Introduction to LASER
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Lecture - 18
Ray Paths in Spherical Mirror Resonators

Welcome to this MOOC on LASERs. In the last couple of lectures, we discussed about spherical mirror resonators and we derived the resonator stability condition. We also looked at the matrix optics approach and the ray transfer matrix for the resonator. So, in today's class, we will use the ray transfer matrix method to actually trace the rays through a spherical mirror resonator.

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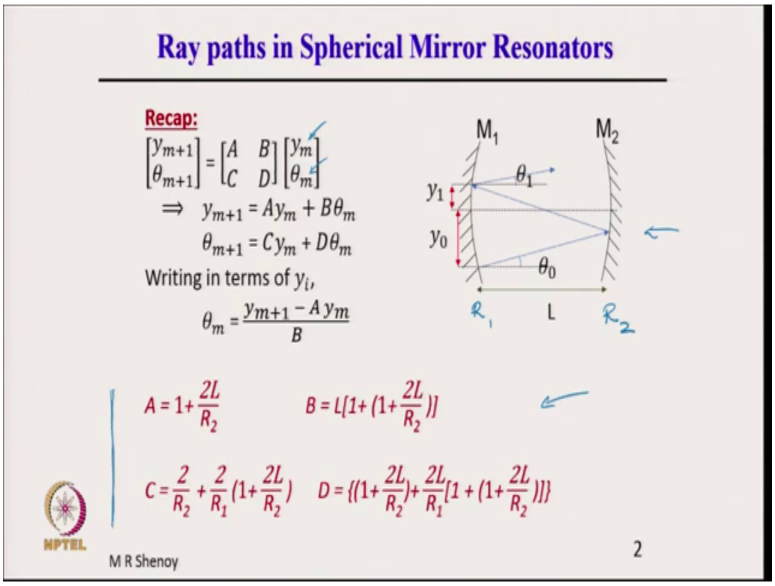

Ray paths in spherical mirror resonator. A very quick recap that we had seen for a given spherical mirror resonator with mirrors M 1 and M 2 with radius of curvature R 1 and R 2 separated by a distance L. We can find a ray transfer matrix A, B, C, D matrix. The elements of which are given here, A, B, C, D. The elements as we note are in terms of the radius of curvature and the separation between the mirrors.

Using the ray transfer matrix if y m and theta m are the coordinates of a ray after m round trips, then the next round trip; the coordinates corresponding to the m plus 1th round trip is given by this. And therefore, we can write y m plus 1 is equal to Ay m plus B theta m and theta m plus 1 equal to Cy m plus D theta m. Note that we know A, B, C, D.

Therefore, theta m can be written in terms of the displacement y m plus 1 and y m. So, the matrix elements A, B, C, D are given here. So, we would need this when we want to trace rays in a given optical resonator.

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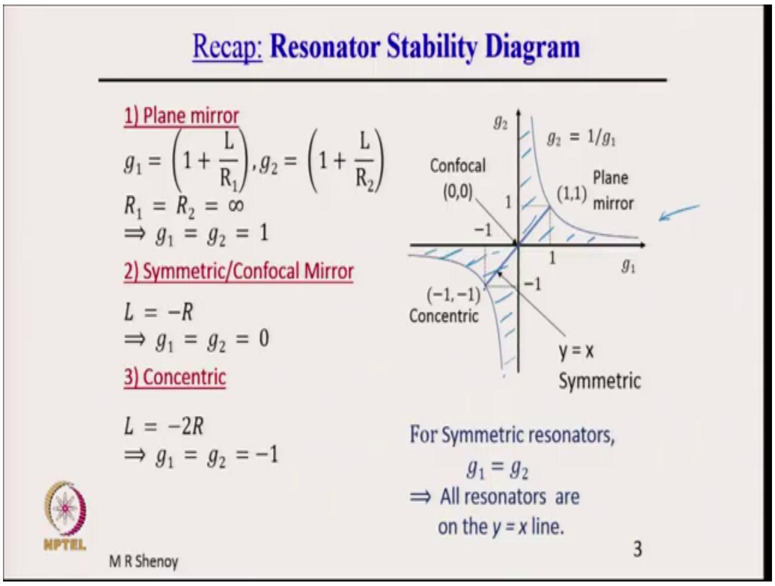

Second point, we also saw the resonator stability diagram which is shown here which is a plot between g 1 and g 2 and we have already discussed that the shaded area between the axis and the y is equal to 1 by x curve represents the stability, stable region. And therefore, for a given plane mirror for example, we have taken these three common examples.

For a plane mirror g 1 is equal to g 2 is equal to 1 and the point is here right on the border, it is on the y is equal to 1 by x curve corresponding to symmetric confocal mirror resonator we have the origin. And for the concentric meter we have traced here. So, today subsequently we will trace these points, we will locate these points for some general arbitrary optical resonators and see whether they are in the stable region or unstable region.

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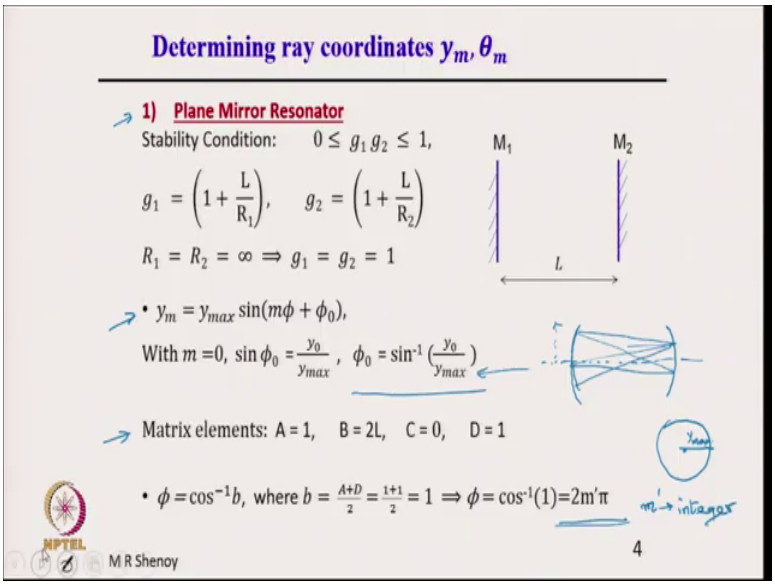

Now, to determine the ray coordinates y m theta m. Let us consider a plane mirror; plane mirror resonator. Therefore, first given any resonator, we must verify whether it meets the stability condition or not before even beginning to start ray tracing. So, g 1 is equal to g 2 is equal to given by these formulae and for plane mirror resonators R 1 is equal to R 2 is equal to infinity. And therefore, g 1 is equal to g 2 is equal to 1.

And therefore, it satisfies the stability condition that g 1 g 2 is equal to 1. Now, the displacement the solution for y m is given by y max into sin m phi plus phi naught; phi naught is given here. So, this is determined by the maximum displacement permitted, usually y max is determined by the extent of the mirrors.

For example if you have two spherical mirrors or plane mirrors, then the rays can be; so, the rays can lie anywhere at best within the mirror and therefore, y max represents the maximum displacement or essentially radius of the mirror. So, if this is the mirror spherical mirror or if I show the front view then the spherical mirror is like this, then the radius of the mirror is the maximum displacement that you can have.

Therefore, this is y max this can be assumed as y max and y naught is so, y 0 is the starting point and therefore, phi 0 is equal to sin inverse y 0 by y max. Now, with m is equal to 0 sin phi 0 is equal to y 0 by y max, now for the plane mirror resonator the matrix elements A, B, C and D. So, we can see here the matrix elements, R is equal to infinity. Therefore, A is equal to 1, B again R here is infinity.

Therefore, we have this term 0, but we have 1 plus 1. So, B is equal to 2 L C. So, here R is infinity R is infinity. So, C is 0 and D. So, this term is 0 because R 1 is infinity, second term is also 0 here therefore, D is equal to 1. So, we have C equal to 0, D equal to 1, B is equal to 2, L and A is equal to 1. So, let us see what we have got. So, here A is equal to 1, B is equal to 2 L, C is equal to 0 and D is equal to 1.

So, why do we need this? Because we need to find out phi in this expression and phi is given by cos inverse of B where B is equal to A plus D by 2, A and D are here. And for the plane mirror resonator, this is equal to 1 plus 1 by 2 is equal to 1, B is equal to 1 which means phi is equal to cos inverse of 1 which is equal to 2 m dash pi; m dash is an integer here. So, I have used the dash just to distinguish because we already use m for the round trip and therefore, m dash is an integer alright.

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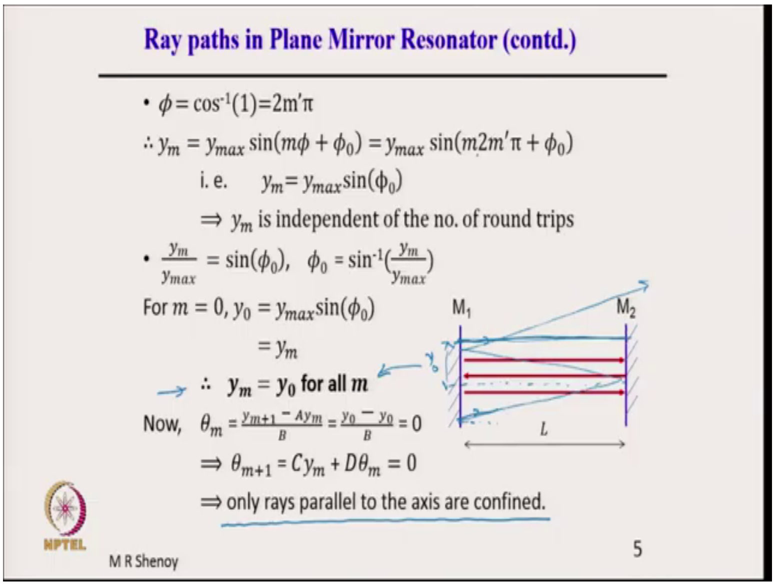

Now, let us continue and therefore, we have phi is equal to cos inverse of 1 is equal to 2 m dash pi. Therefore, y m is equal to y max into sin m phi plus phi 0; for phi we can substitute from here. So, we have substituted sin m into 2 m dash pi plus phi naught. So, this is a even multiple of 2 pi and therefore, this will be simply equal to sin phi 0. And therefore, y m is equal to y max into sin phi 0. Note that y m on the right hand side is independent of m, y m is independent of the number of round trips.

So, we have y m by y max equal to sin phi 0 and therefore, phi 0 we know now and for m is equal to 0, y 0 is equal to y max sin phi 0 equal to y m. What does this mean? This means y m is equal to y 0 for all m. What does this mean? This means, if we start at for some y 0 here, a ray then the ray will go back and forth and y m will be equal to y 0 for all m.

If you let us just see the theta now for example. Now theta m is given by this expression; y m plus 1 minus Ay m by B. So, substitute for A and B so, we get theta m is equal to 0. Therefore, theta m plus 1; if theta m is 0 theta m plus 1 C plus D into theta m is also 0 and therefore, only rays parallel to the axis are confined.

So, there are two points; one if you to trace the rays for example, to trace the rays the procedure followed is you start with a y 0 and theta 0 and then find out what is the next y 1 theta 1 after one round trip after 2 round trip y 2 theta 2 and so on. Now, we have seen that y 0 if you start with y 0 theta is always 0 theta 0 is 0 for all m theta m is 0 for all m which means if you start at y 0, the ray has to go horizontally parallel to the axis.

And therefore, we have only rays parallel to the axis are confined. Once the ray is parallel to the axis this means y 0 is equal to y m for all m the rays any ray that you take will go back and forth along the same line. If a ray makes a slightly different angle theta not equal to 0 then for example, if I show a start a ray making a small angle like this theta 0 like this, then this will eventually go out.

We already know this our basic understanding of reflection tells us that yes indeed it will eventually go out. So, mathematically we have got the same thing that rays permitted are those which have theta equal to 0; permitted here refers to confined rays which are confined correspond to theta equal to 0 and the displacement y m is equal to the initial displacement y 0 for all values of m.

This of course, for a plane mirror resonator we know this. But if you take a general resonator, then this is the procedure to be followed.

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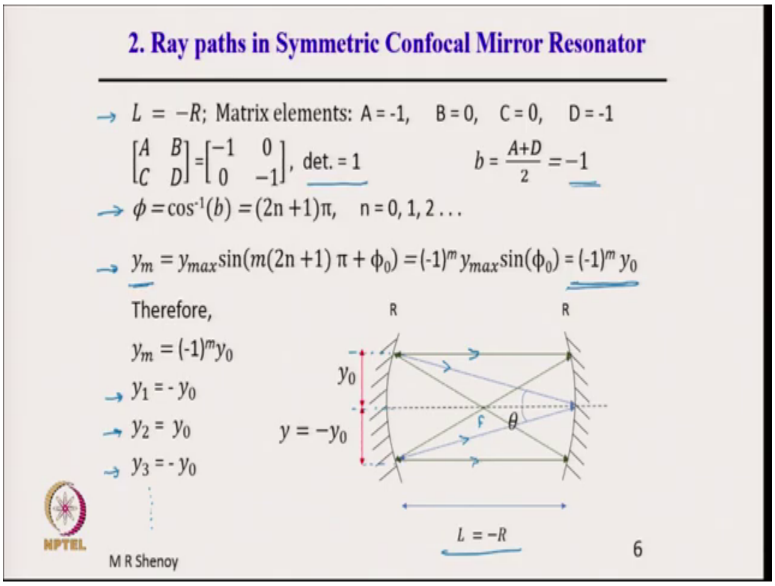

So, let me take another example. So, ray paths in symmetric confocal mirror resonator. From our common sense common understanding we have already traced the rays in confocal mirror resonator. Now let us see the matrix procedure what do we get. So, L is equal to minus R for a confocal resonator L is minus R and therefore, the matrix elements A is equal to minus 1, you can substitute in those A, B, C, D.

We get A is equal to minus 1, B is equal to 0, C is equal to 0 and D is equal to minus 1. And therefore, the A, B, C, D matrix is here and we note that the determinant is equal to 1. Further B is equal to minus 1 plus minus 1 by 2 is equal to minus 1. This means that phi is equal to cos inverse B which is equal to 2n plus 1 pi with n is equal to 0, 1, 2 etcetera. Therefore, y m is equal to y max into sin phi.

So, for phi we have 2n plus 1 pi. So, sin m phi so, m into 2n plus 1 pi plus phi 0. Now note that this has an even pi plus 1 pi. So, odd pi and therefore, this is equal to minus 1 to the power m that is whenever m becomes even this will become 1 and when m is odd, we will have minus 1 as the sin which is outside. So, we have y max into sin phi 0 and that is equal to minus 1 to the power m y 0.

So, y m is equal to minus 1 to the power m y 0. What does that mean? If m is equal to 0, then of course, y 0 equal to y 0; if m is equal to 1, then y 1 is equal to minus y 0. If m equal to 2 y 2 is equal to again y 0 which means if we start with a y 0 which is here, a ray which goes. So, there are two different ratio. Let us consider first this ray a ray which is going like this.

Then after one round trip, y 1 will be equal to minus y 0 that is in the lower half symmetrically about the axis, we will have y 1 is equal to minus y 0. Now, the ray would start again from here because note that this separation is equal to radius of curvature and therefore, any ray coming from the radius of curvature, center of curvature of the mirror will be reflected back along the same line.

And therefore, it will again start going in this direction satisfy the law of reflection here and come back that is y 2 this was y 1 and this is y 2 is equal to again y 0 and the ray will go back and forth. It is second type of ray which we saw that is also possible is a ray which travels from here parallel to the axis. If it is parallel, then after reflection; it will pass through the focal point.

So, this is focal point midway and therefore, from focus it continues here and any point coming any ray coming from the focus will be rendered parallel. So, this is what we had already seen and from here of course, a parallel ray will pass through the focus which means again after 2 round trips, the ray comes back to its original position and mathematically.

So, mathematically it directly gives us y m is equal to minus 1 to the power m into y 0 which means y 1 is equal to minus y 0 that is in the lower half and then y 2 is again equal to y 0 coming back to its original position, y 3 is equal to minus y 0 and so on. So, we are able to trace rays inside a confocal mirror resonator by simply using the mathematical expression that we have; what we need to know is the ray transfer matrix the elements A, B, C and D ok.

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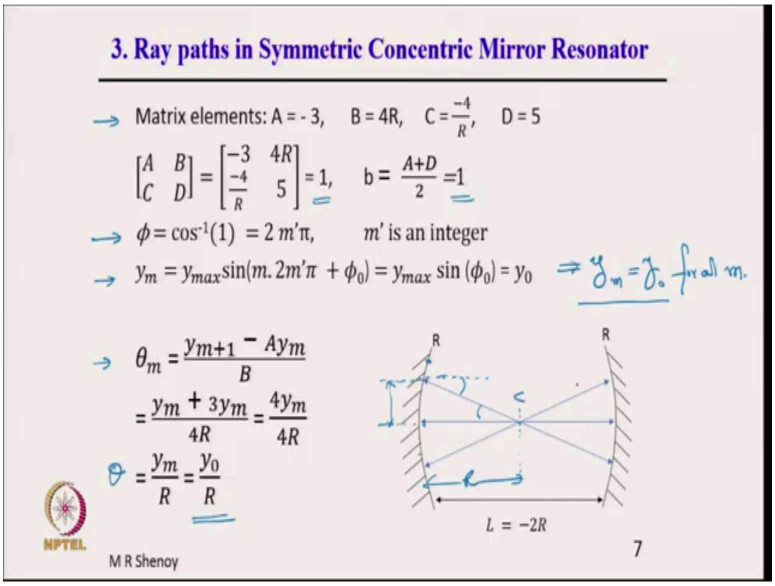

Let us now take symmetric concentric mirror resonator. Again for this from our common understanding we know how to trace rays in such a resonator. Now, the matrix elements if you calculate the matrix elements, then you see that A is equal to minus 3, B is equal to 4 times R, R is the radius of curvature identical; it is a symmetric one. So, R and C is equal to minus 4 by R and D is equal to 5.

Note that the elements are neither 1s or 0s, but the determinant is again 1. So, you can just multiply the two and see the. So, this is minus 15 minus for R R cancels 16 minus minus 16 is equal to 1. So, it need not be 1 0 etcetera, but the determinant is always 1 and therefore, B is equal to A plus D by 2; A is minus 3, D is 5 and therefore, A plus D by 2 is equal to 1. As before phi is equal to cos inverse of 1 which is equal to 2 m dash pi where m dash is an integer and therefore, y m is equal to y max into sin m pi plus phi 0.

So, for phi is substitute 2 m dash pi and therefore, equal to y max into sin phi 0 which is the same as y 0 because you substitute for y 0, y 0 will be y max into if you put m equal to 0 sin phi 0 and m y m is also equal to y 0. This implies y m is equal to y 0 for all m for all m. Now let us look at theta.

So, theta m is given by this formula here and we substitute for A and B. So, A is 3 minus 3 therefore, y m plus minus 3 y m by 4R which is equal to 4 y m by 4R which is y m by R. Now y m is equal to y 0 for all m therefore, this is equal to y 0 by R. So, theta m is equal to y 0 by R.

What is theta m? So, we can see here. So, this is theta that is equal to y 0. What is y 0? Is the displacement from the axis? So, this is y 0 and this is 2R. Therefore, this up to this the distance up to this is R. So, this is R. Therefore, note that y 0 by R is nothing, but tan theta; tan theta is nearly equal to theta and that is how we got theta is equal to y 0 by R. So, if you are moving to a different point because in ray tracing we have to start tracing rays from different position.

So, if you are starting at some y 0 here, then you must have theta is equal to y 0 divided by R and that ray will come back to the along the same path that is the only value of theta which is permitted for confined rays. So, we know now how to trace the rays. So, note that a ray which goes with this theta will come back along the same path because this separation is equal to twice R or this point here is the center of curvature for both the mirrors that is why the name concentric mirror resonator.

So, our common sense plotted plot by common sense is consistent with the mathematics or the other way the mathematics consistently gives us the same ray paths.

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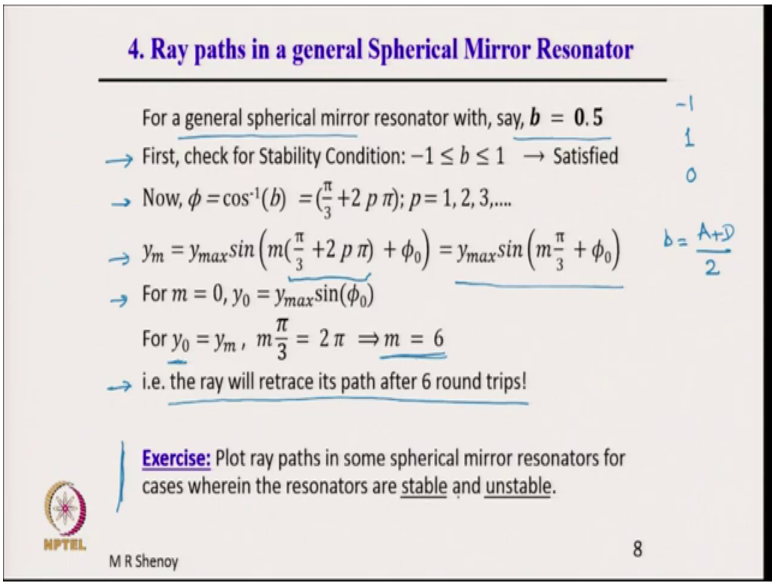

Now, let us take a general spherical mirror resonator. So, these are the common ones which we know plane mirror confocal mirror and concentric mirror. Now let us take a general spherical mirror resonator. So, for a general spherical mirror resonator, let us say b is equal to 0.5.

So, far we had b is equal to minus 1, we had b is equal to 1, we had b is equal to 0. Now let me take some b in between as b is equal to 0.5. So, how to proceed? So, given a resonator, first find out the RTM Ray Transfer Matrix elements a, b, c, d. Once you know a, b, c, d, you know what is b, because b is equal to A plus D by 2 where, A and D are the matrix elements given a resonator you know what is the ray transfer matrix elements A and D. Therefore, you know what is b.

If you know b, then first check for stability condition that is b must be between minus 1 less than or equal to 1 and now phi is equal to cos inverse of b. We followed this exactly the same procedure, you can see here phi is equal to cos inverse of b, b had come out to be 1 in this case. So, now, I have I am following the general procedure that phi is equal to cos inverse of b.

In this case b, I have taken 0.5; you could have taken 0.4 or 0.38 or whatever number that you get. And if it is 0.5, then we see that this is equal to pi by 3 plus 2 p pi because cos inverse of 0.5 is 60 degree plus 2 m pi. And therefore, 60 degrees pi by 3. So, pi by 3 plus 2 p pi where p is an integer.

Therefore, y m so, once you know phi substitute for the expression for y m equal to y max sin m pi plus phi 0. So, we have substituted for phi. And now you see that this product m into 2 p pi is an even integral multiple of 2 pi and therefore, this is simply equal to m pi by 3 plus phi 0 m for m is equal to 0. So, y m is equal to y max plus this if we put m is equal to 0, we have y 0 equal to y max sin phi 0.

For y 0 to be equal to y m that is if the ray has to come back to its original starting point y 0 is the initial displacement, for y m to be equal to y 0, we must have m pi by 3 is equal to 2 pi because, if this becomes 2 pi, then sin m pi by 3 plus phi 0 will be equal to sin phi 0 and this must be equal to pi which implies m is equal to 6.

What does this mean? This means that the ray will retrace its path after 6 round trips. Remember we have taken examples in which ray would come back after every round trip ray would come back after 2 round trips and there are some other examples where I had shown ray would come back after.

So, here of course, ray would come back after every round trip, but there are examples where ray would come back after 4 round trips and in this particular example that I have taken, the mathematics tells that the ray will retrace its path after 6 round trips, but the ray will retrace its path which means the ray is confined. It is a boundary which is confined.

So, this is the procedure followed in tracing rays through spherical mirror resonators. So, here I have given an exercise plot rays in some spherical mirror resonators for cases wherein the resonators are stable and unstable stable means minus 1 less than. So, you have to choose a value of b. If you choose b between minus 1 and plus 1, that will be a stable resonator. And if you choose a value let us say b equal to 1.5, then it will be an unstable resonator.

So, it is an exercise which I recommend you to do that trace rays through stable and unstable resonator. Just as an example, I will show some plots of tracing which is done by an undergraduate student for your benefit.

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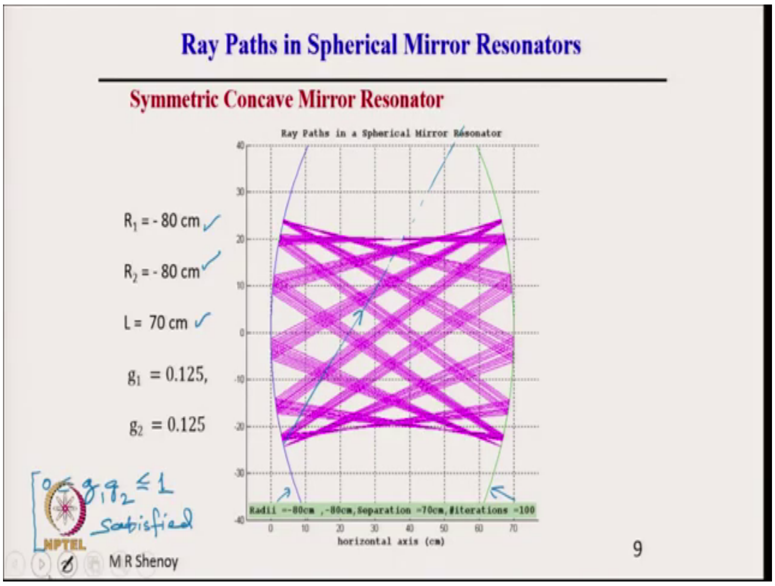

So, see here. So, what is shown is the ray paths. So, what are shown here are ray paths plotted these are actual ray paths plotted with the help of a computer. So, what is shown is so, these are the mirrors. So, here are the mirrors, you can see the curvature. So, these are the mirrors of radius of curvature R 1 is equal to minus 80 centimeter that is actually radius of curvature is 80 centimeter, but these are concave mirrors.

So, symmetric concave mirror resonator. Radius of curvature used in this calculation, this plot are 80 centimeters each with the separation between the mirrors as 70 centimeter. So, the moment the resonator is given you can calculate g 1. So, g 1 comes out to be 0.125, it is symmetric.

Therefore, g 2 is also 0.125 and therefore, we know the stability condition 0 less than or equal to g 1 g 2 less than or equal to 1 is satisfied. So, it is a stable resonator. So, this is satisfied. So, this is a stable resonator and what you see is the ray which are rays which are confined. There are rays plotted with different angles and the rays shown are those rays which are confined.

Obviously, if you take a ray for example, if a ray which starts from here at a deep angle like this. So, this will go out so, but what is important is if the resonator is stable, then one can always find ray paths which are retracing back and forth and remain confined to the resonator.

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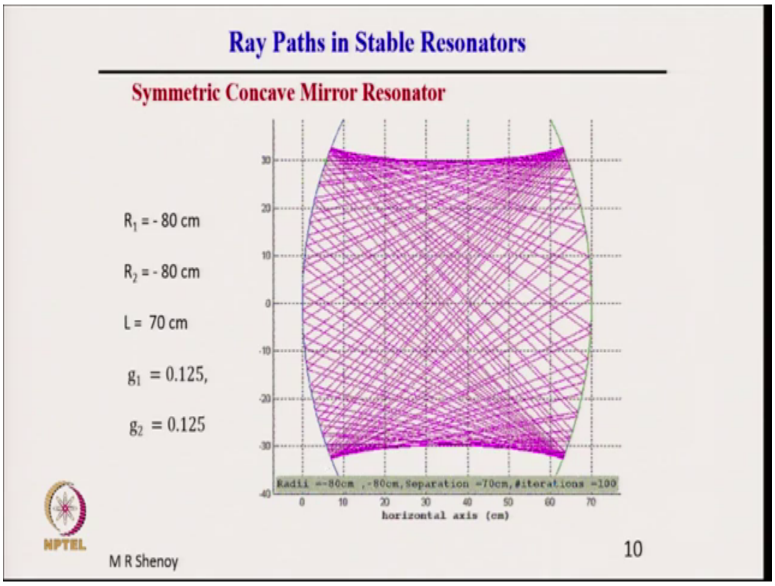

Let us let me show another diagram here. It is the same resonator now the rays are spread out more. So, in the previous case the rays were shown as bunches. So, starting from one point, a bunch of rays were propagated with slightly different angles. Now, it is the same resonator the rays are spread out you can see that the rays are spread now, but the rays are still confined to the spherical mirror resonator. So, these are the mirrors M 1 and M 2 and as before, it satisfies the stability condition.

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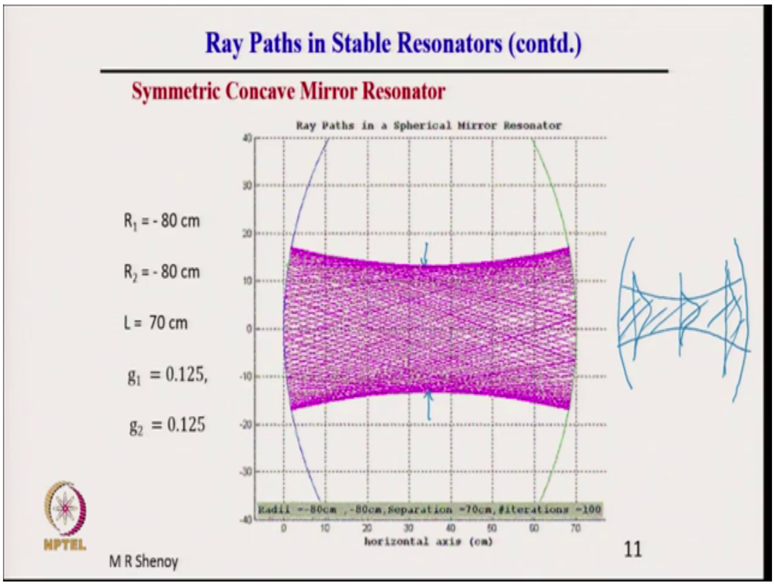

The same resonator again with the different set of a rays which are propagating, but these are ray propagation, but you see that the rays bunch in such a way that it looks as if there is a beam which is fact. We will see later on subsequently, we will consider Gaussian beams in a resonator where we will see that the Gaussian beam will be confined to the resonator in this fashion. So, this is a Gaussian which is propagating back and forth, we have a Gaussian field distribution.

So, this is the waist of the Gaussian. So, we will discuss this in detail and then the beam propagates back and forth inside the resonator. But what is important to see is this, there is no beam here. This is just rays all straight line paths rays which are plotted, but they bunch in such a way that it gives an impression as if there is a wave there is a beam which is propagating with a waist here and its spreading on both the sides.

So, even it simply depends on the number of rays chosen and the angles which are chosen, but the picture clearly shows confinement of rays equivalent to confinement of beams in optical resonators.

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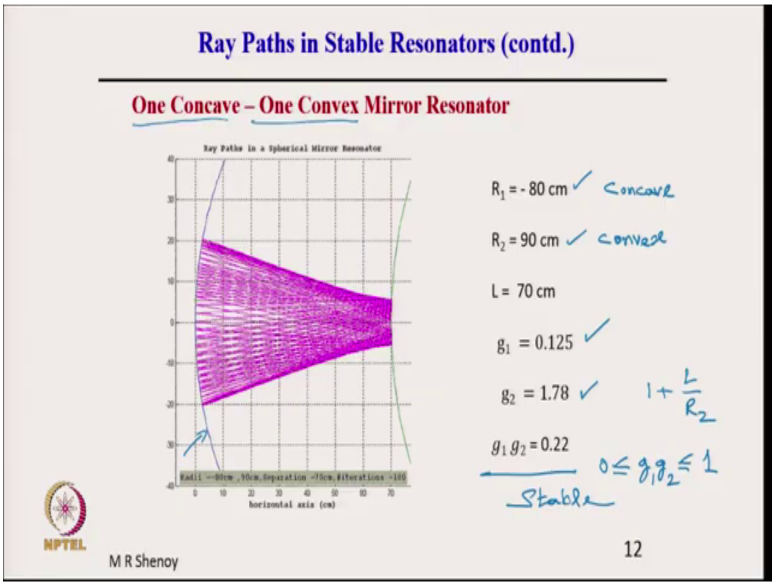

So, what is now shown is a one concave mirror and one convex mirror. I had mentioned that yes, they can also form stable resonators. See the example which is taken R 1 is minus 80 centimeter which means this is concave, the one mirror on the left side here. So, this is concave minus 80 centimeter and this is convex. So, this one is convex 90 centimeter radius of curvature separated by a separation of 70 centimeters.

Note that the g 1 comes out to be 0.25, but g 2 is now 1.78 because g 2 is 1 plus L by R 2. So, R 2 is also positive, L is also 70. So, this is more than 1. So, 1.78; however, the product comes out to be 0.22. So, it still satisfies 0 less than or equal to g 1 g 2 less than or equal to 1 which means it is a stable resonator. So, we clearly see in a stable resonator when the rays are plotted, they are confined; they are confined to the resonator.

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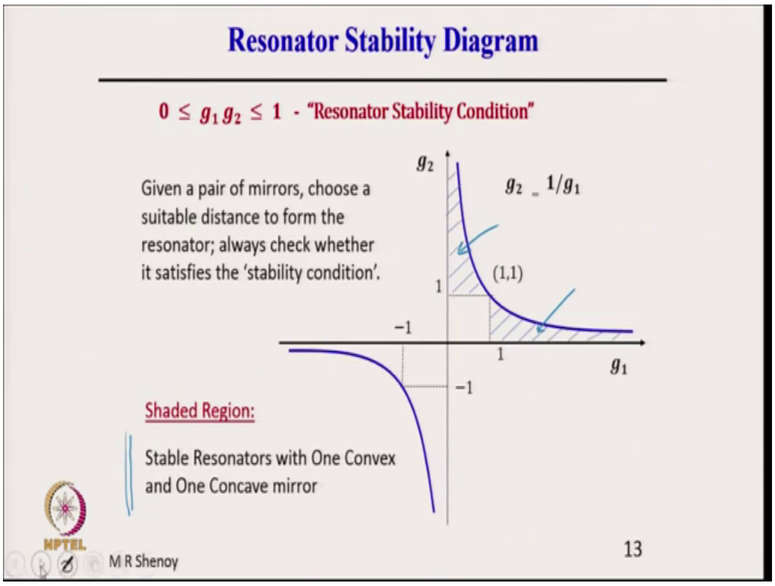

What is shown here is the resonator stability diagram? In this where will be the position of one concave and one convex? So, one concave and one convex please see that if it is concave g 1 is less than 1. If it is convex g 2 is greater than 1 because it is 1 plus L by R 2 and therefore, it will be in those regions where one of the gs is less than 1 and the other g is greater than 1.

So, that is illustrated here. So, the shaded region here where g 2 is greater than 1, but g 1 is less than 1. Similarly here g 1 is greater than 1, but g 2 is less than 1. So, the stable resonators with one convex and one concave mirror are the shaded regions here. So, they represent one concave and one convex. And clearly therefore, one concave and one convex mirror pair can also form stable resonators which is clearly indicated by the diagram.

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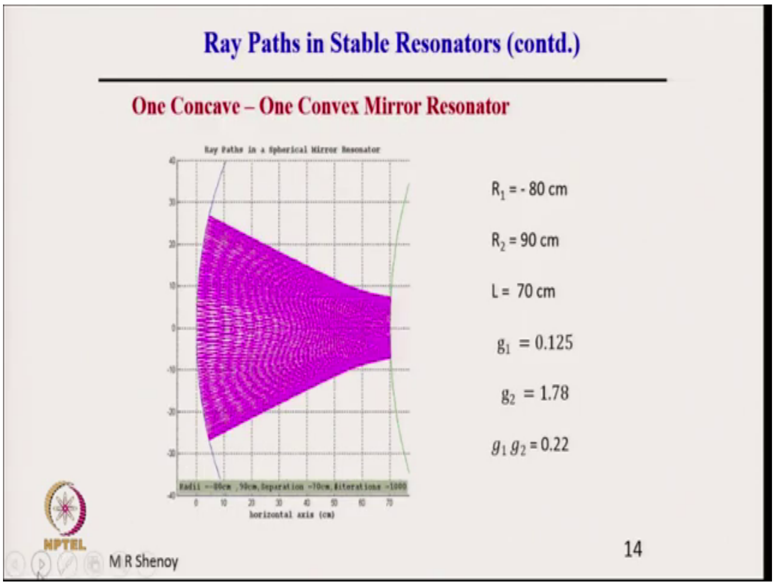

So, here again the same diagram same mirror, but the number of rays are now much larger. So, we can see it as a dense beam which is going back and forth alright.

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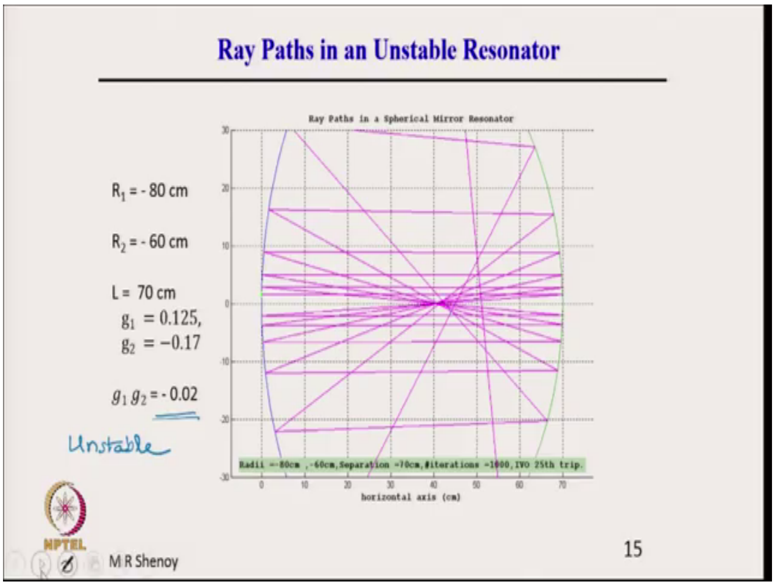

Let me take an unstable resonator. So, ray path in an unstable resonator. So, I have taken R 1 and R 2 these numbers here with separation still remaining 70 centimeter. In all the examples I had kept the separation same, but R 1 and R 2 have been changed. Now we get g 1 is equal to 0.125, g 2 is equal to minus 0.17 and therefore, the product is now negative.

Therefore, it does not satisfy the stable stability condition which means this is an unstable resonator. You see the ray paths which are plotted, they travel for some time, but then they either go from here or go out; or go out in different directions. In other words we cannot find rays which are confined. Initially they can go back and forth for some time, but afterwards you can see that it finally, goes out of the resonator alright.

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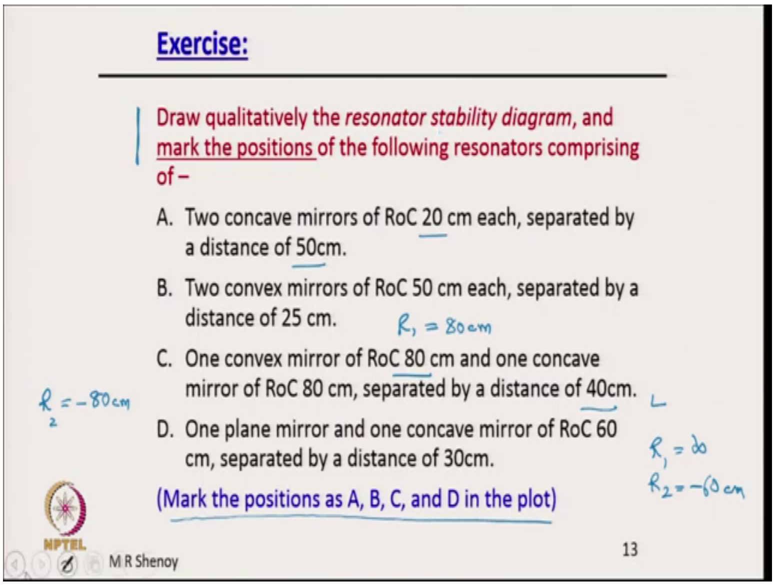

Finally I have taken one simple exercise here. So, draw qualitatively the resonator stability diagram and mark the positions of the following resonators comprising of because right now I have taken actual numbers roc different radius of curvatures and distances; so, distances and RoCs.

So, two concave mirrors of RoC 70 centimeter each which means it is a symmetric resonator separated by a distance of 50 centimeter. B two convex mirrors of 50 centimeter each separated by 25 centimeter, one convex mirror of RoC 80 centimeter and one concave mirror of RoC 80 centimeter. Remember once it set one concave mirror of RoC 80 centimeter immediately we have to write R 2 is equal to minus 80 centimeter separated by a distance 40 centmeter, this is L. So, this is R 1.

So, R 1 is convex mirror. So, it is positive 80 centimeter. One plane mirror and one concave mirror of RoC 60 centimeter which means R 1 is equal to infinity R 2 is equal to minus 60 centimeter separated by a distance of 30 centimeter. So, the question is mark the positions as A, B, C, D in the plot, draw qualitatively the resonator stability diagram and mark the positions A, B, C, D.

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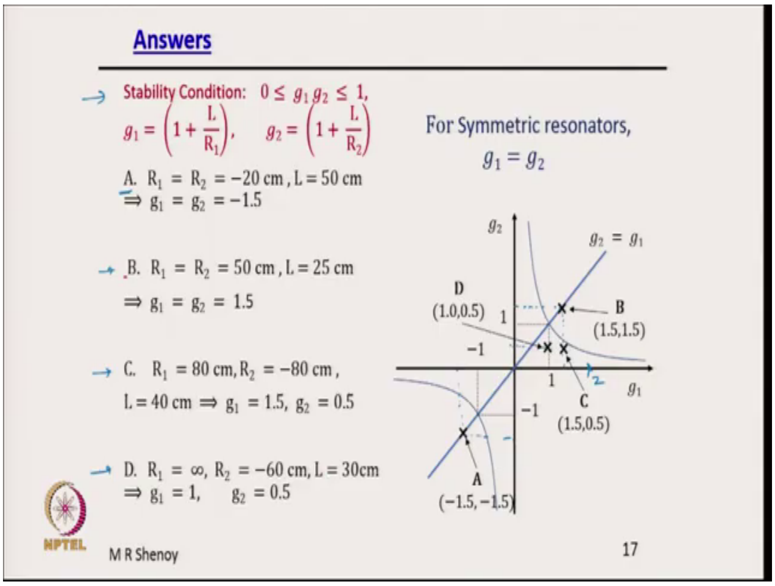

So, I have worked out this example and the answer is here. So, the first the stability condition and for the first point A, R 1 R 2 minus 20 centimeter, L is equal to this. Therefore, we get g 1 is equal to g 2 is equal to minus 1.5 and if you plot them then, 1.5. So, this is one therefore, this position is 1.5 because 2 maybe here. This is 2 and this is 1.5. So, the point is 1.5, 1.5.

So, this is minus 1.5 here. The point A is here minus 1.5, 1.5 whereas, for B it is 50 centimeter and L is 25 centimeter, we get g 1 is equal to g 2 equal to 1.5. So, that is in the positive quadrant and this is in the third quadrant minus 1.5 and minus 1.5, but immediately you note that is outside the stability region.

It is a symmetric resonator both A and B are symmetric resonators, but and therefore, they lie on the y is equal to x straight line that is g 1 is equal to g 2, but outside the stable region. Therefore, the conclusion is immediate conclusion is both A and B are unstable resonators. So, both A and B are unstable. Now we come to see R 1 is 80 centimeter R 2 is minus 80 centimeter.

So, if you calculate g 1 and g 2, g 1 is 1.5 g 2 is 0.5 and therefore, for C, R 1 is 80 centimeter, R 2 is minus 80. We got g 1 is equal to g 2 is equal to this much and if we plot, we see that it is here g 1 is so, it is 1.5. So, g 1 is 1.5, this point is 1.5 and g 2 is 0.5 because this is one therefore, 0.5 is here and this is the point.

For the last one that is D. So, please see here D, one plane mirror and one concave mirror of RoC 60 centimeter. So, R 1 is infinity, R 2 is minus 60. So, R 1 is infinity, R 2 is minus 60 and L is 30 centimeters and therefore, g 1 is equal to 1 and g 2 is equal to 0.5. So, g 1 is 1. So, this dotted line is 1 and this is 0.5. So, the clearly once we look at these position of the point, we can conclude that A and B are unstable resonators whereas, C and D; so, this is D this is C these are stable resonators ok.

So, this is an example just to illustrate given a resonator, the first thing to do is to determine whether they are stable resonators whether it is a stable resonator or not. And if it is a stable resonator, then either one can follow ray tracing techniques or also one can determine the beams the Gaussian beam of the resonator. We will discuss this in the next part of the course.

Thank you.