Introduction to LASER
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Lecture - 14
Spectral Response of an Optical Resonator

Welcome to this MOOC on Lasers. So, today we will discuss the Spectral Response of an Optical Resonator. In the last lecture we introduced open resonators and then we discussed the various properties basically listed various properties and then one particular characteristic namely the resonance frequencies of the resonator we have got an expression for that and now today we will obtain the spectral response of an optical resonator.

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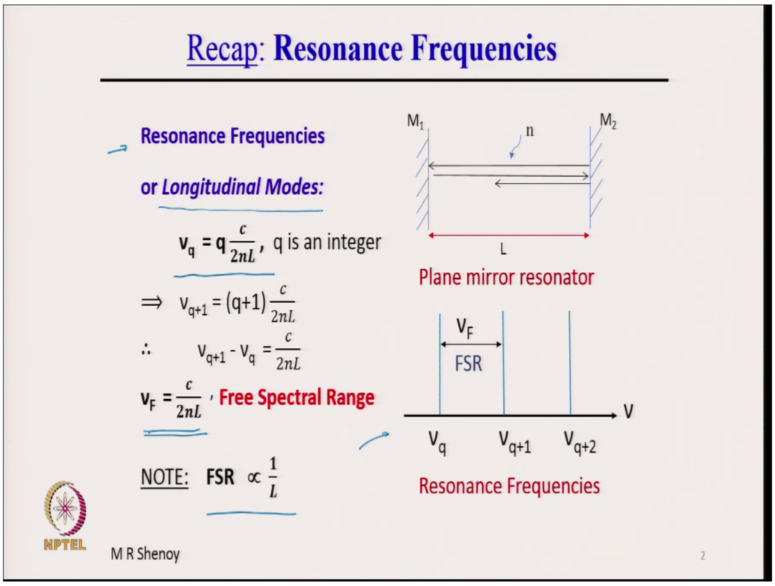

A very quick recap. So, the resonance frequencies or they are also called longitudinal modes of a laser are given by nu q is equal to q into c by 2 n L where q is an integer. We have seen that in the case of laser resonators q is generally very large number 10 to the power of 5 10 to the power of 6 and so, on.

If nu q is given by this if we replace q by q plus 1, we have expression for nu q plus 1 is equal to q plus 1 into c by 2 n L and therefore, the difference that is the separation or difference between the adjacent frequencies is given by nu q plus 1 minus nu q is equal to c by 2 n L and this is called the free spectral range is important to note that the free spectral range is inversely proportional to the length of the resonator.

If the length of the resonator is small then the FSR becomes large. So, that is illustrated here. So, nu q plus 1. So, free spectral range is the separation between adjacent resonance frequencies.

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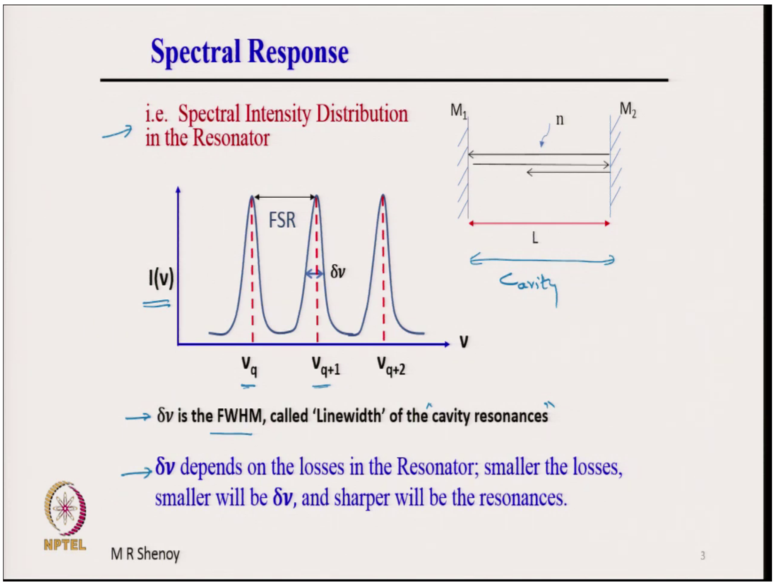

Now, the spectral response refers to the spectral intensity distribution in the resonator. That is spectral intensity distribution means, intensity distribution as a function of frequency of the radiation inside the resonator that is if we plot the intensity inside the resonator as a function of frequency then; obviously, at the resonance frequencies we will get maximum intensity.

Because that is where they resonantly build up that is they add in phase and build up resonantly at the resonance frequencies, but just adjacent to the resonant frequencies or just off resonance it does not suddenly go to 0 there is a certain amount of intensity even off resonance and the complete distribution is called spectral response of the resonator.

And the resonances are characterized by a line width delta nu which is the full width at half maximum FWHM called the line width of the cavity resonances. So, this resonator is also called as a cavity. So, its a cavity where light is trapped and therefore, the resonances are also called cavity resonances. Just to specifically differentiate between other resonances. So, cavity resonances referred to resonance inside a cavity or a resonator optical resonator.

Now, delta nu we will see this later that delta nu depends on the losses in the resonator; smaller the losses, smaller will be the delta nu smaller the delta nu means, narrower will be the resonance sharper will be the resonance and this is very important. And therefore, smaller the losses in the resonator, the resonances will be sharper and this has implications in laser physics. So, we will see these details as we go further.

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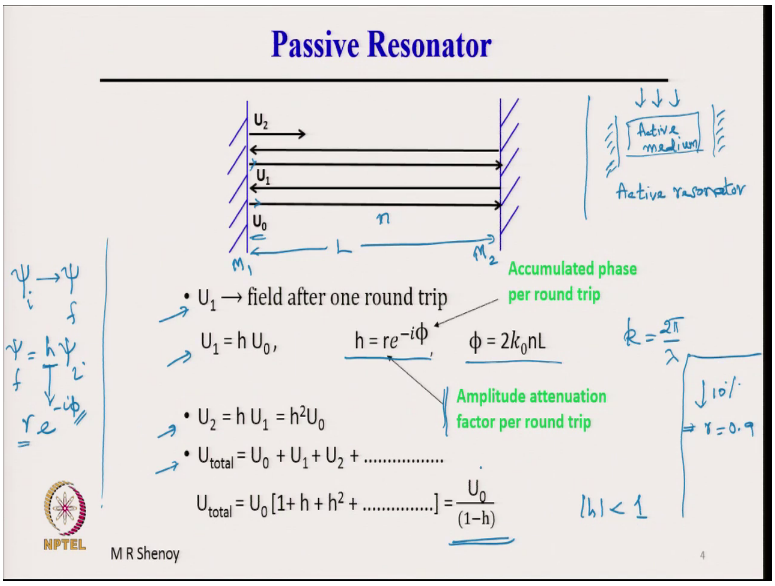

Now, let us consider a passive resonator right now we are looking at the optical resonator which is a passive resonator. So, passive means there are no active medium inside the resonator. So, if you have a resonator with an active medium inside, then this is called an active resonator particularly when the active medium is pumped.

So, if there is an active medium, then this is called an active resonator as opposed to passive resonator. So, this is active resonator this is just for comparison active resonator. So, laser is indeed an active resonator because laser is a resonator which contains an active medium that is pumped and therefore, laser can be called as an active resonator.

So, let us first look at the passive resonator which comprises. So, what is shown is a plain mirror resonator and if U 0 is the field of the wave which starts at a certain time t from mirror M 1 let us say at mirror M 1; M 1 and M 2 are the 2 mirrors. So, the initial field including amplitude and the phase is U 0, then the field propagates to the other mirror gets reflected back and comes back and U 1 represents the field after reflection after one reflection.

So, if U 1 is the field after one round trip. So, U 0 is here, U 1 is after one round trip. So, in one round trip; obviously, there are 2 lens propagation and then 2 reflections one at this mirror and the other at that method. So, we will see those details later. But if U 1 is the field after one round trip, then we can write U 1 as h times U 0 where, h can be represented as r into e to the power minus i phi where phi is the round trip phase.

So, you have a certain field distribution. So, we can call it by some other name. So, let us say psi is the starting field distribution, initial field distribution, then the final field distribution will be given by the initial field distribution plus the propagation phase accumulated plus any propagation losses associated. And therefore, this psi f the final field is written as a factor h into psi i where h contains an amplitude attenuation factor r and an accumulated phase phi.

So, we write this h. So, h is the complex number equal to r into e phi e to the power of minus i phi where phi takes care of the round trip accumulated phase and r takes care of the amplitude attenuation factor in one round trip and that is what is written here that h is equal to r into e power minus i phi. So, this r is the amplitude attenuation factor in one round trip and phi is the accumulated phase per round trip.

So, phi is equal to twice k 0 into n L because the separation is L. So, this separation is L and n is the refractive index of this medium k 0 is the propagation constant or phase constant which is equal to 2 pi by lambda where lambda is the free space wavelength. So, phi is equal to 2 k 0 into n L. In writing this we have neglected the phase at the mirrors phase change at the mirrors. We could also take care of the phase change, but it is not necessary when light undergoes reflection at a dielectric mirror.

Then there is no phase change and therefore, in this consideration we have not considered the phase change on reflection, but otherwise if its a perfect metal coated reflector then there will be a phase change of pi here and there will be a phase change of pi in the other mirror. And therefore, there will be a net additional phase change of 2 pi that that does not affect the calculations that is why right now we have not considered any phase change on reflection. So, we have considered phi as only the propagation accumulated phase in one round trip.

Now, U 2 that is after. So, U 1 it started propagating from here going back and then U 2 just after reflection at mirror M 1 is h times U 1, but U 1 is h times U 0 therefore, it is equal to h squared times U 0 and therefore, the total field after one round trip. So, this is the electric field with the amplitude and phase and the total field can be written as the sum of U 0 plus U 1 plus U 2 and so, on and that is equal to U 0 into 1 plus h plus h square plus etcetera and this is equal to U 0 divided by 1 minus h.

If mod h is less than 1, then the summation adds to U 0 divided by 1 minus h. So, in this case mod h is less than 1 because h mod which is simply mod r and r is an amplitude attenuation factor. So, it is an attenuation factor which means if the amplitude drops down by 10 percent reduces by 10 percent. So, let me show here if the amplitude drops down.

So, U drops down by 10 percent, this implies r is equal to 0.9, r is equal to 1 means there is no change in the amplitude r drops by 10 percent; 10 percent attenuation in terms of amplitude then r is equal to 0.9. So, therefore, mod h is less than 1 and therefore, we can write the sum as U 0 divided by 1 minus h alright.

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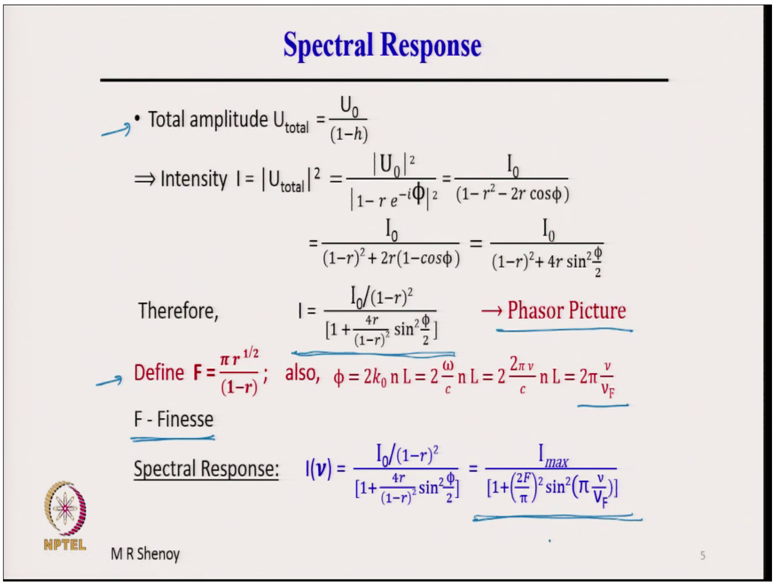

So, the total amplitude here u total is equal to U 0 divided by 1 minus h. Therefore, the intensity is equal to mod square of the amplitude which is equal to mod U 0 square divided by 1 minus r into e power minus i phi that is h mod square which is equal to if we call this as I 0 intensity I 0, then divided by 1 minus r minus twice r cos phi and that can be rewritten as 1 minus r the whole square plus 2 r into 1 minus cos phi and using 1 minus cos phi cos theta is equal to 2 sin square theta by 2.

So, you can substitute and write it in this form here and therefore, the final expression for the response in terms of the phase change phi and the amplitude attenuation factor r is given by I is equal to I 0 divided by 1 minus r whole square the whole divided by 1 plus 4 r by 1 minus r whole squared into sin square phi by 2. So, there are no approximations made here. So, if we now define F is equal to pi r to the power half by 1 minus r where F is called the finesse there are different ways of defining finesse.

But here we define F in this fashion we will see the advantage of defining F in this fashion. And then we also note that find the round trip phase is equal to 2 k naught n into L which is equal to 2 omega by c into n L, k naught is omega by c omega is 2 pi nu by c and therefore, this phi can be written as 2 pi into nu by nu F. Nu is the frequency and nu F is the free spectral range. And if you substitute for phi in this expression here, then we have the numerator is the same in the denominator in place of sin square phi by 2 we have sin square pi nu by nu F.

So, we have now written it in terms of the frequency because we are interested in the spectral response which means intensity as a function of nu. Here it is given intensity as a function of phi now I nu gives the intensity distribution as a function of nu. So, this is called the spectral response. So, this expression gives us the spectral response of the resonator. Now, let us have a look at the phasor picture that gives us some idea before I discuss the details of this result.

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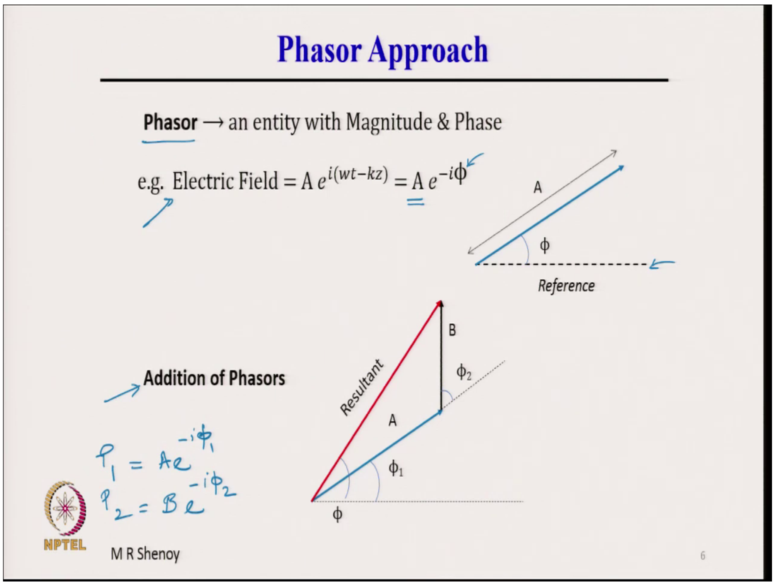

So, let us have look at the phasor approach which is followed in understanding the resonator. So, though for those of you who are not familiar phasor is an entity with the magnitude and phase just like vector is a quantity or an entity with the magnitude and direction. So, phasor is with magnitude and phase.

Typically electric field is a phasor because it has characterized by an amplitude and a phase what is shown is a plane wave here a field distribution here propagating plane wave. So, this can be is written as A e power minus i phi which means it is characterized by an amplitude and a phase phi.

So, this phi could be phi of t its changing with time, but at any instant it is the phasor is determined by its phase at that instant. Now, depending on a reference phase you can represent such a phasor just as vector the length of the arrow is determined by the amplitude

A and the phase angle phi with respect to a reference and therefore, if you have two phasors. So, what is shown is just like vector addition of phasors. So, the first phasor is if you had A into e power minus i phi 1, then phi 1 is the phase of the first phasor.

So, if I want to call this as P 1 then and the second one is P 2 which is equal to some other amplitude B into e to the power of minus i phi 2, then we can add these two phasors as shown here just as we add the vectors, but now the angle is determined by the phase with respect to the reference. So, initially initial phase phi 1 is with respect to a certain reference usually we consider this as 0.

The next phasor is placed at an angle where the phase angle is phi 2 is with respect to the previous phase angle and therefore, the phasor is shown like this and the resultant phasor is obtained by connecting the tip of. So, the starting point to the tip of the second phasor. Just like vector, but here the angles are determined by the accumulated phase or round trip phase. This will become more, clear when I take further examples.

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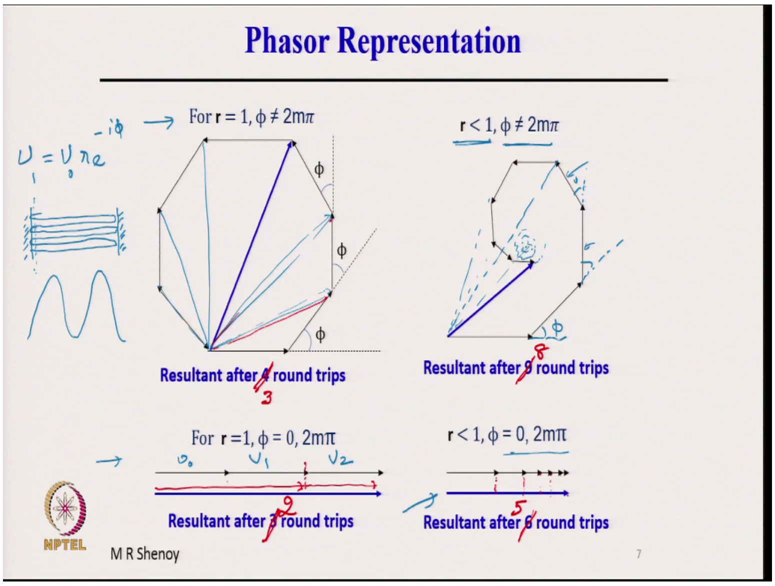

So, let us look at a situation where r is equal to 1 which means there is no amplitude attenuation factor. So, recall that U 1 is equal to U 0 into r into e to the power of minus i phi that is h. If r is equal to 1, the magnitude of U 1 is the same as magnitude of U 0 which means all these magnitudes are the same, but phi the round trip phase phi is not zero or integral multiple of 2 pi when phi is finite then if we start adding them we can see the phasor after two round trips.

So, this is the first initial phasor after one round trip phasor which means after one round trip we are here, the net phasor is here after two round trips the net phasor. So, net phasor means what? So, if the length of the phasor tells you the magnitude and the angle with respect to the original reference tells the net accumulated phase and after three round trips we are here and so, on.

So, we see that in this case in such a situation the phasor goes on evolving into a circle circular path. In other words the amplitude increases and after some time it comes back and then the amplitude goes down. Why does it go down? Because there is a phase difference between the initial field and after one round trip.

So, the phase difference accumulates and therefore, the net amplitude becomes more and less more and less. So, it goes on amplitude goes on increasing decreasing increasing decreasing in spite of the fact that it is inside the resonator. So, what are we showing here? Inside the resonator the field U 0 after one round trip is U 1, after another round trip is U 2, after another round trip is U 3 and so, on.

Therefore, the net field here is given by the resultant phasor. The importance will become clear if you simply assume that r is of course, 1 there is no attenuation, but still amplitude is not building up. Why? Because there is a phase difference phi. Assume that phi is equal to 0 then the first phasor is here. So, this is the first phasor U 0, next phasor U 1 because r is 1 and phase is 0 therefore, U 1 and U 2 and so, on.

So, what about the resultant? Resultant will simply add up. In other words inside the resonator if there is no propagation loss if there is no loss and if the round trip phase is an integral multiple of 2 pi, then the intensity will build up because the amplitudes are simply adding up and the intensity will go on building up.

So, what does this tell? This indicates that the round trip phase equal to integral multiple of 2 pi the condition which I had imposed in the last lecture in determining the resonance frequencies can be seen here, what is this resonance? For building up the round trip phase must be integral multiple of 2 pi.

And it clearly shows that if the round trip is not integral multiple of 2 pi, then there will be a phase lag between the adjacent that subsequent round trip fields and therefore, the net resultant will oscillate between a certain maximum and minimum it will not build up into a large intensity.

However, if we see there is another example which is shown here. So, what is shown here is r is less than 1 which means you can see that the subsequent length of the arrows are smaller because let us say r is 0.9, then this is 0.9 of this, this one is 0.9 of this, this amplitude is 0.9 of the previous one, but it is not just the 0.9 it is also phi is not neither 0 nor integral multiple of 2 pi which means there is a finite phi. So, this is phi, this is phi, all the angle is the same phi, so this is phi.

Now, in this case you can see that the result end was if we had taken after 3 round trips then resultant was here, if it was after 4 round trips result was here, if after 5 resultant is here after 6, after 7, after 8 and after 9. So, the resultant continuously and you will see that after some time the resultant will. So, this will finally, if you go on plotting then this will finally, lead to some point here and the resultant will be that much. There is no building up of the amplitude if phi is not an integral multiple of 2 pi.

But look at the last example, the last example is of course, r is less than 1 as we will see all practical resonators will have r less than 1 there are no resonator which are lossless and the lossless resonators have of no use because we need some output from the resonator and any output any useful output from the resonator is a loss as far as the resonator is concerned. And therefore, all practical resonators will be having finite loss and r is always less than 1.

So, if r is less than 1, but if phi is 0 or an integral multiple of 2 pi, then you can see that the subsequent amplitudes add up, but the amplitudes are dropping down and therefore, finally, you will have a reasonably strong resultant amplitude. Oh just a minute I think ok this is resultant after 2 round trips because this is the first one U 0. So, the initial phasor after 1 round trip is the phasor U 1 therefore, up to this the resultant sum of this is up to after 1 round trip U 0 plus U 1 is the total field or total phasor.

After 1 round trip from here to here and therefore, after 2 round trips it is U 0 plus U 1 plus U 2. So, up to this the total here is resultant after 2 round trips. So, this must be 2 round trips. Similarly, I think let us look at this. So, this is the initial phasor, this is the first. So, this is after 1 round trip, this is the 2nd one and therefore, this is after 2 round trip and therefore, the blue line which is shown here is after 3 round trips. So, this is three round trips.

Similarly, here this is the initial one after 1 round trip, the sum of the phasor is up to this, after 2, after 3, after 4 and after 5. So, the given the shown phasor is after 5 round trips and similarly this must be initial phasor after one round trip, after 2 round trip, after 3, after 4, after 5, after 6, after 7, after 8. So, this phasor is after 8 round trips ok that is correct.

So, this phasor picture illustrates the importance of phase round trip phase being 0 or integral multiple of 2 pi in building up of the resonance. Let us come back to the mathematics now.

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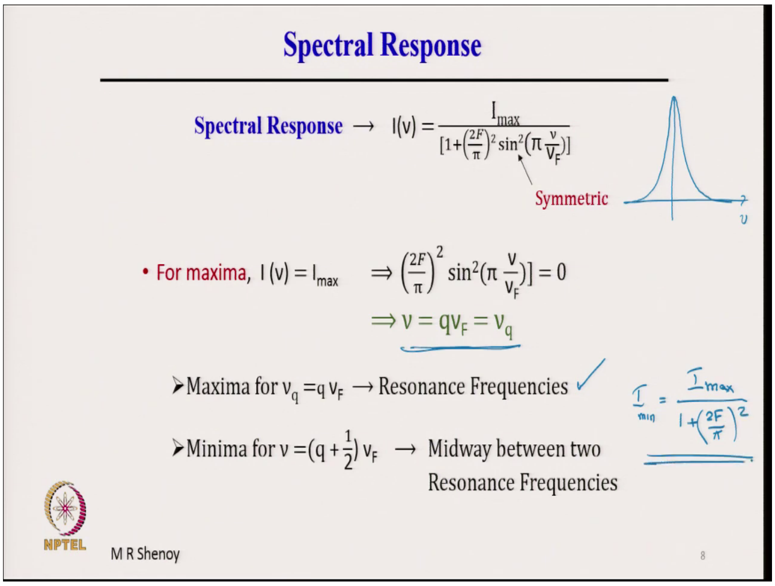

So, we have derived the expression for the spectral response here, note that the response is symmetric because this is a square function here symmetric in nu which means if I were to plot this resonance as a function of nu, then it will be symmetric about the resonance. So, something like this.

So, for maxima, the maxima is when all are positive quantities therefore, if the denominator the second term in the denominator if this becomes 0, then we will have I nu as I max that is what is written here for I nu will be equal to I max for the second in the denominator being 0 which implies that nu is equal to q times nu F is the same condition that we have derived namely maxima occurs for resonance frequencies.

We had already derived this by assuming that the round trip phase must be integral multiple of 2. The minima will correspond to when the denominator second term becomes largest, then we will have minima and when will this be largest at that frequency when the sin squared term becomes 1.

We will have the largest value which is I max divided by 1 by 2 F by pi the whole square. So, in between that occurs at a nu is equal to q plus half that is half integral multiple of a nu f.

So, if you substitute for nu is equal to q plus half times nu F you will see that this sin square term will become 1 and we will have I min is equal to 2 I minimum is equal to I max divided by 1 plus 2 F by pi the whole square.

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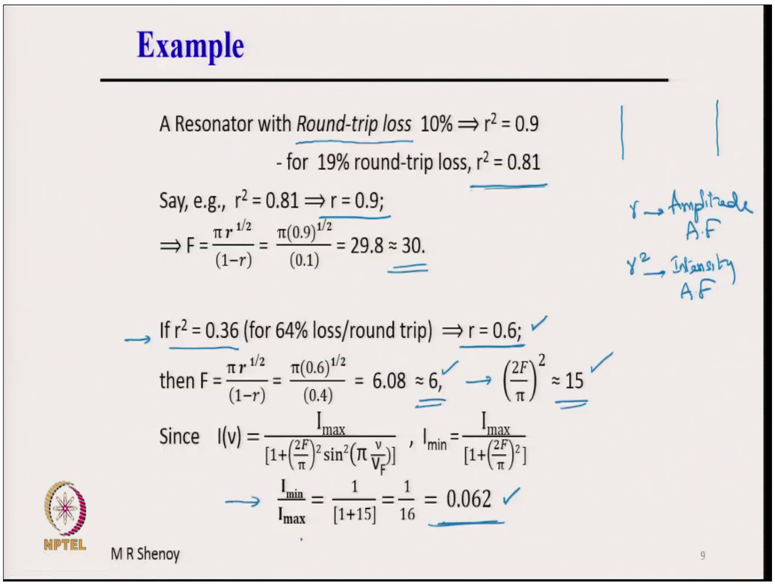

Let us take an example consider a resonator with a round trip loss of 10 percent. Round trip loss of 10 percent what does this mean? This means round trip loss refers to loss of energy in all optical measurements we measure energy or power or intensity.

We do not measure the amplitude of the electric field of light, we measured the measurable parameter is the intensity or the power. And therefore, the loss round trip loss here refers to the energy or power and therefore, if it is 10 percent it means r square is equal to 0.9 because r is amplitude attenuation factor. So, this is amplitude attenuation factor. So, let me write it as AF then r square will be intensity attenuation factor.

So, the intensity attenuation factor and therefore, if the round trip loss is 10 percent if it is given that a resonator has a round trip loss of 10 percent; that means, r square is equal to 1 minus 0.1 which is 0.9. Similarly, if it is 19 percent round trip loss you can imagine why I have taking this example of 19 percent because if 19 percent round trip loss means 1 minus 0.19 that is r square that is r square is equal to 0.81.

Why am I interested in 0.81? Because if r square is equal to 0.81 then r is equal to 0.9. So, we can easily calculate F therefore, F is equal to pi into r to the power half divided by 1 minus r, r is 0.9 which is pi into 0.9 to the power half divided by 1 minus 0.9 which is 0.1 which comes out to be 29.8 or approximately 30. So, this is the kind of number that we have got for the finesse, finesse of the resonator.

Usually the finesse can vary from few tens to 10 million or larger depending on the loss in the resonator. So, a resonator which has finesse of a million let us say are called high finesse resonators. So, the normal resonators may have a few tens or a few hundred as the finesse, if r square is equal to 0.36 that is 64 percent loss per round trip here earlier we had seen 19 percent loss per round trip.

Now, if the loss is 64 percent again I have taken a nice number so, that I get square root of this as r. So, r square is equal to 0.36 which means r is equal to 0.9 and then if you substitute in this expression you will get F is equal to 6. If we know F, then we can calculate 2 F by pi the whole square in this case this comes out to be 15 for F is equal to phi.

So, why I am interested in this term? Because I min is equal to I max divided by 1 plus 2 F by pi the whole square. And therefore, in the spectral response we can determine the minimum as 1 divided by 1 plus 15 which is 1 by 16 which is equal to this much. So, now, my objective is to plot the spectral response for typical numbers.

So, that is why we have considered a case although this is slightly a lossy resonator, but just to illustrate the point, we have considered an example where r is equal to 0.6; r is equal to 0.6 which gives us the finesse as 6, and then it gives us this 2 F by pi whole square as 15 and therefore, the minimum value of the intensity transmission is 0.062. So, let us plot the spectral response for this resonator.

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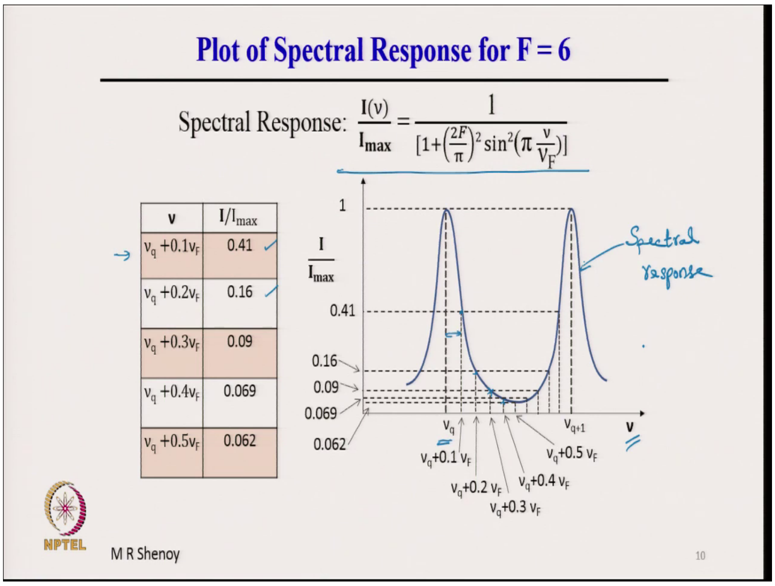

So, a plot of spectral response for F is equal to 6. So, how would you go about this? So, this is the expression for the spectral response. So, we have just a normalized with respect to I max so, that the maximum value is 1. So, the maximum value instead of I max if you normalize with I max, then maximum is 1. So, at nu is equal to q times nu F we have 1 the denominator is 0 q times nu F therefore, d sin square sum pi which is 0 and therefore, the second term in the denominator is 0 and therefore, you have the maximum as 1.

So, this is at nu is equal to nu q we have maximum. Now, you shift slightly to the other side that is you find at a frequency which is 0.1 times nu F. Please see nu F is the separation between two resonances. So, 0.1 times that is this. So, nu F. So, this point is nu q plus 0.1 times nu F. So, if we calculate this intensity we get this as 0.41. So, I would advise you to do this calculation and check and slightly further that is plus 0.2 times nu F, then the intensity drops down to 0.16 and so, on and finally, the minimum which we have calculated which happens midway.

We had already seen that the minima minimum occurs midway we have already written here. So, it is it occurs midway between the two resonances. So, when this is 0.5 q plus 0.5 times nu F and that is what we are seeing here that the minimum is 0.062 that is what we just now calculated here 0.062 is the minimum value. And then I have shown the its reasonably to scale and then you link all these points. So, these are the points which are calculated which are here and then you join them half of width is there.

So, the other half is of course, symmetric. So, you can plot the spectral response. So, this is the spectral response that is intensity distribution inside the resonator as a function of frequency spectral response of the resonator ok.

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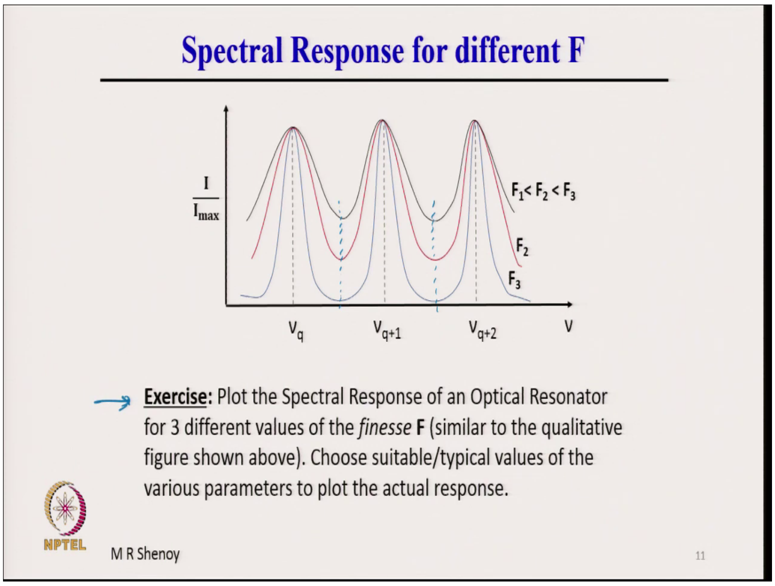

So, there is an exercise given here if you take for different F values, if F is higher that is if the finesse is higher the resonance will become narrower and narrower, we will just work out the expression for the line width of the resonances.

So, for different values of finesse if you plot then you would get typical what is shown is a qualitative graph if you plot this I by I max as a function of frequency for different values of F, you would see that all the peaks will coincide at the resonance frequencies for a given resonator. And the minimas will also occur at the same value of the frequency which is midway between the two resonances. So, here is an exercise that plot spectral response of an optical resonator for 3 different values of the finesse F.

So, we had shown this response for F is equal to 6. If you had taken F is equal to 20 or 30, then this would have become very narrow, if you had taken F is equal to 3, then it would have become like this, it would have been flatter or more, broader and the resonances would have been wider and that is what is shown in this qualitative graph.

So, choose suitable or typical values of the various parameters to plot an actual response. You can do is very easily using a computer or even using calculator you can calculate all the points and simply plot.

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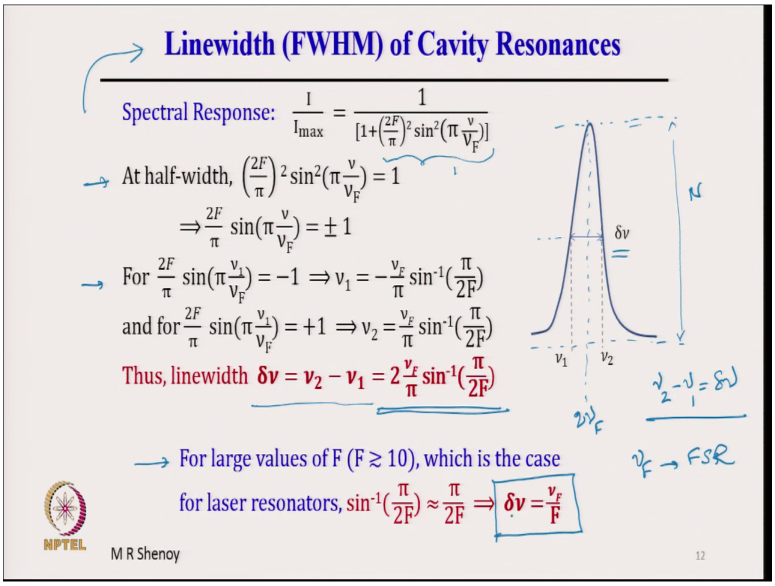

Finally, we come to the line width of the cavity resonances. Line width refers to the full width at half maximum of these resonances. So, it is illustrated here. So, this is the resonance and the frequency resonance frequency is q times nu F is the resonance frequency.

Now, the line width refers to full width at half maximum. So, this is we have already discussed about FWHM. So, if this height is some number and then at n by 2 if you drop down and find out the full width of the resonance, then that gives us what is called line width of the cavity resonances delta nu.

So, mathematically how do we go about it? The spectral response is given by this expression the response will drop down to half of its value from the maximum. Maximum means I max, so which is 1. If you consider I by I max the maximum is 1 therefore, when will this become half?

This the quantity will become half when this term in the denominator becomes 1 then it will be 1 plus 1 2 or 1 by 2 and that is what is written at half width 2 F by pi whole square sin square pi nu by nu F is equal to 1 this implies that we take the square root.

So, you will get plus minus 1 sign and if we take the plus sign this becomes plus minus 1, the plus sign gives because there are two solutions nu 1 and nu F because it says symmetric function and therefore, at the half value corresponds to two frequencies nu 1 on one side nu 2 on the other side of the resonance. So, the minus sign corresponds to this that is at a lower frequency and the positive sign corresponds to the higher frequency. So, this nu 2 minus nu 1 will give us delta u.

So, nu 2 minus nu 1 is delta which is the line width. So, nu 1 is given by minus nu F by pi into sin inverse of pi by 2 f from this expression here and nu 2 is plus nu F by pi into this and therefore, the line width delta nu is equal to nu 2 minus nu 1 is given by this expression. So, rigorously correct expression which gives the line width of the cavity resonances. So, if a cavity is given which means you know already the nu F free spectral range and the finesse F and then you can calculate the line width of the cavity resonance.

For large values of F typically, F greater than are of the order of 10 which is the case for all laser resonators sin inverse of pi by 2 F will be nearly equal to pi by 2 F indeed you can put some numbers for F and see that this will match correct to second or third decimal. So, its nearly equal to pi by 2 F and then once you substitute instead of signing the pi by 2 F pi pi cancels 2 2 cancels and what you are left with is a simple expression for line width namely delta nu is equal to nu F by F.

Therefore, given a cavity given a resonator which means you already know nu F because nu F is by 2 n L. So, given a resonator means L is given. So, you know nu F. If the loss in the resonator is specified, then you also know the finesse there could be some other parameter which specifies the loss we will see shortly that the finesse can be calculated by other measurable parameters such as the cavity lifetime. I will discuss this in the next lecture.

And therefore, the line width of the resonance is given by a simple formula nu F by F that is FSR divided by finesse. So, FSR here divided by finesse. This tells us very clearly which we had already seen qualitatively that as F becomes larger and larger the resonance become sharper and sharper or delta nu becomes smaller and smaller with this picture we will stop here and continue in the next class.

Thank you.