Introduction to LASER
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Lecture - 31
Mode Locking

Welcome to this MOOC on LASERS. So, today we will take up the next topic that is Mode Locking. So, mode locking is one of the important methods to obtain pulsed output from a laser, regular periodic pulses from a laser.

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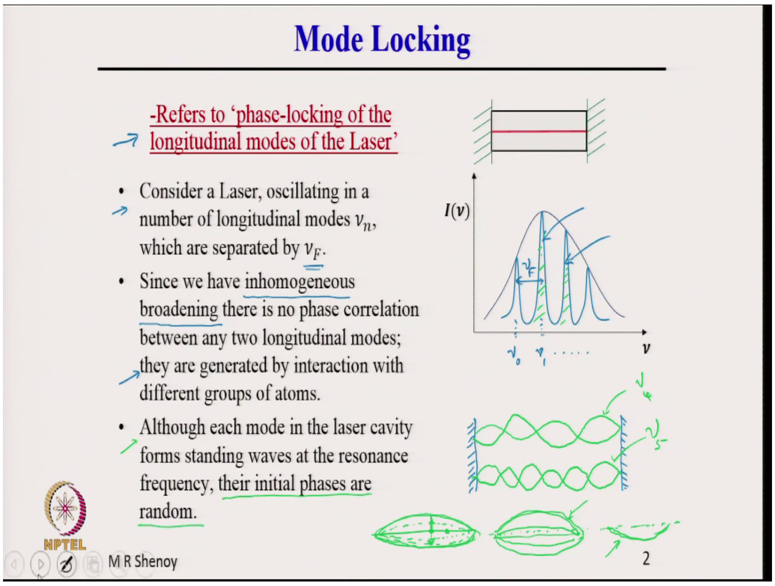

So, mode locking refers to phase locking of the longitudinal mode. So, it refers to phase locking of the longitudinal modes of the laser. Consider a laser oscillating in a number of longitudinal modes, generally large number with frequency nu n. So, n is equal to 0, 1, 2, 3, etcetera which are separated by the free spectral range nu F. So, longitudinal modes and this separation we know is nu F. So, this is nu F, and let us say this is nu 0, this is nu 1 and so on, large number of modes.

You get large number of longitudinal modes in a in homogeneously broadened line shape function. When the laser medium is characterized by an in homogeneously broadened line shape function then such lasers support large number of longitudinal modes. Since we have in homogeneous broadening there is no phase correlation between any two longitudinal modes. This is because these longitudinal modes, the different longitudinal modes are generated by interaction with different groups of atoms.

So, there is no relation between the laser line or longitudinal mode generated here and the longitudinal mode generated here. Of course, so, for example, in the resonator each one of them will have standing waves. So, each one of them each of the longitudinal modes make a standing wave, however, there is no phase relation between them.

So, what do I mean by each one of them make a standing wave? So, we have already seen this. So, every mode forms a standing wave. So, I am showing only 4 of such and the next mode may have more number and qualitatively showing that these are the longitudinal modes. So, this is let us say nu 1 or nu 4 and this is nu 5, let us say the next one nu 5.

Each one of them forms standing wave, but there is no phase relation between these two. That is the initial phase of these modes are different and they are random and there is no relation because this mode is generated by interaction with different groups of atoms and this mode is generated by interaction with different groups of atoms.

So, this can be imagine what do I mean by no phase correlation please try to look at this that in a standing wave let me show one standing wave think of a string standing waves on a string. So, how do you see this loop? There was just one string. So, what happens is the displacement of the string changes with the time. So, if you take a time snap then you will see that the displacement changes with the time.

So, the string goes up and then comes down, up and down that is every point every point on the string here makes a harmonic oscillation like this. And the net result is at one stage the string is like this at one instant and another instant the string is like this and another instant the string is straight another instant like this.

Now, when I say that there is no phase relation what I mean is when one of the string or one of the mode here is in this position in this situation then for the other mode it may be in this, the other mode think of the displacement of the string. So, one is going down another is going up at that instant at the same instant which means, the two are not in phase this one and this one is not in phase.

This is what we mean by saying that there is no phase relation between these longitudinal modes in a normal multi longitudinal mode laser. Unless you force them unless you phase lock pull them all in the same phase which means, when one mode starts building up amplitude in the positive direction the other mode should also be building up like that.

So, that is in simple terms what is this phase correlation that we are talking about, alright. Although each mode in the laser cavity forms standing waves at the resonance frequency their initial phases are random.

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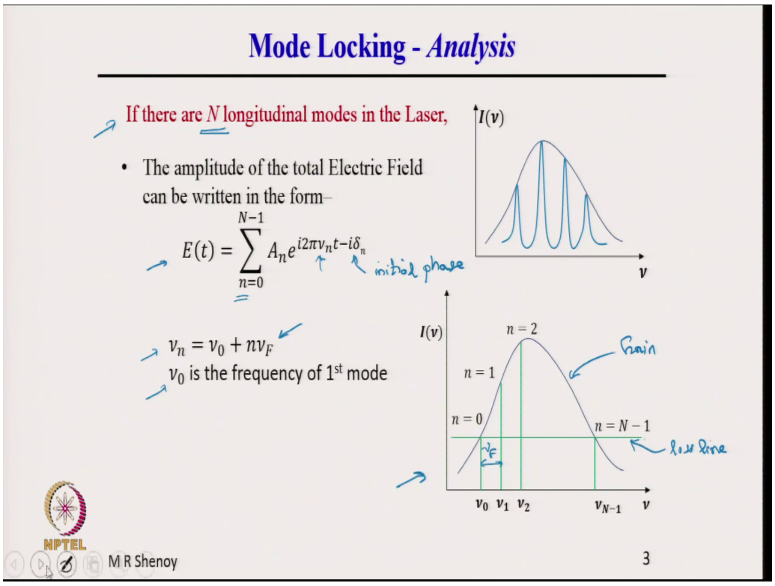

So, let us look at the mathematics it will become very clear. So, let us look at the analysis mode locking. Consider a laser which has N oscillating longitudinal modes in the laser. The total electric field inside the laser can be written in the form E of t is equal to some of electric fields of all the modes.

Each mode has an amplitude A n and a phase term which is e to the power of i 2 pi nu n into t, nu n is the frequency the frequency of the mode. So, nu is the frequency and delta is this initial phase. So, this is the initial phase, so, initial phase. And this delta is different the delta n is different for different modes, so, initial phase.

And we can write the total electric field as the sum of electric fields of all the individual longitudinal modes. Note that nu n that is the frequency of the nth mode is nu 0 plus n F. So, this is illustrated here in this diagram. So, what is shown here? This is the loss line. So, this is the loss line and this is the gain curve ok.

And therefore, all resonance frequencies for which the gain is greater than or equal to loss can oscillate. And in this laser there are N modes capital N, N modes. Therefore, we start with nu 0, nu 1, nu 2 and so on and the last one is nu N minus 1. So, n is equal to N minus 1, n is equal to 0, n equal to 1.

So, it is the same n that we will be using here in the summation. So, n refers to the index of these modes. So, nu 0 is the frequency of the 1st mode. Remember that this separation is nu F, the free spectral gain. Therefore, nu 1 will be nu 0 plus nu F, nu 2 will be nu 1 plus nu F and so on. So, that is what is shown here that nu n is equal to nu 0 plus n times nu F alright.

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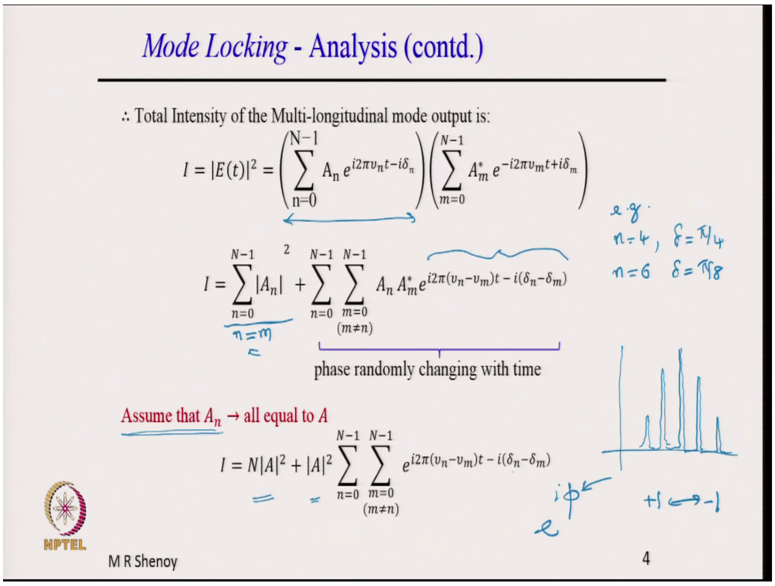

So, let us simplify this. Therefore, the total intensity of these multi longitudinal mode output is I is equal to mod E t square. So, E t is this that is the electric field is this one. So, we have to take mod square, which means you have to have the complex conjugate. So, A n A m star into e to the power of minus i 2 pi nu m t plus i times delta m. So, all i's are now converted to minus sign.

So, m is equal to 0 to N minus 1 n is equal to 0 to N minus 1 and we can simplify this. So, whenever there is n is equal to m we will get mod A n square and the frequency e to the power of i 2 pi nu n t here minus i 2 pi nu n t when n is equal to m. So, all n is equal to m this is product of two series product of two summations and therefore, every corresponding term here will give you mod A n square. So, one term when n is equal to m.

So, whenever n is equal to m, we get mod A n square. So, this term represents mod A n square and therefore, there are N minus 1 n goes from 0 to N minus 1 summation mod A n square. Now, for n not equal to m the summation will comprise of the product of these two. So, the product can be written in this form n is equal to 0 to N minus 1 and m is equal to 0 to N minus 1.

And with the condition that m is not equal to n because m is equal to n has been taken up in the first term. So, the remaining terms are here into A n A m star into e to the power of i 2 pi into nu n minus nu m into t minus i times delta n minus delta m. So, this term the last term is randomly changing this is a randomly changing term; because the phase is randomly changing with the time because delta n and delta m have no correlation

For example, for n is equal to 4 delta could be pi by 4 and for n is equal to 6 delta may be pi by 8. So, this term will go on changing randomly because individually delta n and delta m do not have any correlation. So, just for example, so, delta n could be something, delta m could be something and n and m are different and therefore, this term will represent a randomly changing there is the phase term here will be randomly changing with the summation.

So, as we change the values of n and m to add, so, this term will randomly change. And therefore, so, we can write for example, just for simplicity, if we assume we can understand it better if we assume that all modes have the same amplitude. This is not true actually because we know that the modes have different amplitudes because the modes in the side have smaller intensity and therefore, smaller amplitude like this.

But, just for simplicity to understand it easily let us assume that all the amplitudes A n are equal to A then the first term will give us n times mod A square and the second term because all the amplitudes are equal therefore, this product A n into A m comes out and that is also mod A square into the summation here.

This summation is e to the power some quantity here. So, e to the power some i phi and this phi is randomly changing because delta n and delta m are different. And therefore, this term, what is the maximum value this can give? This can give between plus 1 and minus 1 plus 1 to minus 1. When e to the power of i 2 pi then it is 1 and whenever e to the power minus i pi then it is minus 1 and therefore, this term varies between plus 1 and minus 1.

And therefore, if we look at this sum this sum will vary around n times mod A square. Let me make this clear, let us look at the next line.

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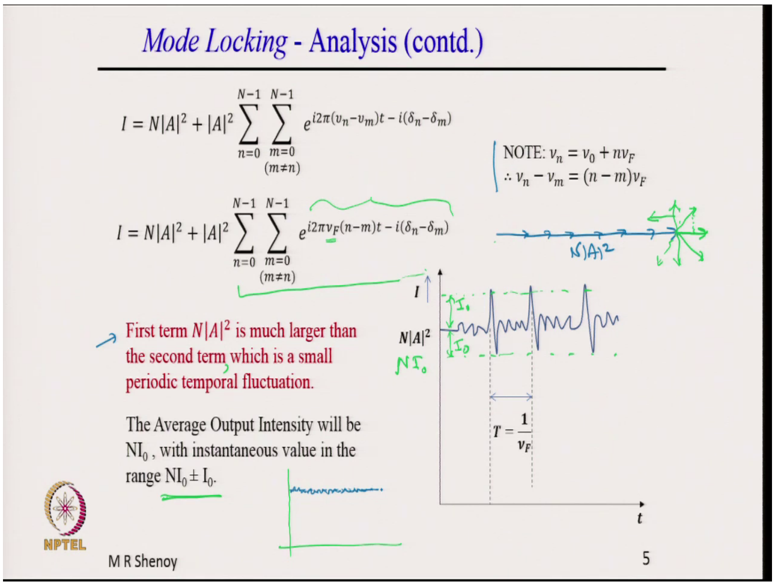

So, here so, it is again shown here. Now, we call that nu n is nu 0 plus n times nu F and therefore, this nu n minus nu F is n minus m into nu F. So, that is why we have written e to the power of i 2 pi nu F into n minus m times t, n and m are integers and i times delta n minus delta m.

So, e to the power this phase is randomly changing with time and with mode number. And therefore, this term here, so, the first term n into A square is much larger because this is only one A square. So, for example, I can show this as phaser. So, let us say this is mod A square. So, mod A square, mod A square, mod A square, mod A square and so on.

So, it is not a phaser the first quantity is not a phaser, but I am just showing that this is n times mod A square. Now, the last A square, this A square depending on this value depending on the n and m value this A square if total phase comes out to be e to the power of i 2 pi it will be 1, ok. Let me change the color here.

And the last A square mod A square when the net phase here is an integral multiple of 2 pi then I will get mod A square adding on to this. If it becomes an odd integral multiple of pi then I will have this subtracting like this, so, in the reverse direction. For any other value this will have a magnitude in between.

That is if I consider this as a phaser because the phase is changing then the magnitude is changing and therefore, the total amplitude will be N times mod A square plus minus mod A square maximum plus minus mod A square. So, the total intensity here will be N times mod A square which is N times I 0, where I 0 is the intensity of one mode intensity value.

Therefore, this is N times I 0 N times mod A square and this can vary plus minus 1. So, this height is also again I 0. This is I 0, this is I 0. So, essentially I can have N times I 0 plus minus I 0 that is the intensity variation. Now, if N is very large note that this quantity will represent the first term N A square is much larger than the second term and the second term is a small periodic temporal fluctuation.

The second term here is a small periodic it is periodic because you see it is e to the power 2 pi nu F into an integer. So, this is an integer. So, this phase is periodically varying. So, the it is a periodic with the t period is equal to 1 by nu F because nu F is here. So, the period is in time. So, the period is 1 by nu F. So, it is a periodic oscillation. I am just showing that in some places it may be maximum and some places it may be minimum, but it is repeating like this.

So, strictly speaking if you have very large value of N, you could essentially represent it like. So, this is the average level and very little fluctuation very the little variations, but that variations are periodic. This is when the phase of individual longitudinal mode are random and not locked they are not oscillating in the same phase. Now, let us see what would happen if we phase lock all the longitudinal modes. How to do it? We will see, alright.

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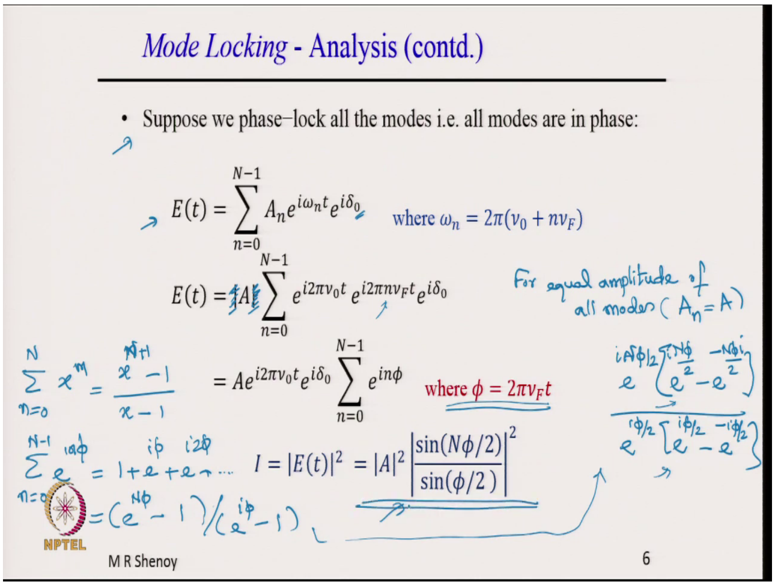

So, suppose now we phase lock all the modes, that is all modes are in phase. Now, the electric field E of t as before given by the same expression, but now it is not delta n there is one delta 0, all of them are in phase. Therefore, there is a constant phase which is here and omega n of course, is 2 pi nu 0 into n times. So, 2 pi has been included in that.

So, we have written it again here E of t is equal to mod A into; why did we write mod A? Mod is not required. So, A comes out as before if we assume so, for equal amplitude of all modes. As before if I assume that all modes have the same amplitude for equal amplitude of all modes that is A n is equal to A for all modes.

So, then we have E of t is equal to A into n equal to 0 to N minus 1 e to the power i 2 pi nu 0 into e to the power 2 pi n nu F. So, we have substituted for omega this one into e to the power of i delta 0. So, this term is constant, the first term is constant. Therefore, the running index n is only in this term second term and therefore, this comes out here and we have n equal to 0 to N minus 1 into e to the power of i n phi, where phi is equal to 2 pi nu F t.

So, these 2 pi nu F t into n; phi is equal to 2 pi nu F t. Now, this can be simplified. The intensity is given by mod E t square. So, now, we have mod A square, all of these phase terms disappear. So, mod A square into mod square of this term. You can show that this is equal to sin N phi by 2 divided by sin phi by 2 mod square.

So, very simple, please try this. So, this is use the formula summation x to the power m, n is equal to 0 to N. You have n is equal to 0 toN, then this will be equal to x to the power m plus 1; so, N plus 1 capital N plus 1 minus 1 divided by x minus 1. So, use this to simplify this.

So, summation e to the power i n phi is equal to for n is equal to 0 to N minus 1, we can write this first one is 0 e to the power of i phi plus e to the power of 2 i phi plus etcetera and this will be equal to so, N minus 1 plus 1 will be n. So, this will be e to the power N phi minus 1 divided by e to the power i phi minus 1.

So, that is x, x in this expression is e to the power of i phi. So, that is what we have and we can simplify this further. So, this can be ok, I should have shown the derivation ok. Let me show it here. So, this is equal to e the power of minus I take N phi by 2 outside N phi by 2 into e to the power N phi by 2 minus e to the power of minus N phi by 2 minus N phi by 2 divided by e to the power i phi by 2 into e to the power i phi by 2 minus e to the power minus i phi by 2. And what do we have?

So, this is sin N phi by 2, the first term here is sin N phi by 2 divided by sin phi by 2. 2 i 2 i cancels. So, we have sin N phi by 2 divided by sin phi by 2 and mod square. So, what I had shown is simply E of t, the summation. Only the summation here is this and when you take mod square both of these.

So, this is e to the power i N phi by 2 i N phi by 2 I have said i, but I have not written i. So, i N phi by 2 i phi by 2 divided by this. So, when we take mod square these disappear, but this is sin N phi by 2 divided by sin phi by 2. So, we have got the answer here ok.

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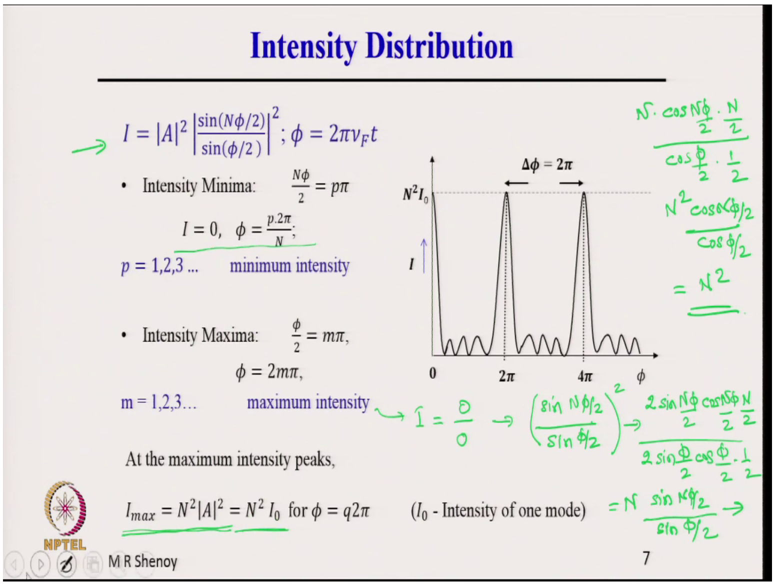

Let us discuss the intensity distribution. So, we have intensity due to N phase locked modes is given by mod A square into sin N phi by 2 divided by sin phi by 2 whole square, where phi is equal to 2 pi nu F t. So, we are here. The intensity minima are given by whenever the numerator is 0, we have an intensity minima. And when will be the numerator 0?

Numerator is 0, when N phi by 2 is an integral multiple of pi and therefore, I the intensity will be equal to 0 for phi is equal to p by N into 2 pi. So, we have simply written phi is equal to 2 we have taken here and N in the denominator. So, p by N into 2 pi. What is p?

p is an integer and N is the number of oscillating longitudinal modes. So, N is also an integer, p is also an integer. p is equal to 1, 2, 3 for minimum intensity. The intensity maxima are given by phi by 2 is equal to m pi that is m pi phi by 2 then sin phi by 2 is 0 and if phi by 2 is m pi N times phi by 2 is also an integral multiple of pi which means it is 0.

So, when the intensity maxima occurs at phi by 2 is equal to 2 pi, we have the expression becomes. So, at this value we have I is equal to 0 by 0 situation. So, its 0 by 0 that is sin N phi by 2 and sin phi by 2. So, both are when this situation comes we have I is equal to 0 by 0 situation. So, how to solve this?

So, we have to then apply the L'Hospital rule and we have to go from here to. So, let me write this; sin N phi by 2 divided by sin phi by 2 whole square, so, mod square. So, let me write it as whole square or mod square does not matter. So, we have to take derivative of the numerator and derivative of the denominator. So, the numerator will be 2 times sin N phi by 2 N phi by 2 into cos N phi by 2.

That is sin square theta is 2 sin theta cos theta d theta. So, N phi by 2 into d theta that is N by 2 this is the numerator divided by the denominator is 2 times sin phi by 2 cos phi by 2 into half. So, 2 sin theta cos theta d theta. So, we have simplified this. So, the 2, 2 cancels and what we have is equal to N times I could leave this to you, but N times 2 sin theta cos theta. So, that is 2 sin theta cos theta is sin 2 theta.

So, sin N phi by 2 N phi by 2 divided by sin phi by 2, 2 sin theta cos theta. So, again sin phi by 2. So, this is after the first derivative. Even now if you put N phi by 2 is phi by 2 is equal to m pi still we are in the 0 by 0 situation. So, this is also 0 by 0 situation. So, we apply again the L'Hospitals rule.

Let us see what happens now. Let me do it here above. So, if I apply again N is already there. So, cos N phi by 2 we have cos N phi by 2 into N by 2 cos theta d theta divided by in the denominator cos phi by 2 phi by 2 into half. So, what we have is n square into cos N phi by 2 N phi by 2 divided by cos phi by 2.

Now, we put phi by 2 is equal to m pi then we have cos 0 by cos 0 situation. So, this will be equal to N square. So, I have shown that intensity maxima correspond to phi is equal to 2 m pi and the maximum intensity I max is n square times mod A square, which is nothing but N square times I 0, for phi by 2 is equal to m pi. What does this mean?

This means that the maxima are N square times. Recall when we did not phase lock, we had N times I 0 plus a small fluctuating term. Now, we have N square times I 0 plus minimas going down to 0. So, look back at the intensity variation here. We had N times I 0 plus a fluctuation. This is when the laser the same laser when it was not mode locked.

Now, when we mode lock the laser, we have an intensity variation which is N square I 0 and we have several minimas here because depending on the N, so, whenever phi is equal to 2 p pi by N then the intensity goes down to 0. So, there is a sinusoidal variation here in the intensity, which is shown here. It is a sinusoidal in terms of period, but the amplitudes will change.

Why are there so many 2s? By 2 phi by 2 phi by 2, 2 sin, sin phi by 2; oh this is 2 sin theta cos theta is equal to sin 2 theta. Therefore, there is no 2 here. So, sin N phi divided by sin phi. 2 sin theta cos theta is sin 2 theta, therefore, sin phi here. And therefore, there is no 2 here, there is no 2, no 2, although the final result does not get affected it is the same N square. So, this is ok.

So, you will have peaks which are separated by secondary minimas and maximas. There are primary peaks, primary maximas and secondary maximas which are here as shown in the figure. Now to appreciate this let us take an example for n is equal to 5. Let us take N is equal to 5 and try to understand this.

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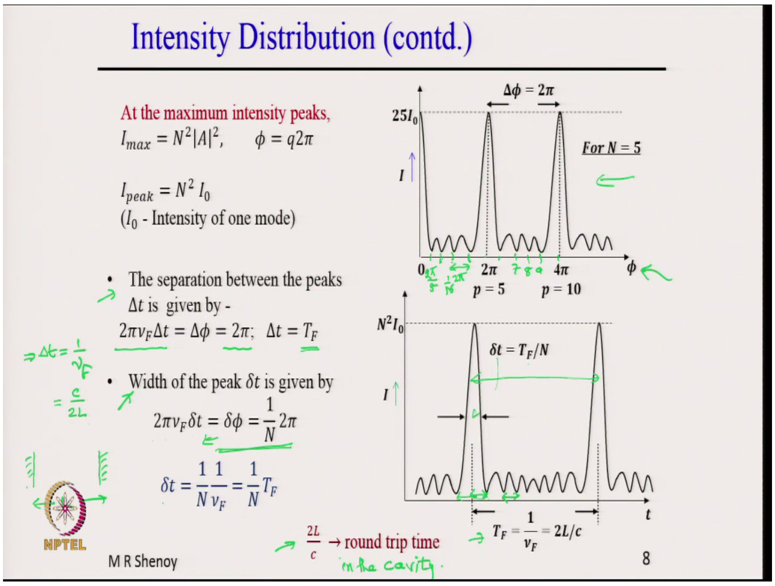

So, for N is equal to 5, so, what I have shown this diagram here is for N is equal to 5. When N is equal to 5, at phi is equal to 0 we have N square times intensity and we have minimas here, minimas going down to 0. So, where are the minimas? The first point is for pi divided by 5. So, this is pi divided by 5 2 pi by 5. This minima is for p is equal to 1, p is equal to 2, p is equal to 3, p is equal to 4 and when p is equal to 5 what we have is.

So, we can see here. When we put p is equal to 5, N is equal to 5 then 5, 5 cancels and we have phi is equal to 2 pi. So, that is what we have shown there. At 2 pi we have the intensity maxima. As we go further this is for p is equal to 6, p is equal to 7, p is equal to 8, p is equal to 9 and this again when p is equal to 10, we have p by N that is 10 by 5 as 2. So, 2 times 2 pi which is 4 pi.

So, this clearly tells that the minimas are here the secondary minimas, but the peaks are N square times I 0 which is the intensity of the pulses. Now, let us see. So, this axis please see is the phase. We have plotted with respect to phi, but in practice we would like to plot it in terms of time because we want to see them as pulses in time how the signal varies.

And then therefore, if the separation between the peaks in phi they correspond to 2 pi, we can see here 2 pi, 4 pi. The separation between the peaks delta t is given by delta phi is equal to 2 pi nu F into delta t because phi is 2 pi nu F into t. Therefore, delta phi is 2 pi nu F into delta t, which is equal to 2 pi, which means therefore, the separation in time this delta t is called T F.

Now, what is that separation? So, this T F is nothing but 1 by nu F. So, if you put 2 pi nu F into delta t is equal to 2 pi then delta t; please see this. 2 pi nu F into delta t equal to 2 pi, this implies delta t is equal to 1 by nu F. What is nu F? nu f is 2 L by c and therefore, this is c divided by 2 L, where L is the length of the cavity.

So, that is what is written here that the pulses or the peaks are separated in time by T F which is equal to 1 divided by nu F is equal to 2 L by c. But what is 2 L by c? 2 L by c is the round trip time. Please see we have a laser cavity of separation L. So, this separation is L and therefore, 2 L is the round trip length is 2 L divided by the velocity of light c is the time taken for one round trip.

So, 2 L by c is the round trip time and therefore, T F the separation in time between the peaks is 1 by nu F is equal to 2 L by c is equal to the round trip time in the cavity in the laser cavity. What about the width of the pulse, so, width of this peak? So, we see that this was corresponding to p is equal to 5 and this was corresponding to p is equal to 6 and therefore, this width we can say that the pulse width delta t here is approximately equal to this separation here or the separation in time between these.

Because actually the bottom of the pulse here that is the here is 2 times the separation, but when we talk of the pulse width normally we talk of full width at half maximum and therefore, we can say that this difference corresponds to this difference here. That is so, width of the peak delta t is given by 2 pi nu F into delta t is equal to delta phi, delta phi corresponding to 2 minimas here delta phi and that delta phi is 1 by N times 2 pi.

Please see that this was 1 by phi times to pi this was 2 by phi times 2 this was 3 by phi times 2 pi 4 by phi times and 5 by phi, which means the separation here is 1 by N times 2 pi everywhere the separation in delta phi. So, delta phi is 1 by N times 2 pi. But what is delta phi?

Delta phi is here. 2 pi nu F into delta t and therefore, delta t the pulse width in time the separation corresponding to 2 minima or in time the separation corresponding this separation is given by 1 by N times 1 by nu F and that is equal to 1 by N times T F. What is T F? T F is the separation between the pulses. So, this is T F, the separation between the pulses. And what is delta t?

Delta t is T F divided by N that is this one this is delta t, T F divided by N. oh This is just inverse; 1 by nu F ok. So, what does this mean? The output from a mode locked laser comprises of intensity peaks which are N square times the intensity of each mode which are separated in time equal to the cavity round trip time and the width of the pulses are given by this separation divided by N. So, larger the N, smaller will be the width of the intensity peaks ok. So, let us discuss this a little bit more.

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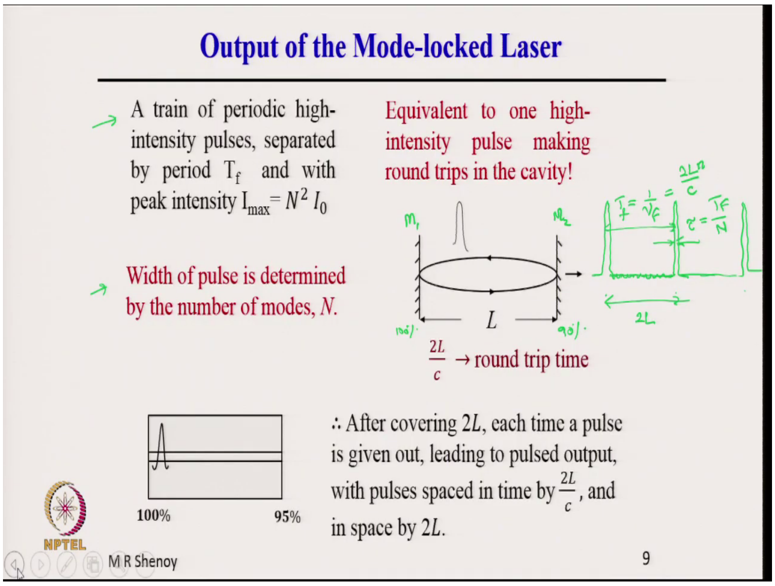

So, the output of the mode lock laser it comprises of a train of periodic high intensity pulses separated by period T f and with peak intensity I max equal to N square I naught. We can see here for example, the output comprises of pulses. Let us say this is 100 percent reflecting and this is 90 percent or 95 percent reflecting.

Then from this end we get periodic pulses like this, the pulse separation, the separation between them. Please see when N is very large when I had taken N is equal to 5. So, it is 25 times I 0. If N is 100, this will be 10000 times and therefore, the minimas here will be very very very small compared to the peaks and that is why literally there is almost flat. Actually, it is not flat, there is there is a fine minimas a periodic variation here.

But the peaks are so high and this is separated by T f, which is equal to 1 divided by nu F equal to 2 L by c. If it is in a medium of refractive index N then we should also add 2 L into

N. So, as shown here again 2 L by c is the round trip time and we are getting pulses separated by this time and here this separation the full width at half maximum. So, this delta t or tau so, if I call this as tau is equal to T f divided by N.

So, larger the number of modes narrower will be pulse width. So, if we see this as the output then it says equivalent to; so, this output picture tells us that this must be happening because inside we have a partially reflecting mirror here m 2. So, m 2 is a partially reflecting mirror, m 1 is a 100 percent reflecting mirror. Therefore, it is equivalent to one high intensity pulse making round trips inside the cavity.

Whenever the pulse strikes this second mirror m 2, a part of it goes out; 10 percent of the energy goes out in the form of a pulse. Again after one round tip, 10 percent of the energy goes out. So, if we look at the outside we see periodic pulses coming out with the time separation equal to round trip time of the cavity and the width of the pulses is determined by the number of modes.

Therefore, after covering 2 L that is round trip each time a pulse is given out leading to pulsed output with the pulses spaced in time by 2 L by c and in space by 2 L. In space this difference is 2 L, so this is 2 L because simply multiplied by the velocity of light that will give us in the spatial distance separated by 2 L.

So, we have a pulsed output from a mode locked laser and the output pulse separation is fixed unlike in the case of q switched laser where we could determine the rate at which we want the output pulse. Although we did not have control on the pulse width, but the rate at which the pulses come out is determined by us because it depends on the time when we switch the loss high and low.

So, so long as the loss remains high there will be no output. When the loss is suddenly switch to low we get output. And therefore, the pulse repetition rate is controlled by the external modulator used to q switch the laser. But, in this case the output the rate at which pulses come is determined by the cavity length and the round trip time taken by the pulse in the cavity ok.

Let me stop at this point and we will discuss the methods of mode locking in the next class.

Thank you.