Introduction to LASER
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Lecture - 07
Line Broadening Mechanisms - 2

Welcome to this MOOC on Lasers. In the last lecture we have discussed the Line Broadening Mechanisms the broad classification of homogeneous line broadening and inhomogeneous line broadening. In the last class I also discussed the lifetime broadening and we have derived an expression for the life time broadening which come out to be a Lorentzian.

So, today we will take up further the Line Broadening Mechanisms and in particular we will discuss the Doppler broadening and derive an expression for the line width due to Doppler broadening ok.

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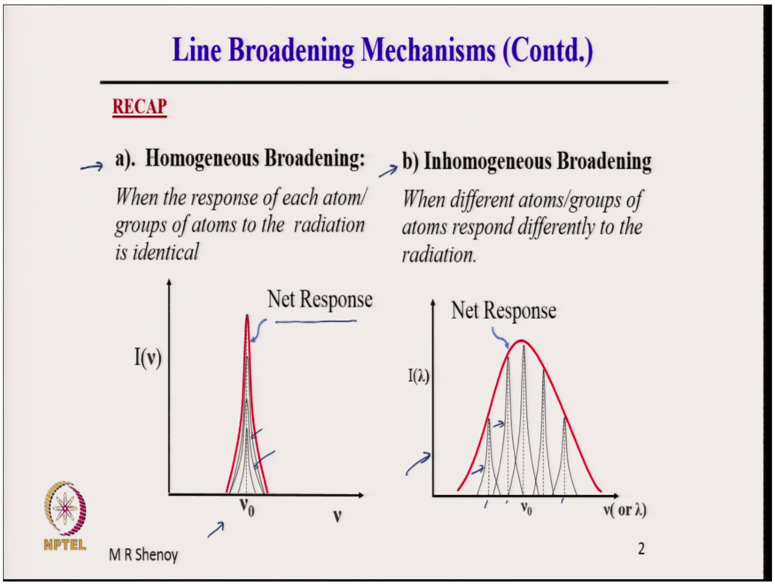

So, very quick recap there are two types of broadening mechanisms; broad classification homogeneous and inhomogeneous. Homogeneous broadening occurs when the response of each atom or groups of atoms to the radiation is identical centered around a resonance frequency.

As you can see here in this diagram all the different curves which are shown here are the response of different groups of atoms, but all of them are centered around a resonance frequency nu 0 and therefore, the cumulative effect will also be a resonance centered around nu 0 and that is the net response which we shown here with the red line.

In the case of inhomogenous broadening; different groups of atoms. So, these responses which are in the insect here correspond to response of different groups of atoms they are centered around different frequencies here as you can see. And the net response is represented by the envelop which is the peak of which corresponds to nu 0, which is the atomic resonance frequency. We will discuss this in a little bit more detail in this lecture.

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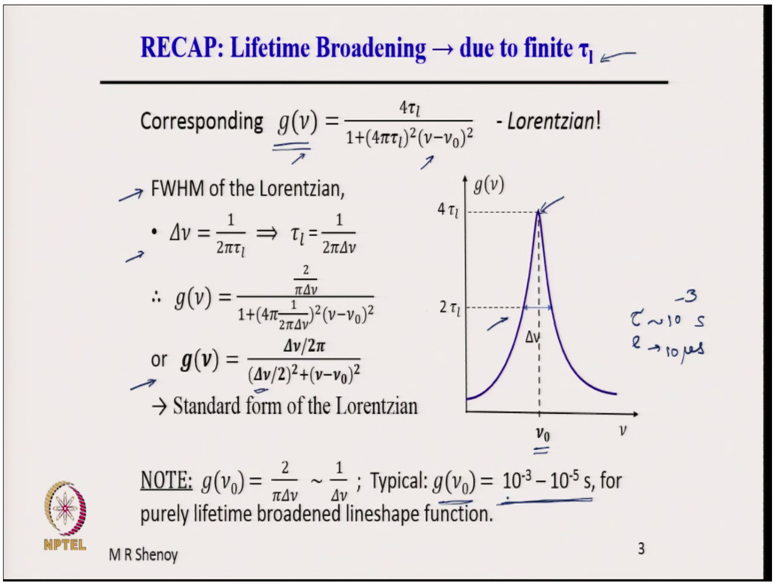

Now, a very quick recap of the life time broadening which is due to finite lifetime of the level tau l and the corresponding line shape function g nu we have derived this in the last lecture is given by 4 tau l divided by 1 plus four pi tau l whole square into nu minus nu 0 the whole square which is a Lorentzian function, it is a symmetric function it peaks at nu is equal to nu 0.

The second half of this term become second term becomes in the denominator become 0 at nu is equal to nu 0. And we have seen the full width at half maximum which is called the line width of the resonance is the full width at half maximum, the maximum is 4 tau l half of it.

So, half maximum and the full width full width around the resonance frequency nu 0, hence the short form FWHM.

FWHM or the Lorentzian we have seen that it is given by delta nu is equal to 1 divided by 2 pi tau l. And if you use this to write a tau l is equal to 1 by 2 pi delta nu and substitute in the expression here for g nu then we can get an expression for g nu which is of this form; g nu is equal to 2 delta nu by 2 pi divided by delta nu by 2 whole square plus nu minus nu 0 the whole square, this is a standard form of the Lorentzian.

Please note that delta nu here is a fixed number it is not a variable delta nu is the FWHM which is here; characteristic of a resonance. So, this is the standard form of a Lorentzian. And an important point that can be noted is g of nu 0, that is when nu is equal to nu 0 the second term is 0 and we have g of nu 0 is equal to 2 by pi into delta nu. 2 by pi is approximately 0.64 and therefore, it is of the order of 1 by delta nu.

That is the peak response which is here g of nu 0, nu equal to nu 0 is 1 by delta nu that is it is 1 by the FWHM of the resonance. So, narrow over the resonance higher will be the value of g of nu 0. And we know that delta nu is equal to 1 by 2 pi tau l and therefore, g of nu 0 which is of the order of tau l which is typically 10 to the power of minus 3. If you were to put some value for g of nu 0 this is the kind of numbers that you would get 10 power minus 3 to 10 power minus 5.

Because we know that the lifetime particularly of the laser levels is typically of the order of 1 millisecond that is 10 power minus 3 second to 10 microseconds typical lasers have the tau l in this range. And therefore the g of nu 0 as the same range because that is equal to 2 pi tau l. Just to have an idea about what kind of numbers we are talking of and note that the unit is second.

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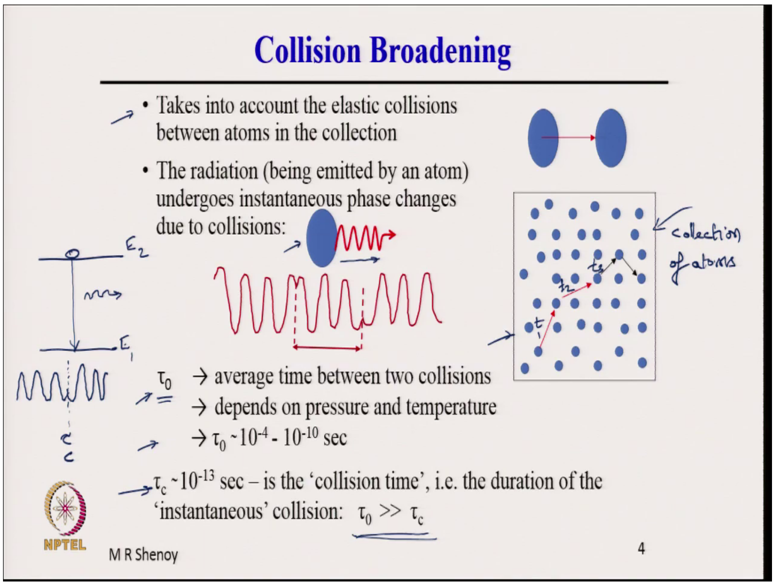

Alright, another important type of broadening which is also homogenous broadening is collision broadening. This takes into count elastic collisions between atoms in the collection, if you consider a collection of atom so here collection of atoms, Then atoms are continuously in a state of motion for example, you take a gas then atoms are continuously moving and they undergo elastic collisions with other atoms in the collection.

For example, if a particular atom if it was radiating as shown here so for example, the situation is this, let say this is a corresponds to a two level systems we have the ground state E 1 and a state E. The atom which was here in the exited states makes a downward transition by spontaneous emission let say. During this transition it gives out radiation there is a finite time taken for this transition and therefore, the emission process is over a finite duration.

Now, during this duration of emission if the atom undergoes suddenly a collision then the emitted radiation undergoes a sudden face change. It was a pure sinusoidal as so it was going like this it was emitting this process and suddenly it meets with a collision with another atom then there is a sudden change in the face. So, this point is where it underwent collision.

So, this is the point and there is a certain collision time tau c is the collision time, this almost instantaneous we know that it is almost instantaneous. Nevertheless, you can calculate and see that tau c is of the order of 10 power minus 13 seconds is the collision time which is the instantaneous collision time.

But the important point is tau 0 which is the time between mean time between collisions. What is shown here in this diagram, an atom moving here could collide with another atom here, it may next collide with another atom here this atom may move and collide with an atom here.

So, it is the time taken so for example, this time is sometime t 1 between this collision next time between the next collision that is from here to here it took some time before it collided. The second collision it may take a time t 2 at time t 3 and; obviously, they will be different.

And the mean collision time is tau 0; tau 0 if we call as the average time between two collisions; of course, it depends on the pressure because if there is a higher pressure then the atoms are in closer vicinity and depends on the pressure and temperature of the medium.

But typically tau 0 in a collection of atoms is of the order of 10 power minus 4 to 10 power minus 10 seconds. The point to note is tau 0 is much much greater than the instantaneous collision time tau c and therefore, in this case the sudden collision leads to a phase change in the emitted radiation.

So, there was a sinusoidal which was being emitted like this and during this process of transition from here to here it underwent a collision which is equivalent to a sudden phase change. Phase change need not be pi it could be some smaller number it could be some different number, but there is a sudden phase change.

And because of this sudden phase change we now no more have a continuous sinusoid from minus infinity to infinity or even from 0 to a certain time a large long time we do not have a continuous sinusoid. There is a sinusoid with the sudden phase changes in between.

And whenever there is a phase change then there is a corresponding bandwidth or spectrum a finite spread in the frequency spectrum associated with these sudden phase changes. And that spread is called collision broadening; broadening of the response due to collisions which actually lead to phase changes of the emitted radiation. So, let us see it a little bit more.

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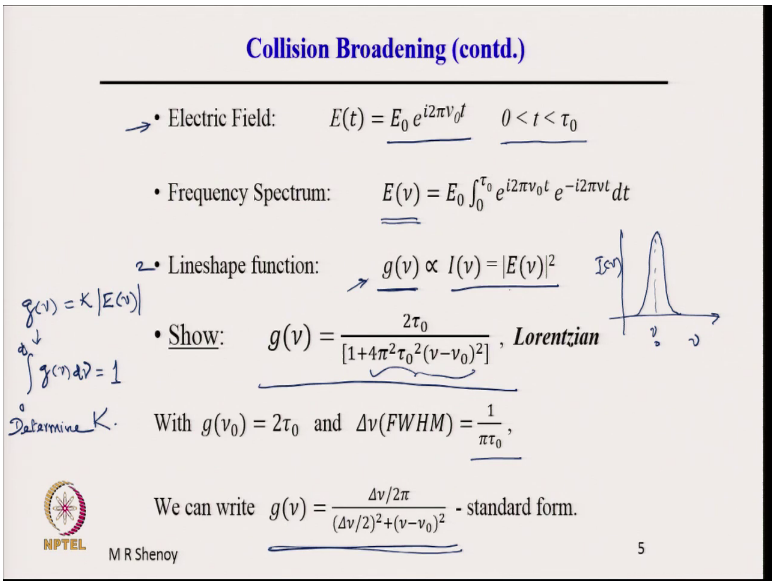

So, the electric field therefore, now only the steps I have outlined here I will not make the whole derivation. The electric field now this is a sinusoid, but it is a perfect sinusoid only during this period 0 less than or equal to tau less than t less than tau 0 mean collision time; in between this time it is perfectly a sinusoid.

And therefore, the corresponding frequency spectrum just as we did for the lifetime case is given by the Fourier transform. So, E 0 into e power i 2 pi nu 0 t into e power minus i 2 pi nu t d t. And there because it is a finite integral we will see that it leads to a finite spectrum.

And therefore, the intensity spectrum intensity spectrum is equal to mod of E nu square here, and we know that the line shape function which is proportional to the intensity spectrum. Intensity spectrum we are talking here of the spontaneous emission intensity; so, intensity spectrum I of lambda or I of nu versus nu around a resonance nu 0.

The intensity spectrum is proportional to g nu because g nu gives us the strength of interaction at any frequency. Strength of interaction in the case of emission we are referring to in the case of spontaneous emission we are referring to the strength of emission. So, wherever there is a stronger emission we have a larger value for g nu or we have a higher value for the intensity. And therefore, g nu is proportional to i nu which is equal to mod E nu square.

And, if you integrate this proportionality you can write as therefore, g nu is equal to; so g nu is equal to some constant K into mod E nu square so we can write this here. And then using the normalization condition of g nu that is 0 to infinity g nu d nu equal to 1 we can determine as before d nu equal to 1, as before determine K, determine the constant K.

And then you will get that g nu is equal to is given by such an expression, note that this is also a Lorentzian. Again centered at nu is equal to nu 0. The g of nu 0 is simply equal to 2 tau 0 because at nu equal to nu 0 the second part is 0 and we simply have g of nu is equal to 2 tau 0.

And the full width at half maximum can be shown to be 1 by pi tau 0 and again if you write delta nu is in terms of tau 0 you can get the same standard form of the Lorentzian here as well. Although the expression for delta nu is 1 by pi tau 0 we get the same standard form of the Lorentzian. So, please take this as an exercise and work it out.

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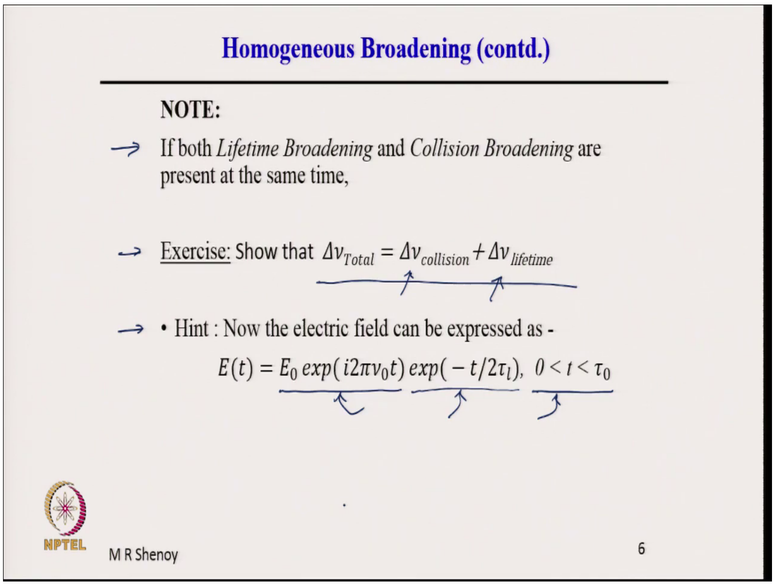

If we simultaneously have both lifetime broadening which is inevitable natural broadening lifetime broadening also called natural broadening because it is because of the finite lifetime of the levels. And collision broadening are present at the same time then, one can show that delta nu total is equal to delta nu collision plus delta nu lifetime. Both of them both the mechanisms are completely independent this is due to finite lifetime of the level and this is because of collisions. And therefore, the total delta nu the net delta nu will be some of these 2.

So, you can try to work this out those of you can do an exercise. So, please work this out and the starting point would be now the electric field is the pure harmonic that is sinusoidal wave, but now multiplied by due to the lifetime finite lifetime there is an exponential DK the damping term. And due to collisions there is a range of time over which it is sinusoidal. So, this sinusoidal is only over this range.

And then because of the lifetime there is a damping term that is why we can write the electric field in this form and proceed the same way to get the spectrum of in the presence of both. And you can see that it will be equal to the sum of lifetime broadening and collision broadening.

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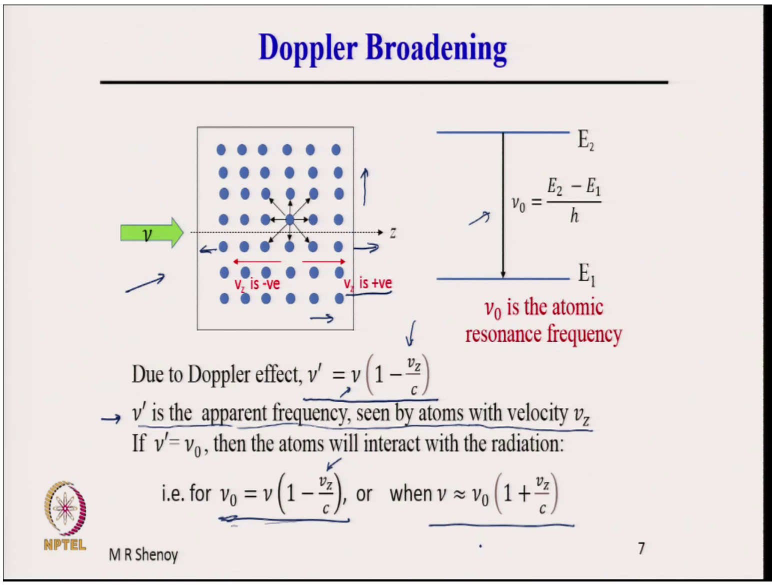

Now, we will take up a inhomogeneous broadening and as I mentioned will take up the Doppler broadening. I have already discussed in the last lecture while indicating the classification that this part we have discussed namely a radiation of frequency nu if it is incident along the z direction.

This is a collection of atoms where the atoms are right now shown as if they are in a fixed lattice, but the atoms could be moving in this direction the atom could be as shown here by the arrows the atom could be moving in any of this direction even in solids. Because in solids atoms are held in place due to elastic bonds and atoms are always in a state of oscillation or agitation.

And therefore at any given instant the atom may be moving from one side to the other side, and then it will have a finite velocity component in the direction of the incoming radiation. If nu 0 is the resonance frequency corresponding to the energy difference between the two levels E 1 and E 2 it is the atomic resonance frequency.

Then due to Doppler effect an incoming frequency nu would be seen as nu dash is equal to nu into 1 minus v z by c so this is due to Doppler effect. An incoming frequency nu is seen for example, if v z is negative which means an atom is moving towards the radiation, in this direction v z is negative z forward direction is positive. So, if an atom is moving in this direction then v z is negative.

And therefore, an atom moving towards the incoming radiation we will see it as a higher frequency. Therefore, we can write it as nu dash is equal to nu into 1 minus v z by c, note that v z with sign v z has components negative and positive those which are moving atoms moving in this direction v z is positive, as shown here by the red arrow and what is written here and those which are moving opposite.

And of course, atoms which are moving in a perpendicular direction v z equal to 0 and they will see no change in the apparent frequency of the incident radiation. So, nu dash is the apparent frequency seen by the atoms so this is important seen by the atoms which move with a velocity component v z therefore.

If nu dash is equal to nu 0 whenever the frequency seen becomes equal to the atomic resonance frequency nu 0 here then the atoms will interact with radiation. This is very important that is, when nu 0 equal to nu into 1 minus that is nu 0 equal to nu dash. So, we have replaced here nu dash by nu 0.

So, whenever nu 0 happens to be this then those atoms with velocity component v z will interact with the incoming radiation. So, this can be approximately written because v z by c is a very small number so we can write nu is equal to nu 0 into 1 plus v z.

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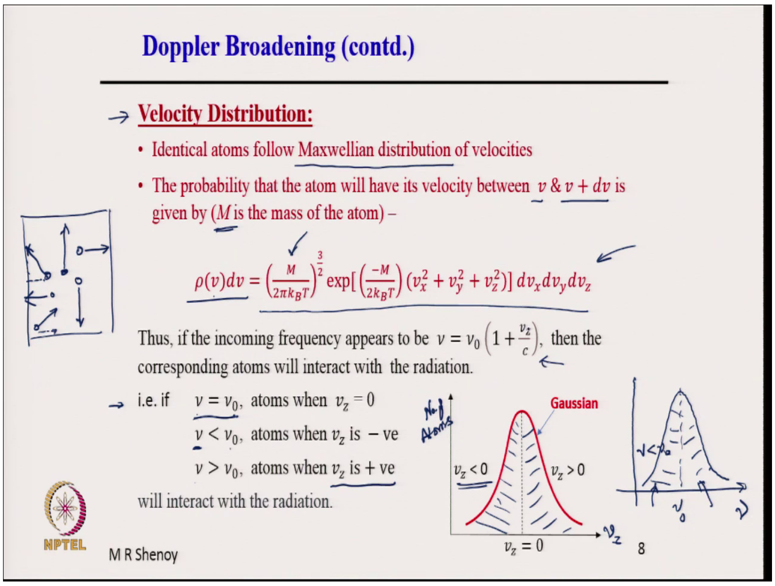

Let us, discuss this logically now. Before we proceed first let what is the velocity distribution, if we consider identical atoms in a collection or in a atomic system then they are given by the Maxwellian distribution of velocities. The probability that the atom will have its velocity between v and v plus dv is given by rho v dv; rho v dv here represents the probability that an atom has its velocity between v and v plus dv is given by this expression the expression here.

And where M is the mass of the atom it might be written somewhere no, yes M is the mass of the atom and k B is the Boltzmann constant and T is the temperature of the system or temperature of the atomic system. Thus therefore so this is the distribution which is given.

So, we would see that distribution will be like this, it is a Gaussian distribution you can see that if around v z equal to 0 similarly it will be around v y equal to 0 and v x equal to 0, we will have this is the distribution what is shown is the number of atoms; so, the number of atoms so this is the number of atoms. So, number of v z equal to 0 because there are atoms if we consider this collection look back, but we can see here.

So, there are atoms which are moving in this direction v z equal to 0, there are atoms which are moving in this direction v z equal to 0, there are atoms which are moving in this direction v z is positive, there are atoms which are moving in this direction v z is negative, there are atoms which are moving in this direction if this will have a v z component here which is negative.

Atoms which are moving in this direction v z is this component this is positive. So, the distribution about v z equal to 0 is shown here so this is v z. So, about 0 we have all the atoms so this distribution tells us the number of atoms which are having velocity less component v z less than 0. And similarly on the other half this shows us atoms which have velocity component greater than v z equal to 0 that is greater than 0.

Therefore, if the incoming frequency appears to be nu equal to this then the corresponding atoms will interact with radiation. What does it mean? Please see this logic. When atom for atoms where v z is 0 that is all the atoms which are having v z equal to 0, traveling in perpendicular direction they will interact with the incoming frequency nu if nu is equal to nu 0.

Whereas, these atoms the atoms which are here in the shaded region with v z less than 0 will interact if the incoming radiation is less than nu 0, why is that? Because even though the incoming radiation is nu it will be seen as a frequency higher and therefore, it will interact with radiation of frequency less than nu 0. And those atoms which have v z positive will interact with the radiation of frequency nu which is greater than nu 0.

If I were to plot here a graph corresponding to the number of atoms as a function of frequency now nu, then also I would get around nu 0 the atoms will interact on both the sides. So, this is the distribution of atoms interacting with radiation. So, here on this side it is nu the incoming radiation is less than nu 0.

So, all these atoms which are on the left half will interact with radiation if v z is less than 0, because they will see it as a higher frequency due to Doppler effect, they will see it as nu equal to nu 0 although the incoming frequency here nu is less than nu 0 and here nu is greater than nu 0.

And therefore, the net summary is that there is a range of radiation over which range of frequencies over which the atoms would interact with the incoming radiation. Therefore, the given collection of atoms would interact with radiation over a range of frequencies; although the atomic resonance actual resonance frequency is E 2 minus E 1 by h is equal to nu 0.

So, it will interact with the frequencies nu less than nu 0 as well as frequencies nu greater than nu 0. And therefore, the net effect is a range of frequencies with which the atomic system interacts; interacts means it may be emitting radiation over a range of frequency or it may be absorbing radiation over a range of frequencies.

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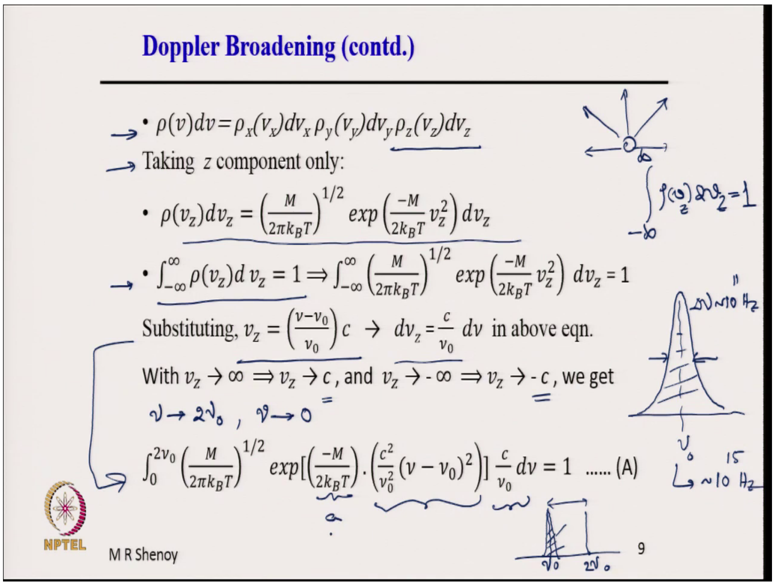

Now, the mathematics is quite simple so let us. Therefore, rho of v dv is equal to rho of v x d v x because the 3 components of velocity are independent and therefore, we can write it as rho x the probabilities can be separated out and taking only the z component then rho v z d v z can be written like this. So, we had exponential term with minus M by 2 K T into v x square plus v y square plus v z square that is for the rho v.

Now, we have taken only this part rho v z d v z so that will take only v z square. So, we can look back the expression which is here. So, this had here v z square into this into d v z v y square into d v y into this and so on. So, therefore it is now split into 3 components and therefore, we write this.

Similarly, as if we integrate that is the probability of having particles with velocity v z velocity component v z between minus infinity to infinity if we integrate this must give 1.

Because if we have a particle it, if it is moving in this direction its v z is positive, if it is moving in this direction its v z is negative, if it is moving in this direction its v z is 0.

And therefore, the range over which if you want to find out the probability from minus infinity to infinity then it will be equal to 1. That is somewhere you will find d v z equal to 1; that means, minus infinity to infinity we substitute this here we have this expression equal to 1.

Now, we do a change of variable a simple mathematics v z is this because it is given by this is given by the expression here, so we can simply transpose this. So, v z will be equal to nu minus nu 0 divided by nu 0 into c. So, d v z is equal to c by nu 0 into d nu d v. So, d v z is equal to c by nu 0 into d nu and when v z tends to infinity, v z tending to infinity velocity tending to infinity means what the highest velocity possible is c and therefore, we write v z goes to c.

And v z going to minus infinity means v z goes to minus c; when v z goes to c we have the nu going to, so when v z goes to v then nu goes to 2 nu 0. We can see in this expression here when v z becomes c so if this has to become c then this must be equal to 2 nu 0 so that 2 nu 0 minus nu 0 is 1 u 0, 1 u 0 nu 0 cancels so we have c. So, nu goes to 2 nu 0 and when v z goes to minus infinity; that means, when v z goes to minus c we have nu going to this is nu going to 0.

And therefore, the nu integration limit after this is here 0 to 2 nu 0 m by 2 pi K B T to the power half into all of this now written; so, this is the v z square which is replaced by here; so, v z is here so v z we have replaced v z square is replaced by this and this is the d v z which is here and this is equal to 1.

Now, this frequency nu, what is what are we looking at? We are looking at the range of interaction so this is nu equal to nu 0. We first we know that the atomic system would interact over a range of frequencies. What is this range of frequencies? We know from practice that this delta nu is of the order of typically 10 power 11 Hertz or 10 power 10 to 10 power 12 Hertz. The nu 0 is of the order of for visible light for example, this is 10 power 15 Hertz.

So, if you say 2 nu 0, if I want to show on the same axis the 2 nu 0 will be; so let me show here. So, nu 0 is here 2 nu 0 is here 2 nu 0 and we are looking at this resonance, that is delta nu is here. The spread around nu 0 we are looking at the spread around nu 0 this width is much smaller compared to this separation. The point I am making is the integral the integrand which is here will be 0 beyond a certain frequency range and therefore, instead of writing from 0 to 2 nu 0 we can as well write it as from 0 to infinity.

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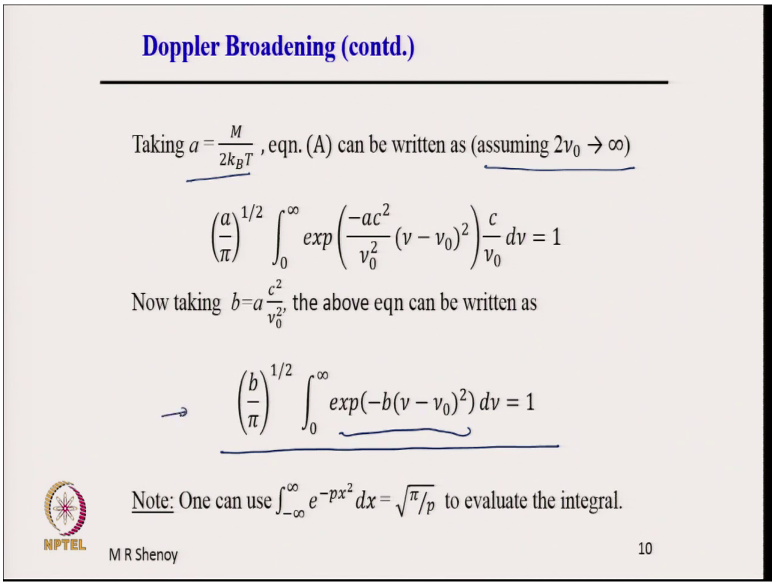

And therefore, we write this as 0 to infinity. So, assuming 2 nu 0 going to infinity we write this as 0 to infinity exponential a into; where a is this. So, a is if you look at the expression here so this is a. So, e to the power minus a into this so that is what we have written, e to the power minus a into this into this expression here. And if we take a into c square by nu 0 square as b then this equation can be written in this form.

This is of course it is an integrable equation using this expression you can write this as. So, 2 times this can be written as this or minus infinity to infinity can be written as 2 times 0 to infinity because this is an even function this is an even function and therefore, using this integration you can integrate this, but we are not interested in integrating right now; so let us see what is the logic that we want to give.

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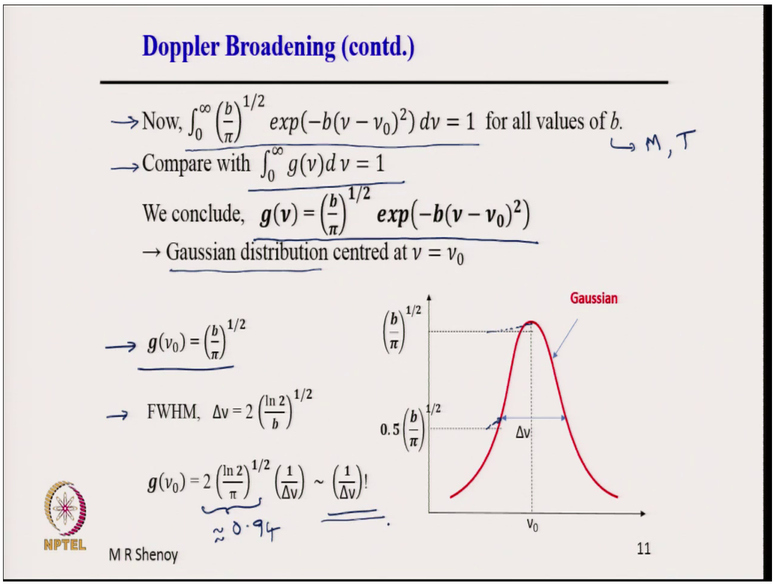

We have expression here 0 to infinity square root of b by pi exponential minus b into this is equal to 1 for all values of b remember that b contains atomic mass m, b contains the temperature t. If this expression; and we also know that 0 to infinity g nu d nu is equal to 1.

If this expression has to hold good for all values of b then the integrand must be equal. In other words therefore, g nu must be equal to this please see this is the normalized line shape function, definition of the normalized line shape function this is the interaction the integral which is specifying the interaction which we have got through the velocity distribution.

And if this integral has to hold good for all values of b then it must be equal to g nu, because g nu d nu integral 0 to infinity is equal to 1. And therefore we conclude that g nu is equal to the integrand, which is clearly it is a Gaussian distribution centered at nu is equal to nu 0, the value is maximum when nu equal to nu 0 this is exponential 0 which is 1 and the maximum value g of nu 0 is given by square root of b by pi that is this value, oh! this must have got shifted.

So, this peak value is here so this is the maximum value and the full width that half maximum is found at half the maximum 0.5 into square root of b by pi and you can determine that FWHM is given by delta nu is equal to 2 into square root of l n 2 by b. And therefore, g of nu 0 if you again substitute for b in terms of delta nu then you get g nu 0 is equal to 2 times square root of l n 2 by pi into 1 by delta nu. If you see this term these are all constants this will come out to be 0.94 approximately 0.94; that means, this almost 1 and therefore, g of nu 0 is equal to 1 by delta nu.

Just as before in the last Lorentzian also we have seen that the peak value g of nu 0 is of the order of 1 by delta nu where delta nu is the full width at half maximum or the line width of the resonance; exactly like that here also we see that the line width of the resonance is inversely proportional or 1 by line width gives you the peak value; that means, narrower the resonance smaller the extent of line broadening larger will be the value of g of nu 0.

This is very important because if you remember that the gain coefficient is directly proportional to g of nu, larger the value of g of nu will means larger will be the value of gain coefficient for amplification by stimulated emission.

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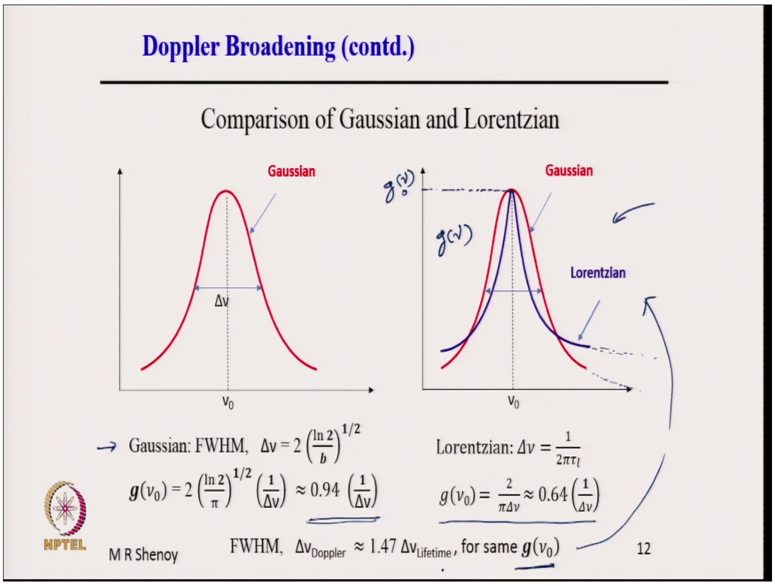

Alright, let us have a just a comparison of these Gaussian and Lorentzian the distributions. So, this looks like a typical bell shaped distribution Gaussian and what we have shown in this graph is the Lorentzian, keeping g of nu 0 constant both of them if we make g of nu 0 constant then the graphs would look like, the variations would look like this.

The Lorentzian has a higher value of the pedestal this goes to 0 at infinity of course, asymptotically, but higher value. Gaussian drops down rapidly and goes down to 0, but Lorentzian has a narrower line width. So, we can see here what we have already derived Gaussian FWHM is given by this therefore, g of nu 0 is approximately 0.94 into this whereas, for the Lorentzian g of nu 0 is 2 by pi into delta nu which is approximately this.

Therefore, for the same value of g of nu 0 that is what we have shown in the figure we have kept this is g of nu 0. So, this is g of nu and this value is u of nu what is plotted, this is the distribution is g of nu. And the peak values for the same value of g of nu 0 the Doppler that is the Gaussian has a wider is more than the delta nu due to lifetime 1.47, this is just out of curiosity.

So, to conclude this part we have seen that population inversion is the necessary condition for amplification by stimulated emission and we have obtained an expression for the gain coefficient gamma of nu and we have seen that the gain coefficient gamma is directly proportional to the line shape function g of nu line shape function describes the strength of interaction; interaction here refers to emission and absorption.

So, strength of emission is described by the line shape function and that is why we have gone into a little bit more detail to understand, what determines the line shape function. There are various mechanisms which determine the line shape function which are called line broadening mechanisms.

Now, that we know the line shape function we know the line shape function means what we know the numerical values of g of nu at any value of nu and at the line center nu 0. Once you know the numerical value of g of nu you know the numerical value actual number for the gain coefficient gamma of nu and if you know the gain coefficient then we know actual amplification factor, how much amplification would take place when radiation passes through a medium.

Therefore in the next part we will see what are the schemes for achieving population inversion. We know population inversion is the necessary condition, but how to achieve population inversion? This we will discuss in part 2 that is scheme of amplification.

Thank you.