Introduction to LASER
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Lecture - 17
Resonator Stability Condition

Welcome to this MOOC on Lasers. In the last class, we started discussion on spherical mirror resonators. So, initially we looked at spherical mirror resonators and ray propagation in these resonators, based on our qualitative knowledge of ray propagation and spherical mirrors. We then picked up matrix optics and we will try to obtain, today we will obtain the Resonator Stability Condition using metrics optics.

Matrix optics is valid under paraxial approximation; that means rays which are traveling close to the axis of the optical system or rays which make very small angle with the optical system. We have also seen the sign convention for matrix optics.

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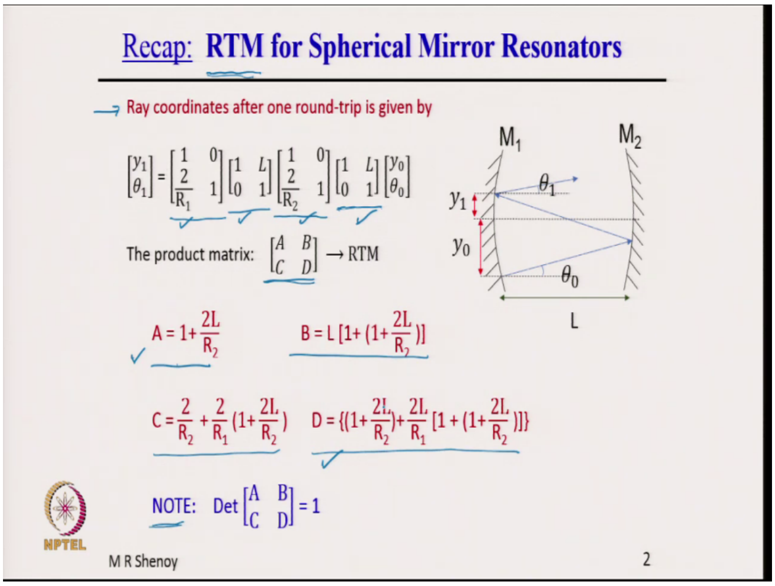

Now, let us quickly recap, a quick recap of what we had done. So, we obtain the ray transfer matrix RTM, here ray transfer matrix for a spherical mirror resonator. What we have done is, the ray coordinate after one complete round trip involving four operations. So, we start with y 0 theta 0. So, y 0 is the displacement from the axis here and theta 0 is the angle that it makes with the horizontal, in this case the optic axis of the system.

So, the ray starts from here, it makes a propagation through a distance L, undergoes reflection. So, the corresponding 2 by 2 matrices, which represent the operation are propagation through a distance L; reflection at the spherical mirror of radius of curvature R 2, back propagation through a distance L and then reflection at mirror M 1 of radius of curvature R 1.

The product matrix if we designate it as A B C D; A B C D is called the ray transfer matrix. The coefficient if we multiply these, the coefficients A; the first coefficient is this one A e A, B, C and D. This is the A B C D product matrix, elements of the A B C D product matrix.

Note that the determinant A B C D is 1; we know that the determinant of a product matrix is equal to the product of the determinant of the individual matrices. And therefore, you can see that the first matrix, the determinant is 1; second matrix determinant is 1, third determinant is 1, fourth determinant is 1 and therefore, the product matrix also has a determinant which is equal to 1. So, we will make use of this at a later stage.

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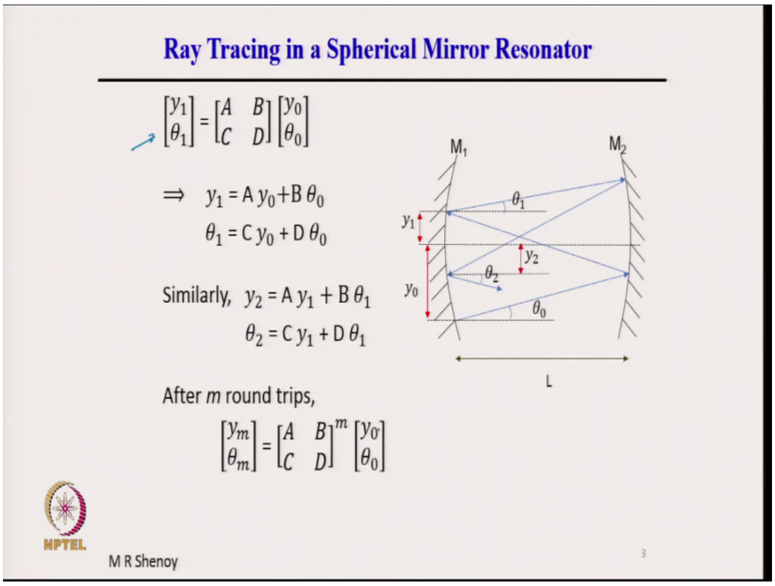

Now, let us look at ray tracing in a spherical mirror resonator. So, as discussed, the ray coordinates y 1 and theta 1 after one round trip is given by the matrix equation here; that means y 1 is equal to A into y 0. So, row into column plus B into theta 0. And similarly theta

1 is equal to C y 0 plus D theta 0. You can take it to the next round; then y 2 and theta 2 are again given by a similar expressions. And after m roundtrips, y m comma theta m, the two coordinates are given by A B C D matrix to the power m into y 0 theta 0 that is quite straightforward.

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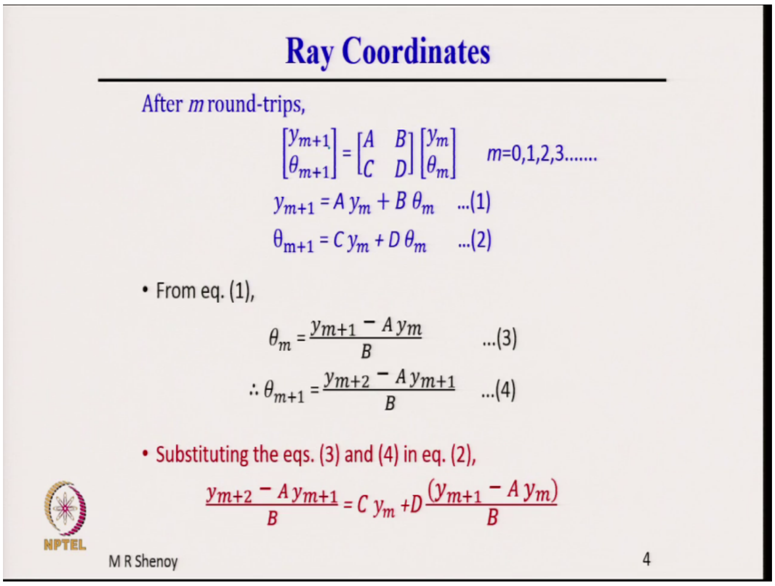

And therefore, after m round trips, the coordinate y m plus 1; the new upgraded coordinates, y m plus 1 theta m plus 1 is still related to y m and theta m through the A B C D matrix.

So, we write y m plus 1 is equal to A y m plus B theta c m and theta m plus 1 is equal to C y m plus D times theta m. Our objective would be to eliminate one of these thetas and write an equation in terms of the displacement alone. And therefore, from equation 1, theta m is equal to y m plus 1 minus A y m by B. And therefore, if we upgrade this m replace m by m plus 1;

then we have theta m plus 1 from the same equation. Now, m plus one will be replaced by y m plus 2 minus A times y m plus 1 by B.

Substituting equations 3 and 4 in equation 2; if we substitute, note that theta is written in terms of the displacement y. And therefore, if we substitute in 2 for these two thetas; then we will get the equation in terms of displacement only and that is what we have done, equations 3 and 4 in equation 2 we get. So, this is theta m plus 1 equal to C y m plus D into theta m.

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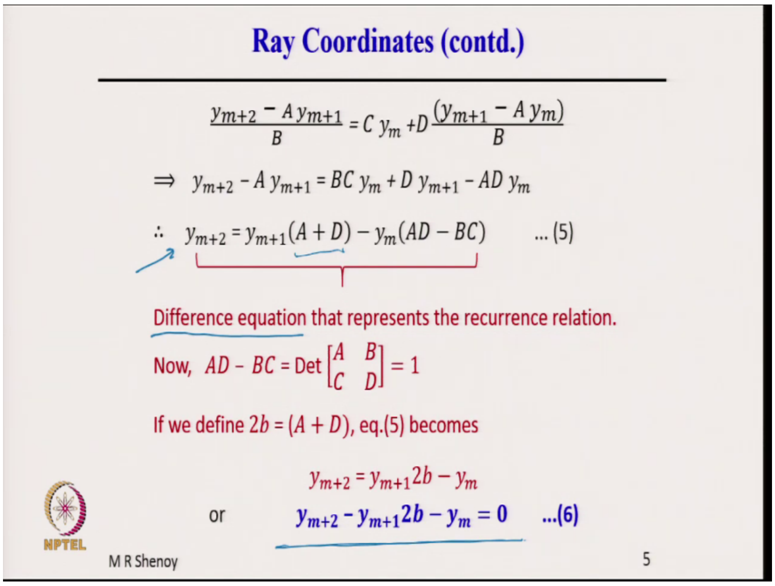

So, let us simplify that further. So, that gives y m plus 2. So, this is the equation, you simplify this; this comes out to be y m plus 2 minus A y m plus 1 is equal to as written here. And therefore, y m plus 2 is equal to y m plus 1 into A plus d.

So, we have taken these two terms together, minus y m into A D minus B C; this represents a difference equation. A difference equation that represents the recurrence relation; what it means is, if we know for any value of m, y m and the next y m plus 1, then we can find out what is y m plus 2.

Now, irrespective of the angle of propagation, we know the once the resonator is given; that means A B C D are known and therefore, we have a recurrence relation which is of this form.

Note that A D minus B C is the determinant which is equal to 1 and therefore, we write; further if we define A plus D as equal to 2b then we can write equation 5 as y m plus 2 equal to y m plus 1 into 2b, this is written as 2b. We will know why it is written as 2b and this is 1 minus y m or we have y m plus 2 minus y m plus 1 into 2b minus y m equal to 0. So, we have got an equation in terms of the ray displacements from the optic axis.

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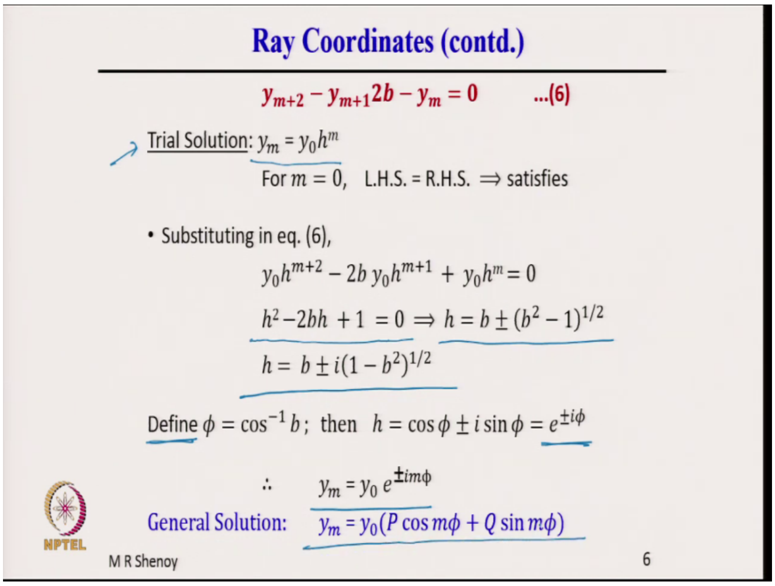

So, let us look for trial solution of this equation; we want to look for solutions of this equation. So, we start with a trial solution. So, a trial solution can be written as y m is equal to y 0 into h to the power m. Note that this satisfies for m is equal to 0; for m is equal to 0, it is y 0 equal to y 0, that is LHS is equal to RHS and that satisfies.

Now, if we substitute this in equation 6, then we have y 0 into h to the power m plus 2, y m plus 2. So, replace m plus 2; then we get y 0 into h to the power m plus 2 minus 2b into y 0 h to the power m plus 1 plus y 0 h to the power m is equal to 0. So, h to the power m is common throughout. So, it can be eliminated to get the equation h square minus 2bh plus 1 is equal to 0.

The solution of this is a quadratic equation and therefore, the solution h is given by b plus minus square root of b square minus 1 or h is equal to; we want to write this 1 and b square interchange.

So, h is equal to b plus minus i into 1 minus b square to the power half. Now, if we define phi is equal to cos inverse phi; if we define phi is equal to cos inverse phi, we know why we are going to define, because if we put this b as cos phi, then we have 1 minus cos square phi, 1 minus cos square phi is sin square phi and b is cos square phi here and therefore, cos phi. So, immediately this will come in the form of e power i phi.

So, that is why define phi is equal to cos inverse b, then h is equal to cos phi plus minus i times sin phi or e power plus minus i phi. This is h and therefore, y m is equal to y 0 into e to the power plus minus i m phi; this was our starting trial solution y m and we get that y m is equal to y 0 into e power plus minus i m phi. So, that is an harmonic solution. So, the general solution can be written as y m is equal to y 0 into P cos m phi plus Q sin m phi. So, written in terms of sin and cos is the general solution.

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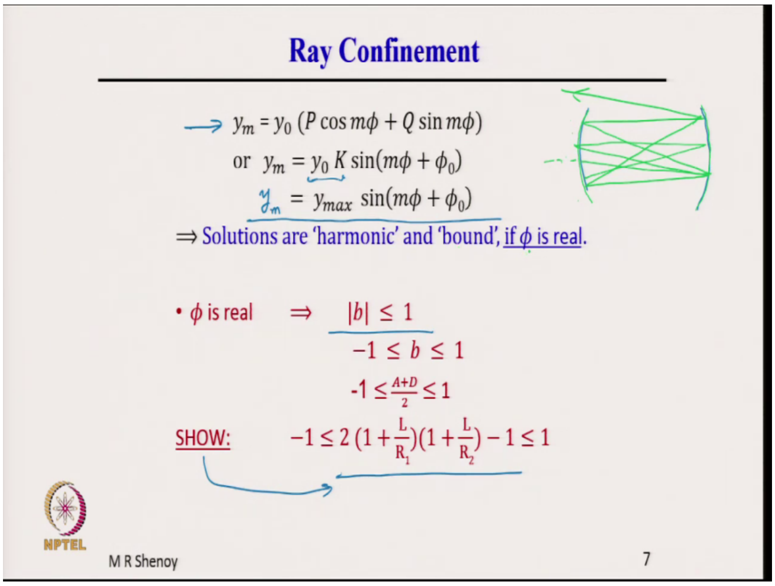

Now, let us go further and discuss. So, y m therefore, is equal to y 0 into P cos m phi plus Q sin m phi, which can also be written in the form; we can combine these and write in the form of y 0 into some constant K sin m phi plus phi 0. And y 0 into K if we designate this as y max; then we have y m is equal to. So, we have y m is equal to y max into sin m phi plus phi naught.

So, this is an harmonic solution. So, long as phi is real. So, the solution y m are harmonic and bound, if phi is real. Phi is real, what is phi? Phi is cos inverse b and therefore, phi is real implies mod b must be less than or equal to 1; that is mod b must lie between minus 1 and plus 1, so that cos phi takes values between minus 1 and plus 1 for all real angles phi. And if phi is real; that means, b must lie between minus 1 and plus 1.

Now, what is b? B is A plus D by 2; we know A and D for the given resonator. And therefore, A plus D by 2 must lie between minus 1 and plus 1. We can show that this A plus D by 2; we know the elements A and D, so please look at the elements A and we are here the element A, we know the element A and we know the element D for a given resonator comprising of mirrors of radius of curvature R 1 and R 2 separated by a distance L and therefore, we know A and D. And therefore, if we find out A plus D by 2; then we get this as 2 times 1 plus L by R 1 into 1 plus L by R 2 minus 1

So, this must be between minus 1 and plus 1. I have written this as show here refers to A plus D by 2 is equal to this, show this. So, in fact I have shown this towards the end of the lecture, so I will discuss this there again.

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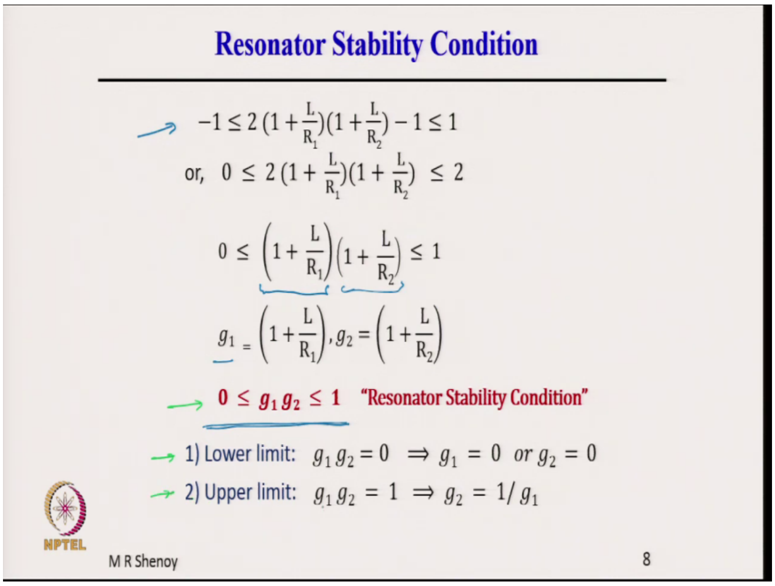

Now, here again the same step is written, minus 1 less than or equal to 2 into 1 plus L by R 1 into 1 plus L by R 2 minus 1 should be less than or equal to 1. Therefore, if you add 1 to all the sides; then you have 0 less than or equal to this and if you divide by 2, we have 0 less than or equal to 1 plus L by R 1 into 1 plus L by R 2 less than or equal to 1.

If we define this as g 1 and this as g 2; then the equation, the inequality is written as 0 less than or equal to g 1 g 2 less than or equal to 1; this is called the resonator stability condition. Now, why is this called resonator stability condition? What does this condition represent? This condition represents that we get; so the condition represents that the solutions are harmonic and bound. What solution are we talking of? Solution for the displacement, recall the mirror here.

So, here are the spherical mirrors, the rays are traveling; let me change the color, the rays are traveling like this. So, going back and then forth and then coming here and maybe then it is making like this, traveling like this and so on. Now, the solution, solution here refers to the displacement y m. The displacement y after m round trips or for any value of m, if it remains bound and well bound to the mirror, that is well within the mirrors; that means y has to oscillate, it has to once become, once it goes up and then it comes down, up, down.

So, y is essentially oscillating and therefore, it is harmonic and it is bound; because it is bound to the limits of the mirror. So, if it is not bound, that means if the ray for example, goes like this; then it is no more bound to the mirrors or it is an unbound ray or the rays are not confined. So, we are looking for rays which are confined to the resonator; which means the solution must be harmonic and bound.

For that phi needs to be real, which means mod b must be less than or equal to 1 and then that led to the condition that 0 less than or equal to g 1 g 2 less than or equal to 1, and this is called the resonator stability condition. It is a very important condition; it tells us that for a given resonator, you can immediately determine the coefficients g 1 and g 2 1 plus L by R 1 and 1 plus L by R 2; R 1, R 2 are known, L is known.

And therefore, for a given resonator we first determine g 1 and g 2 and then see whether the product lies between 0 and 1. And if it lies between; that means in that resonator you can always find rays which are bound and confined, harmonic and bound, that means the rays going back and forth will remain confined to the resonate.

Let us discuss this a little bit more and therefore from this equation, the lower limit is 0; that is the product g 1 g 2 is equal to 0 is the lower limit, which means either g 1 is equal to 0 or g 2 is equal to 0. And the upper limit is g 1 g 2 is equal to 1; that means g 2 is equal to 1 by g 1 or g 1 is equal to 1 by g 2. So, these are the limits and therefore, looking at these limit, we try to plot a resonator stability diagram.

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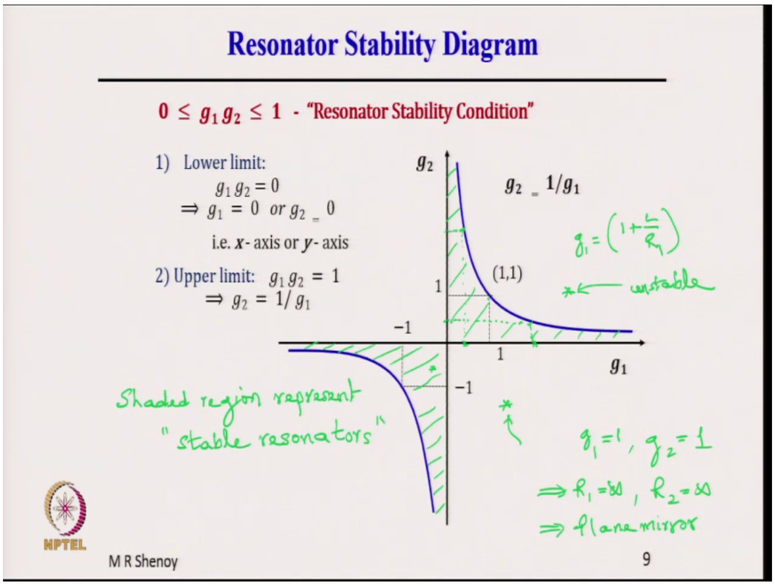

So, that is what is shown in this slide. So, the lower limit g 1 g 2 equal to 0. So, g 1 is equal to 0 or g 2 is equal to 0; the resonator stability diagram is a diagram, where one of the g, that is for example, g 2 is along the y axis and g 1 is along the x axis. So, g 1 is equal to 0 is a limit, which means x axis is a limit. Similarly, g 2 is equal to 0. So, g 1 equal to 0, means g 2 this is the axis; g 1 is 0 all along this axis and for the x axis, g 2 is equal to 0.

So, these are the two bounds. The other bound is the upper limit is given by g 2 is equal to 1 by g 1; which means if for any given value of g 1, any given value of g 1. So, 1 by g 1 is the upper limit for g 2, g 2 can take; if g 1 takes a certain value, please see what is g 1? G 1 is 1 plus L by R 1. So, for the given resonator if the L and R 1 are such that, g 1 remains, so this is g 1. g 1 is a value somewhere here then it says that, R 2 can be such that the smallest value of R 2, so that the maximum value of g 2 is given by 1 by g 1. And if g 1 is here, then the maximum value of g 2 is given by 1 by g 1, which is this value.

So, therefore, the curve g 2 is equal to 1 by g 1 specifies the upper limit; the lower limit is given by the axis here and the upper limit is given by the blue curve here, 1 by x curve. Similarly, for negative values, so negative values of g 1; then 1 by g 1 is given by this value that is g 2 and therefore, the whole area which lies between the limits. So, this area represents the possible resonator combinations which are stable; what does this mean?

This means if I have a resonator for which g 1 and g 2 is a point which is somewhere here in the shaded region, then this resonator is a stable resonate. If for the given resonator g 1 g 2, that is the x coordinate and the y coordinate is such that the point is here; then this is a unstable point or unstable resonate unstable.

So, for any given resonator, if we check the g 1 g 2 coordinates and if the g 1 g 2 comes here; then this is unstable, this is outside if it comes here this is also unstable. The area enclosed by the axis and the 1 by x curves here represent the stable region. So, the shaded region, therefore the shaded region represent stable resonators, stable resonators that is the importance of this resonator stability diagram, stable resonator.

So, for a given resonator whether it is stable or not is determined by this. What do we mean by stable resonator? A resonator in which we can find some rays which are confined for ever, that is a stable resonate. If you cannot find any ray, any type of ray propagating back and forth which will not remain confined, then it is a unstable resonate. So, now, we know a simple formula which indicates that, whether a given resonator is stable or not.

Now, in this diagram therefore, if we take plain mirror resonators; there are two points which are already marked here 1, 1. So, what does 1, 1 correspond to? So, g 1 is equal to 1 and g 2 is equal to 1; that means this implies R must be infinity if this is equal to 1 that means R 1 is equal to infinity and R 2 is equal to infinity. What is infinity? This implies these are plain mirrors; a plain mirror resonator , so plain mirror resonator. So, 1, 1 point here represents a plain mirror resonator, which is right on the edge of the stable region.

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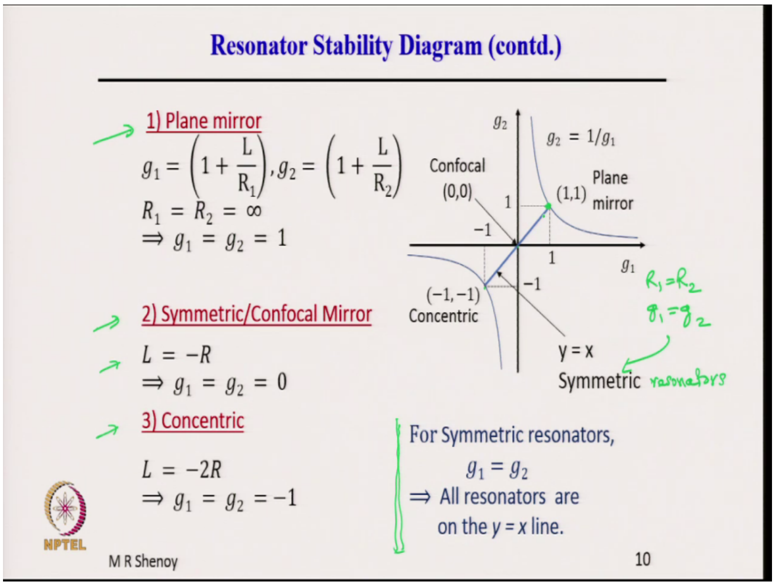

Let us look at other resonators, some more examples. So, here it is, the plain mirror resonator has g 1 is equal to g 2 is equal to 1 and therefore, we have this equal to 1. So, this is the point which corresponds to a plane mirror resonator. Now, for all symmetric resonators g 1 is equal to g 2. So, g 1 is equal to g 2 for all symmetric resonators; which means they are on the y is equal to x line which is shown here, y is equal to x, that all symmetric resonators, so symmetric resonators.

What is a symmetric resonator? A symmetric resonator is a resonator which has g 1 is equal to g 2 means, R 1 is equal to R 2; whenever R 1 is equal to R 2, that is if we have two mirrors of identical radius of curvature, whether it is plain mirror or spherical mirror, then g 1 is equal to g 2 and we call this as symmetric resonators, it is discussed here.

For symmetric resonators, g 1 is equal to g 2 and all resonators are on the y is equal to x line. Now, let us take another example, a symmetric confocal mirror resonator; for a confocal mirror resonator, in the last class I had shown that, L is equal to minus R minus sign comes, because the symmetric resonator comprises of two concave mirrors. For concave mirror the radius of curvature is negative and therefore, there is an additional minus sign; because distance L is always positive and therefore, L is equal to minus R, R itself is negative.

So, this gives g 1 is equal to g 2 is equal to 0. Look at the formula here, when L is equal to minus R; then we have 1 minus 1, therefore g 1 is equal to g 2 is equal to 0. And therefore, in the diagram confocal mirror is here 0, 0; g 1 equal to 0, g 2 equal to 0, the position of the confocal mirror marked on the stability diagram. If we take a concentric mirror resonator, then L is equal to minus 2 R; therefore if you substitute L is equal to minus 2R in this expression, we get this as minus 2 R here and therefore, this is minus 1.

So, L is equal to minus 2R, therefore this whole term is minus 2. So, 1 plus minus 2 is minus 1. So, g 1 is equal to g 2 is equal to minus 1 and that point is here. It is on the same y is equal to x; because we are looking at symmetric confocal and symmetric concentric mirrors. And of course, the plane mirror resonator comprising of two plane mirrors, all of them are on this y is equal to x line.

The confocal mirror is at the center, at the origin and concentric mirror is at here, at the point minus 1 minus 1. So, these are some of the common resonators which are marked on this; but if we take any arbitrary resonator with arbitrary separation which is different from R or 2R, then we can calculate g 1 and g 2 and mark the position of the resonator in the stability diagram. If it falls in the shaded region, that is the stable region; then the resonator is stable, otherwise it is unstable.

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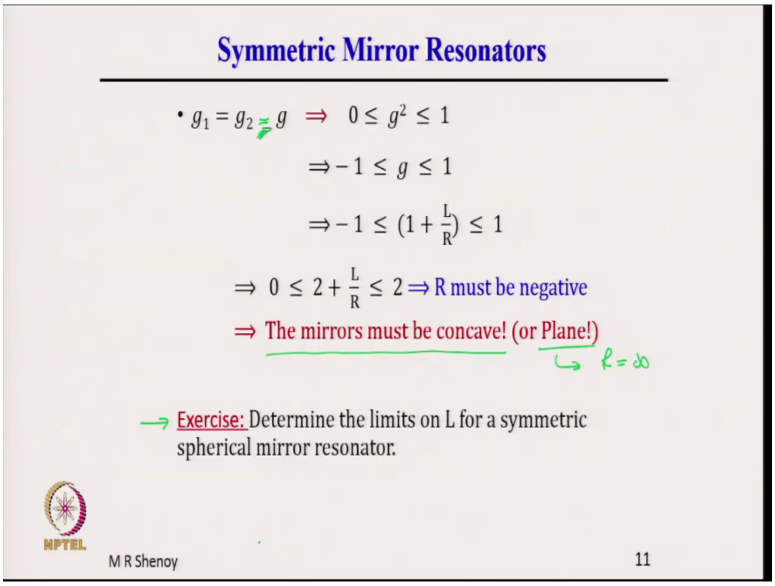

So, for the symmetric mirror resonators, g 1 is equal to g 2 and therefore, equal to let us say equal to g equal to g. Therefore, we write 0 less than or equal to g square less than or equal to 1; which implies g will lie between minus 1 and plus 1 that is minus one less than or equal to 1 plus L by R less than or equal to 1 or this implies 0 less than or equal to. So, we have added plus 1 to all the sides; 0 less than or equal to 2 plus L by R less than or equal to 2.

This implies that R must be negative, because this quantity can be less than or equal to 2; that means this quantity can be less than or equal to 0 and therefore, R must be negative or R can be infinity. If R is infinity, then this is 0; so this inequality is satisfied, otherwise R must be negative, so that we have this quantity less than 2. R is negative implies the mirrors must be concave mirrors or plane; this is the case where R is equal to infinity and therefore, L by R is 0.

So, this is R is equal to infinity. And therefore, when we have symmetric mirror resonators; they always comprise of two concave mirrors or two plane mirrors, usually two concave mirrors.

Now, with this equation here, there is a small exercise which is given; that is determine the limits on L for a symmetric spherical mirror resonator. So, you can find out what is the maximum value of L possible and what is the minimum L possible, so that they are in the stable region, alright.

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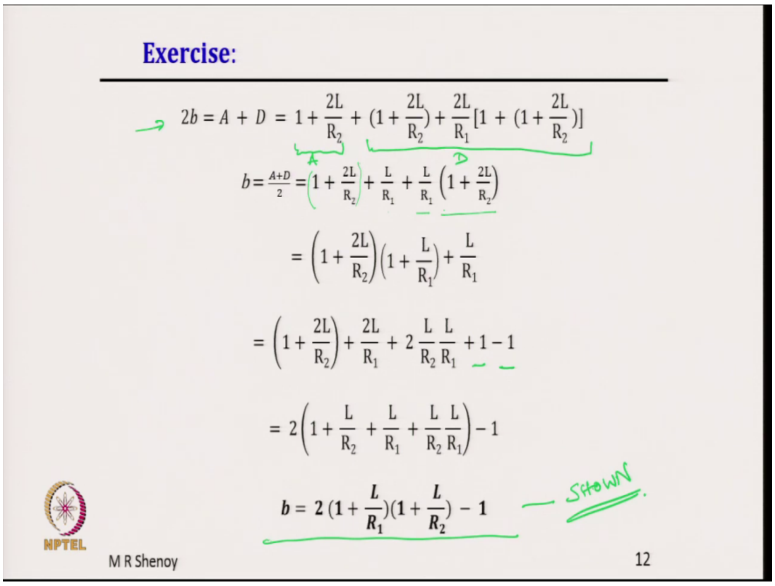

Finally, this is what I mentioned that, I said that I will show 2b is equal to A plus D, where this is the element A and all the rest of the element which is here is D. So, this is A and this is D and therefore, b is equal to A plus D by 2. So, you can simplify this 1 plus 1 is 2. So, there is 2 everywhere. So, A plus D by 2 will give us 1 plus 2L by R 2L by R 1 plus L by R 1 into this, which you can further simplify to this 1 plus 2L by R 2 is taken here together into 1 and then 1 plus L by R 1, because there is L by R 1 into 1 plus 2L by R 2.

So, this is taken together and then we have 1 plus 2L by R 2 into 1 plus L by R 1 plus this term L by R 1. Now, an important small trick is to add 1 and subtract 1, add 1 and subtract 1; then we can write this in the form 2 into this minus 1 and therefore, b is equal to 2 into 1 plus L by R 1. So, this product is 1 plus L by R 1 into 1 plus L by R 2. So, b is equal to this. So, this is what I had asked you to show. So, now, here I have shown.

So, that is that brings us to the end of the discussion on the resonator stability diagram. In the next lecture, I will pick up specific examples and see that for a given resonator, you can always calculate g 1 g 2 and find out whether they are stable or not and then we can also trace the rays using the matrix optics. So, we will do this in the next class.

Thank you.