Introduction to LASER
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Lecture - 10
Laser Rate Equations: 4-Level System

Welcome to this MOOC on LASERs. In the last lecture we discussed about Rate Equations in a 3-level system and today, we will take it up further to the 4-Level System.

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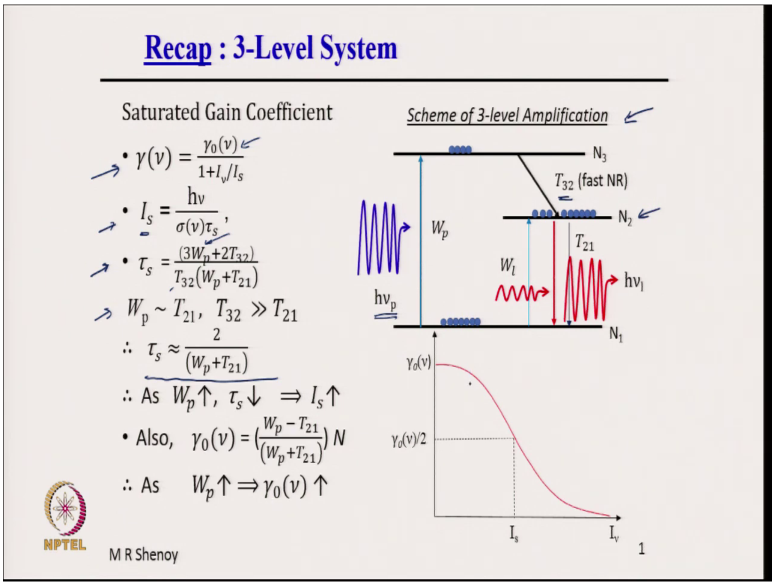

First, a very quick recap of the 3-level system. So, the scheme of amplification is shown here, that is the 3-level amplification scheme where you consider 3 atomic energy levels of the system. So, the ground state and excited state, upper excited state.

A pump of energy h nu p which raises atoms from the ground state to the third excite, second excited state and from there they make rapid transitions to the level e 2 with population N 2 here and then this is the level which is the meta stable state or which is the level that has a longer lifetime, because the level has a long lifetime.

Atoms tend to accumulate here and as qualitatively shown at steady state the number of atoms N 2 can be greater than number of atoms N 1; if we pump it sufficiently hard, because the transition T 32 is a fast usually non radiative transition. The number of atoms at level 3 or N 3 is much smaller usually very small, because as soon as the atoms reach there they go to level 2 by a fast non radiative transition. So, the scheme illustrates and we had also obtained an expression for the gain coefficient and the expression for the saturated gain coefficient is here, gamma of nu is equal to gamma 0 of nu gamma 0 of nu is the small signal gain coefficient.

So, this is the small signal gain coefficient divided by 1 plus I nu by I s, where I nu is the intensity of the radiation which is being amplified and I s is the saturation intensity. The expressions for I s are given here h nu divided by sigma of nu, sigma of nu is the cross section and tau s is the saturation time defined in this way. So, we have made this derivation and we have seen this.

Now, since T 32 is much larger than T 1 so, here T 21, that is here T 32 is much larger than T 21, this is fast, this is slow, T 21 is slow then we can simplify this. Further, W p is of the order of T 21 we have seen that at threshold W p is equal to T 21 and above threshold W p may be slightly greater than T 21, but it is of the same order as T 21.

So, W p is of the same order as T 21, but T 32 is much greater than T 21 and therefore, this W p here, 3 W p here can be neglected in comparison to T 32 and then T 32 and this T 32 cancels and therefore, we have tau s is equal to is approximately equal to 2 by W p plus T 21.

This clearly shows that as W p increases that is the pumping rate increases tau s drops down and I s goes up, because in the expression for I s saturation intensity tau s is in the denominator. So, similarly for gamma 0 of nu is given by this expression and as W p increases, gamma 0 would also go up and therefore, the variation of gain for different pumping rates would be as shown in this diagram here.

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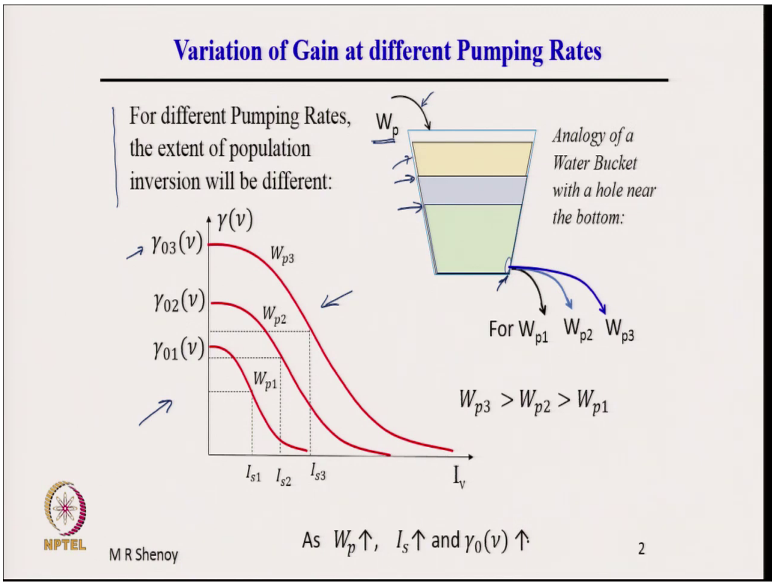

So, this clearly shows that as the pumping rate increases the small signal gain coefficient gamma 01 gamma, 02 gamma, 03 will be higher and the corresponding saturation intensities are also higher. So, they are qualitatively schematically shown in this diagram how the gain curve changes with the pumping rate. So, for three different pumping rates we have shown that the extent of population inversion will be different at different pumping rates and therefore, you have different curves.

Now, this extent of population inversion will be different can be easily recognized or can be easily identified if we take a simple example of with analogy with a water bucket with a hole near its bottom. So, there is a water bucket here. So, this is the water bucket in which there is a small hole near the bottom, if you pour water into the bucket water would go out from the hole and when the rate at which water is poured into the bucket becomes equal to the rate at which water goes out of the bucket we will have a steady state.

For example if we pump at a rate of W p 1 that is a certain rate at which water is pumped into this bucket then the level will build up to a certain level, the level will go up to a certain height so that the pressure with which water goes out of this hole will be such that the quantity of water going out rate at which water goes out will be equal to the rate at which water is poured into the bucket.

If we start pouring at a higher rate then naturally the level has to go to a higher level now, higher height so that water goes out with higher pressure, so that the rate at which water goes out increases. So, the point is depending on the pumping rate, the water level will be different in the bucket to reach steady state that is the rate at which water goes out will be equal to the rate at which water comes in then we have a steady level in the bucket.

Same thing is true here. In the atomic system if you pump harder then the extent of population inversion will be different. The extent of population inversion can be compared with the level of water here and therefore, the small signal gain coefficient will be different and then the gain varies as per the expression for the saturated gain coefficient and as we have discussed as W p increases I s also increases and gamma 0 of nu increases alright.

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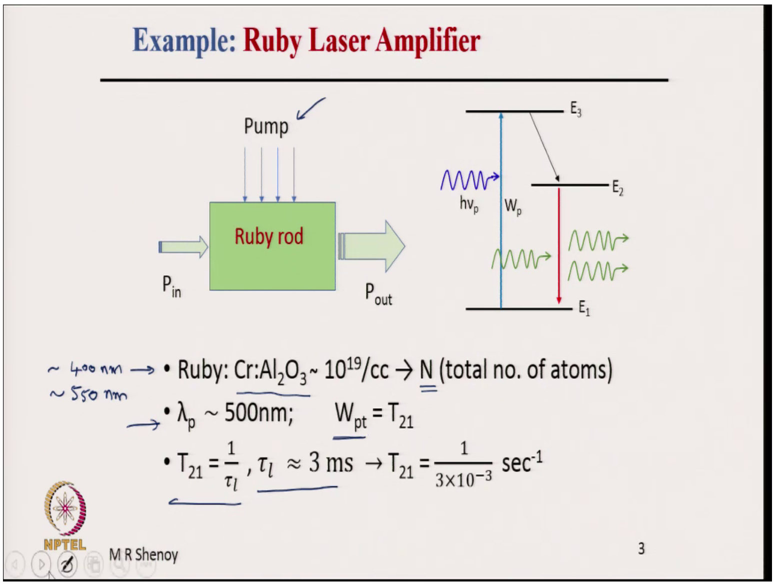

Let us take example of the ruby laser amplifier. We had mentioned that ruby laser is a 3-level example of a 3-level system and let me take the example and calculate what kind of pumping power is required to have lasing action in a ruby laser.

So, what is shown schematically is the ruby laser. There is a ruby rod ruby is a chromium doped alumina. The color, the pinkish red color of ruby comes from the chromium doping and typical doping concentration is 10 to the power of 19 per cc and that is our N that is the total number of atoms.

So, these are the atoms which are participating in the interaction. It is the three level which we are discussing are the levels of the chromium ion. So, in the presence of a pump, pump is generally a flash lamp. So, this could be a xenon flash lamp or krypton flash lamp and ruby has pumping bands of blue and green around 400 nanometer or so 400 nanometer or 550 nanometer so approximately, 500 nanometer so that is why we have taken lambda P of the order of 500 nanometer.

So, there are actually two absorption bands as we have seen earlier. So, the pump wavelength could be of this order and near threshold W pt is equal to T 21 and T 21 we know, because it is T 21 is equal to 1 by lifetime. Lifetime is a measurable quantity. So, for ruby it is typically 3 millisecond. So, we know the value of T 21 which means we know the pumping rate, the threshold pumping rate W pt.

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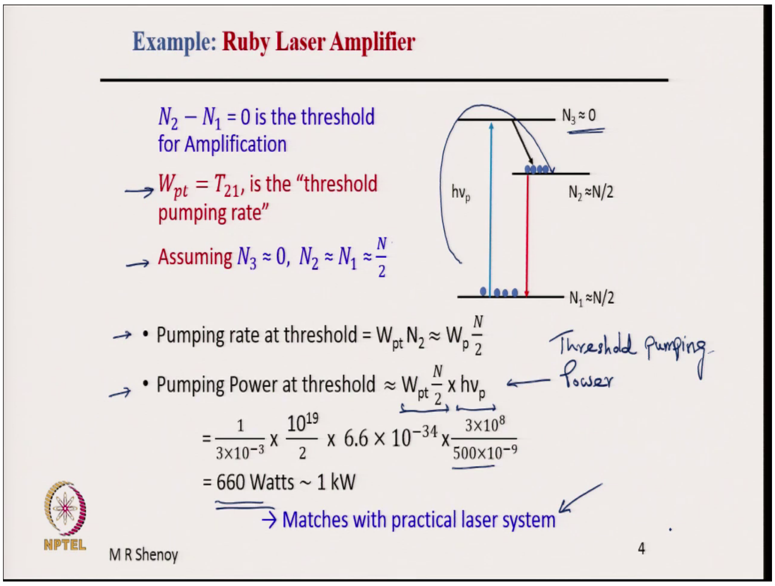

So, let us see what kind of power is required to reach the threshold. So, N 2 minus N 1 equal to 0 is the threshold for amplification that is just before N 2 becomes greater than N 1 and corresponding to that the pumping rate is W pt is the threshold pumping rate. Assuming N 3 is equal to 0, N 3 is the upper level.

Please see the mechanism that the atoms are excited to this level, but from here they rapidly go down here so that at any instant the number of atoms in this excited level, third excited level or the pump level N 3 is approximately 0 or much smaller compared to these and if N 3 is 0 then N 2 nearly equal to N 1 equal to N by 2.

N is the total number of atoms at threshold N 2 minus N 1 is 0 therefore, N 1 must be nearly equal to N 2 must be equal to N by 2 that is 50 percent of the atoms will be here and 50 percent of the atoms will be here, neglecting the presence of any atom at N 3. Therefore, the pumping rate at threshold is given by W p t into N 2. W p t into N 2 gives us the number of atoms to be raised per unit time to achieve this kind of, achieve the kind of a population inversion or just threshold when N 2 is equal to N 1.

Therefore, the corresponding pumping power is given by multiplied by, so this is the number. So, this is the number of atoms to be raised multiplied by the energy of photon raised per second therefore, energy per second will be the power. So, this is the power. So, this is the threshold pumping power. Threshold pumping rate multiplied by the photon energy of 1 photon gives you threshold pumping power maybe, I have written somewhere pumping power.

So, we are now substituting W p t we had shown that is equal to T 2 1 which is equal to 1 by tau l this is N N by 2 is N 2 multiplied by h Planck's constant into nu P the pump wavelength which is C by lambda; lambda we have assumed that 500 nanometer and therefore, this comes out to be 660 and 60 Watts or of the order of 1 kilo Watt. So, this matches with that of practical systems.

Typically, ruby lasers require a threshold pumping power of the order of kilo Watts. So, a very simple calculation tells us that the numbers are alright it is of the order of 1 kilo Watt. We will see that if we go to a 4-level system the pumping power required is much smaller that is, because we are creating population inversion between two excited levels. It is normally we know that the number of atoms in the ground state is much-much higher than those in the excited states.

So, if you want to create population inversion between a ground state and an excited state more than 50 percent of the atoms from the ground state have to be raised to the excited state and therefore, the pumping power required is very high in a 3-level laser.

So, in a 4-level laser the population inversion is created between two excited states therefore, even if you have a number like 100 in one level and 90 in the lower level. You still have population inversion, that is the advantage of a 4-level system. So, will the practical lasers such as helium neon lasers, Nd YAG lasers are 4-level lasers and we will take up the 4-level system.

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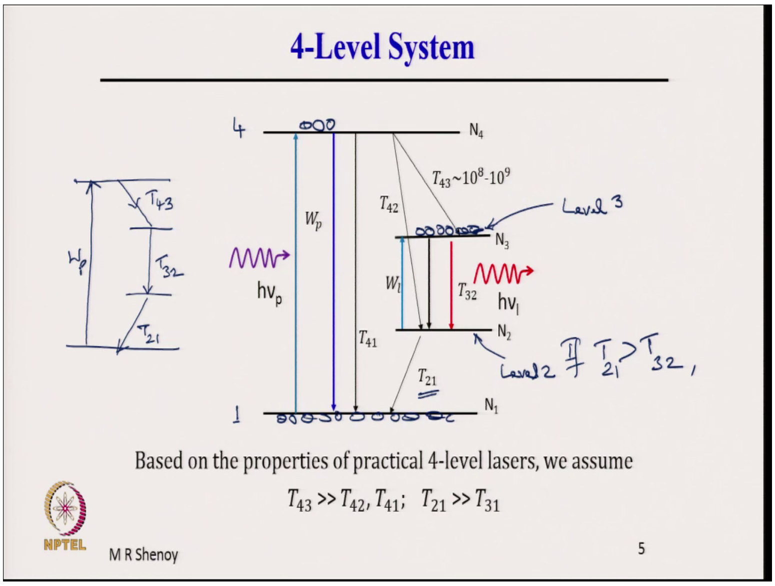

So, the 4-level system; the scheme is this that this is the ground state from where atoms are raised to an excited state level 4. So, level 1. So, this is level 1, 2, 3, and level 4. From there the atoms rapidly come down to level 3. So, this is level 3 and if the lifetime of level 3 is large or if level 3 happens to be and a meta stable state then atoms can get accumulated in this level here.

So, we have large number of atoms here; the ground state and then atoms are raised to the upper state here, from there they rapidly come down to this state and if the level has a long lifetime then atoms can accumulate here. When they come down to level 2, if T 21 is a fast transition that is whatever comes here, immediately goes down to the ground state that is if T 21 is greater than or much greater than T 32.

Then we will have net accumulation here and we will have population inversion between level 3 and level 2. So, this is level 2. This is what I mentioned that you can create population inversion between two excited state by raising much smaller number of atoms.

So, this is the scheme of amplification in a 4-level system. So, although there are transitions from 41, T 41, T 42, and T 43 there are 3 transitions which 3 types of transitions spontaneous transitions that can take place from level 4, but in practice T 4 3 in practical lasers, most of the practical lasers T 43 is much larger, the rate is much larger compared to T 42 and T 41.

T 42 T 41 are of the order of 10 power 4, 10 power 5, 10 power 6, but 10 power, T 43 is much larger of the order of 10 power 8 or 9 and therefore, in practice we can neglect T 41 and T 42 and therefore, if we neglect those then our transition, our levels, our transitions will remain only this.

So, we are raising atoms here, then they come down by this way only. The other two transitions T 42 and T 41 we can neglect and then this comes down here and finally, goes back. So, this is T 21, this is T 32, we are showing spontaneous transitions and this is T 43 and this is the pump W p. So, this is the 4-level scheme, it does not mean there are no other transitions, but other transitions can be neglected in view of the practical numbers alright.

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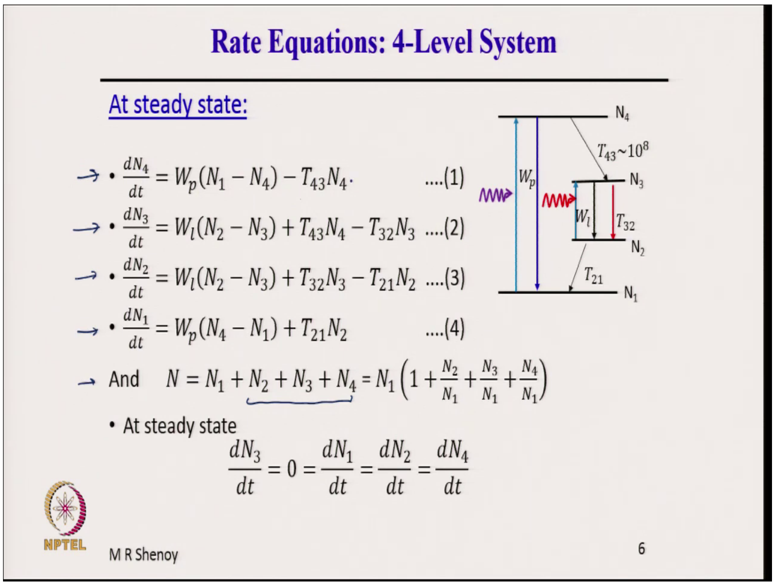

So, now we straight away come to writing rate equations for such a 4-level. You can see here I have shown only T 43 and T 21 the other two namely T 41 and T 42 I have not shown at all and therefore, we can write the rate equations at steady state d N 4 that is the population of level 4 d N 4 by dt rate of change of the population is W p into N 1 minus N 4. We can now directly write, because we know that W p is the pumping rate here.

So, it raises atoms from N 1 and therefore, W p into N 1 is the number of atoms which are reaching here W p into N 4 is the number of atoms which go out and therefore, the net will be, W p into net addition will be W p into N 1 minus N 4 plus atoms are also going down through spontaneous transitions here which is T 43 into N 4. The negative sign says that N 4 is decreasing, because of this transition and positive sign wherever we have says the number is increasing, because of that transition.

If we write d N 3 by d t then we have W p of course so we I have W l here. So, N 3 will increase, because of W l N 3 will increase, because of this N 3 will decrease, because of W l into N 3 here and N 3 will decrease, because of the spontaneous transitions. So, we have two positive terms and two negative terms. So, one positive term, one positive and two negative terms. So, similarly we can write the rate equations for all the levels.

Further, we also have N is equal to N 1 plus N 2 that is the total number N is equal to N 1 plus N 2 plus N 3 plus N 4. In practice of course, most of the atoms are in the ground state therefore, at any time N 1 is much-much greater than N 2, N 3, N 4. In fact, even if you add all of these this will also be much smaller compared to N 1 and this can be written as N 1 into the same way that we had done for the 3-level system. At steady state as before d N 3 by dt each one of them is equal to 0.

So, recall the bucket where at steady state although water is getting poured into the bucket, water is going out of the hole, there is a steady state and therefore, there is a certain level which is maintained in the bucket. Exactly like that there is a certain number of atoms maintained at any given instant at any time whereas, it is a dynamic equilibrium photons are exciting atoms continuously to the upper state from where they get de excited, but there is a steady state which is maintained alright. From equation 1 that is the 1st equation here.

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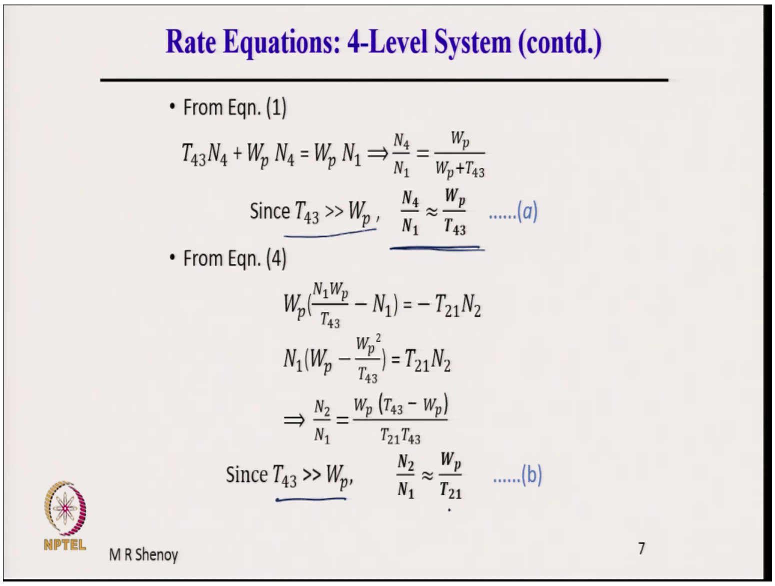

So, if we put d N 4 by d t equal to 0 then this equal to this one and this contains only N 4 and N 1 therefore, we can get the expression for N 4 by N 1. So, that is what is written here N 4 by N 1 is equal to W p divided by W p plus T 43 and as indicated earlier, T 43 is much faster, because it is a high speed non radiative transition and therefore, we can write N 4 by N 1 is equal to W p by T 43 these are approximations which are practically correct approximation.

So, instead of retaining large number of terms, we use these approximations right at the beginning so that we get a small compact expression which is not very different from the rigorously correct expression. Similarly, from equation 4 now, we look at equation 4 that is the fourth equation here this also has only two terms therefore, if you put d N 1 by d t equal to 0 we can get the ratio as shown here. W p into this equal to this one which means it is here written we have substituted for N 4 by N 1 from here W p by T 43 and we have an expression for N 2 by N 1 is equal to W p into T 43 minus W p into this.

As before since T 43 is much greater than W p N 2 by N 1 is nearly equal to W p by that is this is much greater. So, we have neglected then T 43 T 43 cancels and we have W p by T 21.

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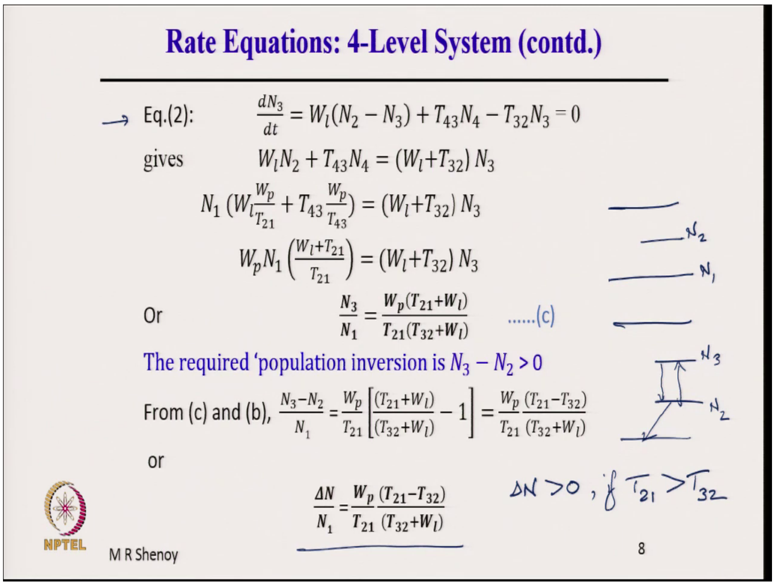

Now, let us take equation 2 that is the 2nd equation there we had written 4 equations and the 2nd equation d N 3 by dt is equal to this. So, let us look at this. From equation 2, which is written here when you put that equal to 0 you can write this way and simplify it further and we get an expression for N 3 by N 1. Now, we have an expression for N 4 by N 1, N 3 by N 1, and N 2 by N 1. Why did we want this? Because we wanted to use the equation that N capital N that is the total number is equal to N 1 into 1 plus N 2 by N 1 plus N 3 by N 1 plus N 4 by N 1.

So, we have got all these terms now so that we can relate the population inversion to the total number of atoms N alright. So, from c and b; so c equation here and b in the previous here, so equation b, we can write. If you subtract one is N 3 by N 1 other is N 2 by N 1. So, N 3 minus N 2 by N 1 comes out to be this and which is equal to this. There are no approximations, only approximation we have made is T 43 is much greater than W p or delta N.

So, N 3 minus N 2 now we call as delta N. In the 4-level, in the 3-level system we so in the 3-level system we called N 2 minus N 1 as delta N, in the 4-level system we call this as delta N. So, this is N 3 minus N 2 as delta N, because delta N here, represents the extent of inversion, the magnitude of inversion, N 3 minus N 2 is delta N. So, in N 3 minus N 2 by N 1 is equal to so, this expression.

So, we have got the expression here. Note that delta, N for delta N to be greater than 0 we need T 21 to be greater than T 32. All other numbers are positive and therefore, delta N that is greater than 0 or population inversion is possible if T 21 is greater than T 32 that is the rate at which atoms go from here to here, if this is faster than the rate at which atoms come from here to here there will be population inversion. Mathematically now, we see that yes we have got the same expression. So, this is what we discussed qualitatively that the requirement is T 21 must be greater than T 32.

So, we have to look for a laser medium where the lifetimes or the transition rates are such that T 21 that is from the lower excited states the transition rate is much faster compared to the upper excited state or if the upper excited state is a meta stable state ok. So, that I have rewritten here.

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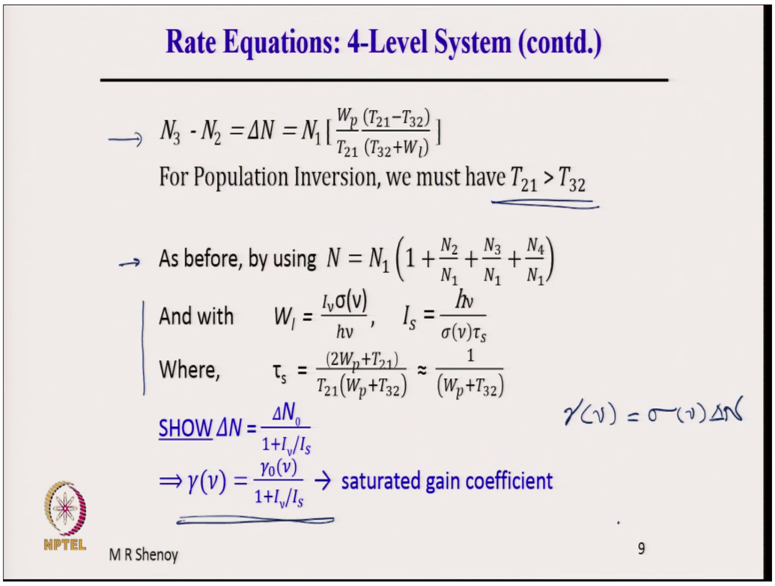

N 3 minus N 2 is equal to delta N into this therefore, for population inversion we must have T 21 greater than T 32. Note that there is no condition on W p, the pumping rate. In a 3-level system we had a condition on W p that is the pumping rate. There was a pumping rate has to satisfy this condition, but now there is no, whatever be the pumping rate so long as you have T 21 greater than T 32 you will be able to achieve population inversion.

So, as before we can use this expression we have got a, b, c, N 2 by N 1 N 3 by this expressions we have in terms of the rates and W l is equal to I nu into sigma nu the cross section by h nu, this is the definition of W l similarly, for W p and I s is h nu divided by cross section into tau s where tau s is this. So, with these expressions we can show as before. In the 3-level case I had shown that delta N we can be written as delta N 0 which depends only on the pumping rate W p divided by 1 plus I nu by I s.

We can write exactly similar expression here only the expression becomes slightly lengthier, because there are many more transition rates in the 4-level system, but you can get the same expression as delta N 0 divided by 1 plus I nu by I s and therefore, gamma of nu is equal to we know that gamma of nu is equal to sigma of nu into delta N. The gain coefficient is given by this and therefore, once delta N is in this form then you will have gamma of nu is equal to gamma 0 of nu divided by 1 plus I nu by I s which is called the saturated gain coefficient.

So, we have got almost similar expressions, same procedure, but note that in a 3-level system there was a condition on the minimum pumping rate which is required to achieve population inversion. So, we calculated that in the case of ruby laser that threshold pumping rate is of the order of kilo Watt. Here, we do not have such condition, what the requirement is T 32 should be slower or T 21 should be faster than T 32 for population inversion alright.

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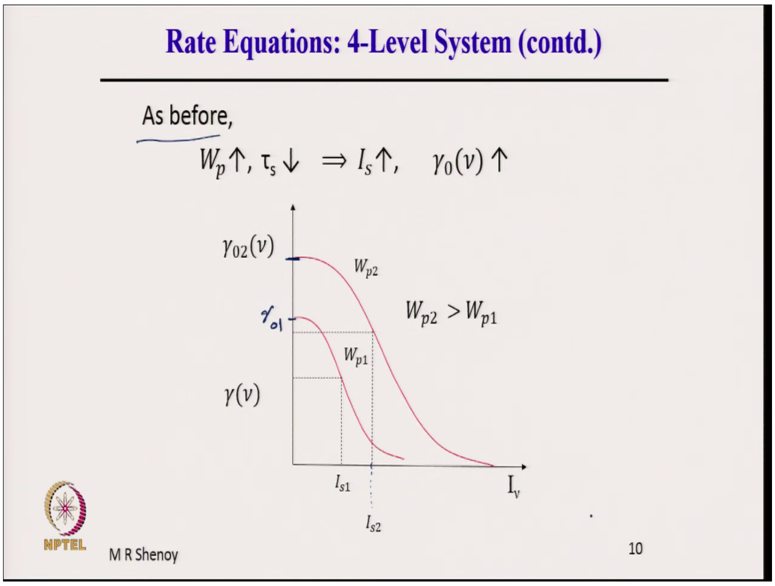

So, as before written, as before the discussion is the same. W p, if W p increases tau s decreases, we can see the expression tau s is here. So, if W p increases tau s decreases. This W p is negligible compared to the T 21 here, T and therefore, as W p so, its written here as if W p increases tau s decreases and that implies I s saturation intensity increases and also gamma 0 of nu increases.

So, just as in the case I discussed it earlier, in this lecture that for the 3-level system, for different pumping rates we had different gain curves that is the small signal gain coefficient increases. So, this is for W p 1. So, gamma 01 here, this value here and this is gamma 02 small signal gain coefficient, which goes on dropping down as the intensity increases, the saturation intensity is shown here. This is the saturation intensity for the second case. So, the saturation intensity also increases.

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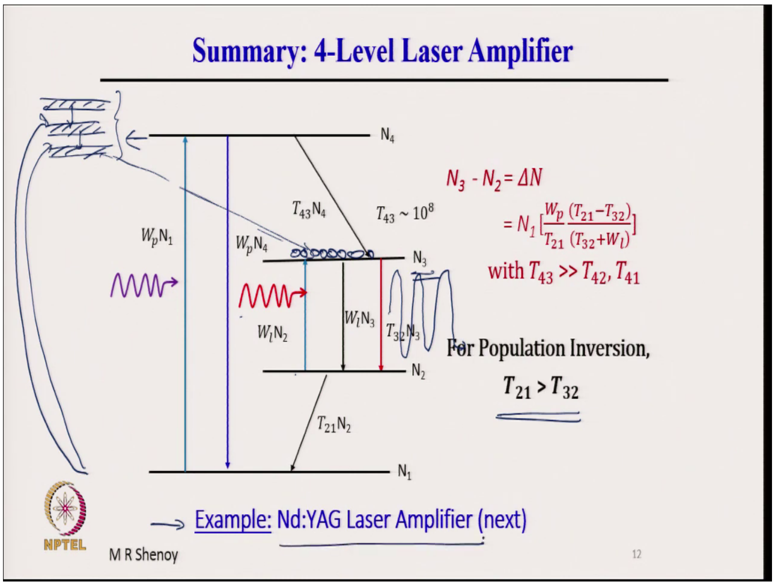

So, now we come to the summary of this 4-level laser amplifier. So, we have a 4-level system which comprises of 3 excited states and the ground states. This is the schematic that we show, but as I explained in the case of a 3-level system, the excited state this itself may comprise of several states or maybe bands, we will see this, this may be several bands. We will see this in next, we will see the example of a Nd YAG laser where I will show that if the 4th-level which I am indicating here as level 4 may comprise of 2 or 3 bands and atoms are pumped to these bands.

So, atoms may be raised to these bands, because if you take usually if we will see in the next lecture that Nd YAG laser is pumped by krypton flash lamp. These rays of course, diode pump solid state lasers will also be discussed later, but if you use krypton flash lamp it has several lines which will raise atoms to these bands and these bands within themselves have very rapid transitions from here to here and then from this level it comes down to the lower level which is here. In other words, the level 4 itself may comprise of several levels, but we do not call this as a 5 level or a 6-level laser.

This is still a 4-level laser and the pump band all of them are represented by level 4 from where atoms make rapid transition to a meta stable state. This level 3 is a meta stable state where the atoms have to accumulate. I am repeating this part in this summary that atoms are excited to an upper state or states from where they rapidly decay to a lower state, a lower excited state and if that state happens to be a meta stable state with the long lifetime then atoms tend to accumulate at that level.

The next excited state; if it has a fast transition to the ground state then we can always achieve population inversion for any pumping rate between these two levels. This is the mechanism of achieving population inversion in a 4-level system and a signal which corresponds to the energy difference between level 3 and level 2 will get amplified. Once population inversion is created then this signal will get amplified by stimulated emission.

So, the necessary condition for population inversion is T 21, this transition must be faster compared to T 32. We will take up in the next lecture the example of a Nd YAG laser amplifier which is widely used and which is a very nice amplifier to discuss. The picture will become more clear.

Thank you.