as we said we will be using the so when we say p mu p mu which actually means p 0 p 0 plus p 1 p 1 plus p 2 p 2 plus p 3 p 3 and that is equal to m square c square and summation over repeated index is as we have been considering as our convention.

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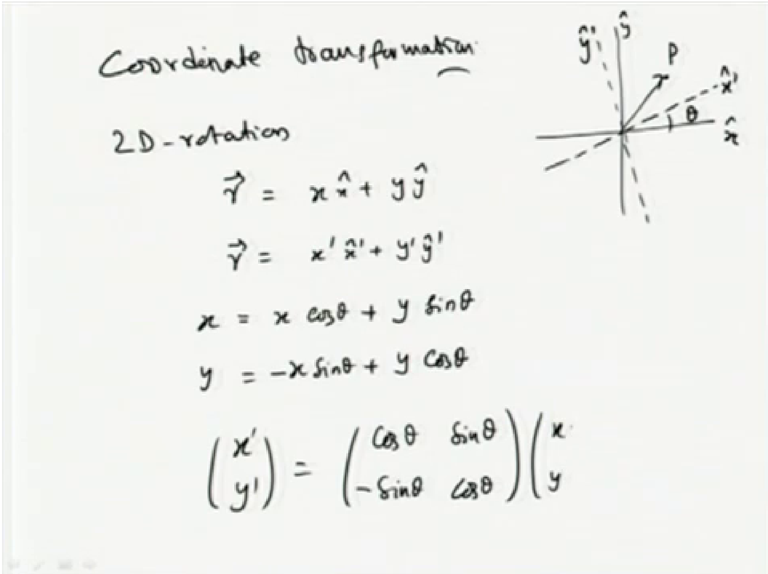

Now, let us look at the transformation of these objects coordinate transformations. Let us consider a coordinate transformation. First consider rotation in two dimension 2D-rotation, rotation in two dimension. So, we will consider actually only two coordinates x and y along x cap and y cap directions. So, let me draw that in a proper way. We have a horizontal axis which we denote by x-axis; and we have a vertical axis which we denote by a y cap, and a vector r position vector of point p is denoted by say r which is equal to x x cap plus y y cap.

Question is if now we rotate the coordinate system, so that we have a new axis x prime and y prime caps, how do you represent this point in x prime and y prime as theta? Where; theta is the angle of rotation. So, we are only considering a rotation about the z axis, the perpendicular axis perpendicular to the plane x y plane. Same vector r is equal to x prime x prime plus y prime y prime cap. And we will see we know these thing that coordinates if you look at x is equal to x cos theta plus y sin theta, and y is equal to minus x sin theta plus y cos theta. I will leave this as an exercise, you must have done it many times in your elementary classes or I can write in a matrix form x y equal to cos theta sin theta minus sin theta cos theta multiplying sorry x prime y prime equal to multiplying x and y.

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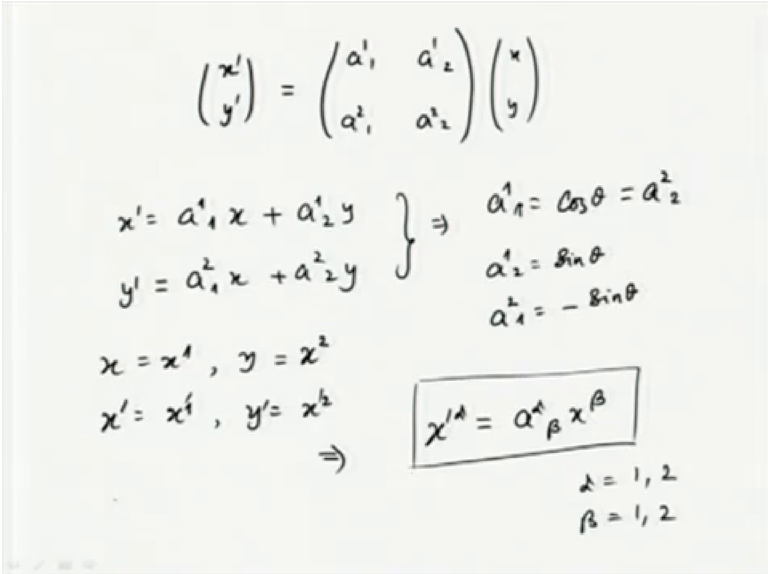

Let me write it in a way that will be more suitable or generalized easier for us to generalize. I will write it as a 11 a 1 2 a 2 1 a 2 2 x y. So, that x prime is a 1 1 x. So, there are one and prime I will make a little distinction, so prime a 1 1 x 1 a 1 2 x 2. And y prime this is at the moment just x and y x prime y prime is equal to a 2 1 x plus a 2 2 y. And the whole thing gives us a 1 1 equal to cos theta which is also equal to a 2 2 diagonal elements. a 1 2 is equal to sin theta, a 2 1 is equal to minus sin theta.

Now, let us denote x as x 1, y as y 1 sorry y as x 2 and similarly x prime as x 1 prime or y prime as x 2 prime x prime 2. This then leaves us a compact notation x prime alpha equal to a alpha beta x beta, where alpha equal to runs from 0 to sorry 1 tend to it can take values 1 and 2 and beta can also take values 1 and 2. So, if I take x alpha equal to 1, beta is summed over. So, it is repeated and therefore, it is summed over. So, if I take x 1, so x alpha equal to 1, x prime 1 equal to a 1 1 x 1 plus a 1 2 x 2. Similarly, for x prime 2, this is a compact notation. So, we will we have a compact way of denoting the transformation of coordinates under rotation in two dimension.

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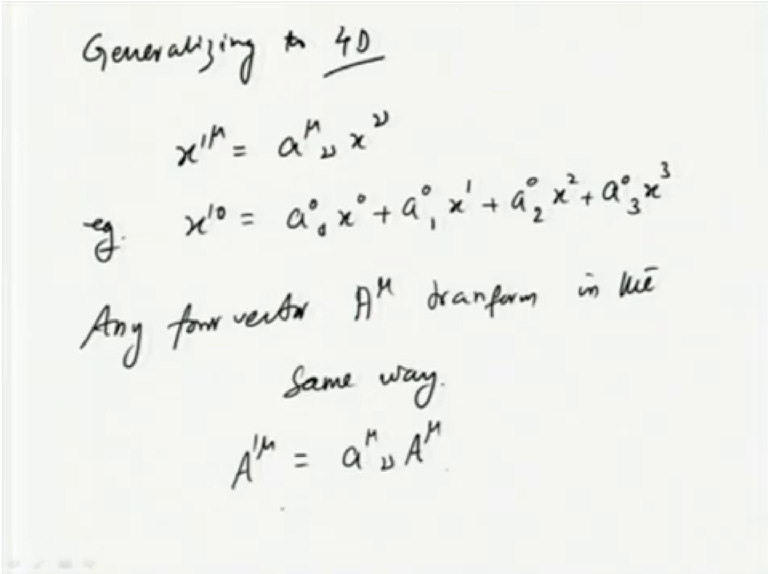

We will generalize this now to the case of four dimension. Generalizing to four dimension, we have we can write x prime mu equal to a mu nu x nu very similar to x prime alpha equal to a alpha beta x beta. And here as we have mentioned we have to take mu and nu values 0, 1, 2 and 3. For example, mu equal to 0, we have x prime 0 equal to a 0 1 x 0 plus a 0 2 x 2 plus a 0 3 x 3 plus a I missed the 0 0 a and 0 0 a 0 1 x 1 and a 0 2 x 2 plus a 0 3 x 3 similarly, four other values of mu.

And any four vector transform in a similar way in fact, in the same manner under a particular special transformation coordinate transformation. If x prime x mu changes to x prime mu which is equal to a mu nu x nu with a particular set of a mu nu then any vector any object that transform in that way is with the same set of a mu nu are called vectors r vectors. So, that is one way of identifying whether a particular object is a vector or not. Whereas, the scalar quantities and the special coordinate transformation not in which or do not transfer transform we know do not change.

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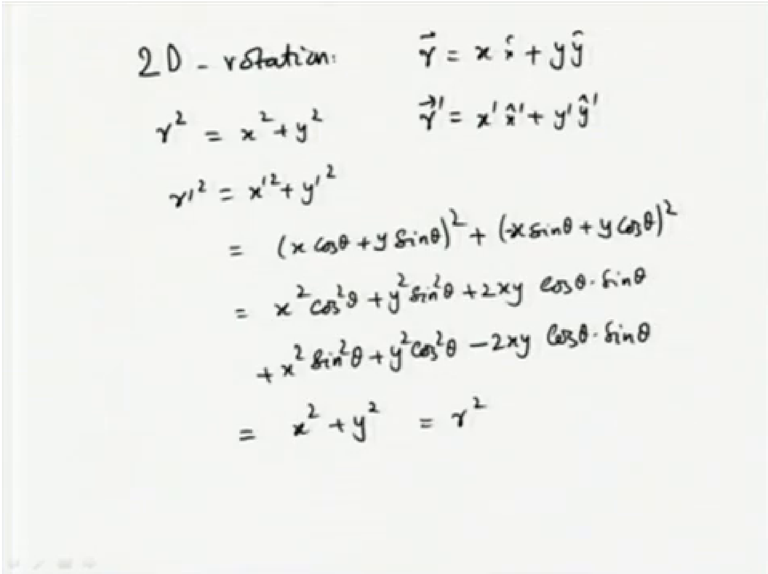

What is an invariant quantity that you can think of take again 2D-rotation. If you take r square just go back to the earlier picture, if you take r square which is equal to x square plus y square right, the vector we denoted by x x cap plus y y cap are the same thing in a rotated coordinate system was x prime x prime cap y prime y prime cap. So, this is also equal to x prime square plus y prime square, but x prime square is equal to x square plus so all right. So, let us see this is the vector r let us say this is the I will denote this by r prime it is also the same vector in case; write it exactly as x cos theta plus y sin theta whole square plus minus x sin theta plus y cos theta whole square.

This is equal to x square cos square theta plus y square sin square theta plus 2 x y cos theta sin theta plus x square sin square theta plus y square cos square theta minus 2 x y cos theta sin theta adds up to x square plus y square. So, this is the same as this. So, this is this preserves the length of the position vector; and this is an invariant quantity under the special transformation which we consider which is rotation.

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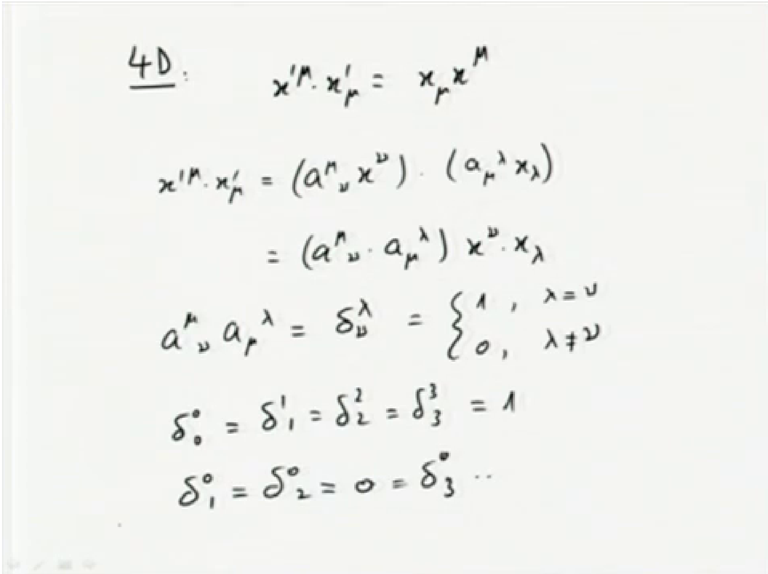

Now, so we considered let us consider the four-dimensional case x prime mu x prime covariant mu the dot product is equal to x mu is what we want to demand because we want to consider the dot product of these two like r dot r r square is an invariant here x square is an invariant quantity. We want to have that to preserve this length. Then we write x prime mu x prime mu equal to the contravariant one a mu nu x nu, covariant one there is nothing to really describe there, but you can see that they transform like a mu let me take a second another index lambda x lambda.

Since, nu is a dummy index which is summed over, I cannot take the same index in the same term for another dummy index. Therefore, I use another index lambda here if I take that as nu when I expand this give specific values to nu then that will be very confusing. So, this is the right way to do this.

So, whenever you see a dummy repeated index or there should not be any more index than that, but the other way is that in any term this is one particular term that we are writing, there should not be more than two indices the same. Repeated indices cannot be more than two in a term. And if there is a repeated index usually it is summed over, here it is summed over for example. So, this is equal to a mu nu a mu lambda x nu x lambda all right.

Now, this says that we have to have if you want this to be x nu nu or x lambda lambda then we have to have a nu mu lambda mu nu a mu lambda equal to the Kronecker delta nu lambda, which is equal to one for lambda equal to nu and 0 for lambda not equal to nu. For example, if you consider delta 0 0 that is equal to or delta 1 1 delta 2 2 delta 3 3, all these are equal to 1; delta 0 1 delta 0 2 etcetera all are equal to 0.

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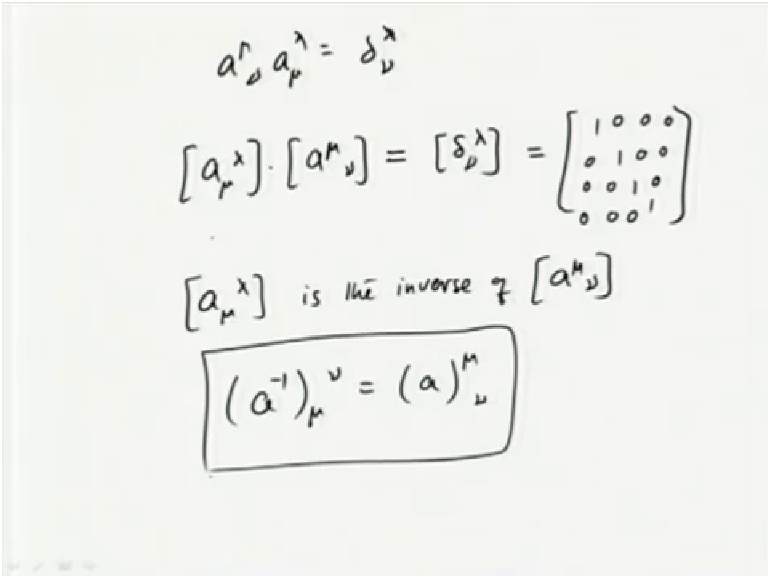

So, let me club this together in a matrix form. So, what I mean is so let me write it down as a mu nu a lambda mu lambda is equal to delta lambda in matrix form I can write it as a mu lambda matrix the corresponding matrix a mu nu matrix is equal to some unit vector delta nu lambda. Which means and this gives you the mu lambdas component or lambda mu component of the matrix a.

So, write it as this is an inverse of this has to be these two are inverse of each other, because this delta fang matrix if I write as matrix it is basically a unit matrix of dimension 4 by 4 delta lambda (Refer Time: 46:13). So, we can say that a mu lambda matrix is the inverse of a mu nu matrix. Or we can say that in proper way of writing it, a inverse mu nu is essentially equal to a mu nu that is mu nu element of inverse matrix a is the same as mu upper index nu lower index of the matrix a. So, this is something which we will use to continue our discussion tomorrow.