Basic Quantum Mechanics

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Module No. # 08

Angular Momentum - II

Lecture No. # 05

Addition of angular Momenta: Clebsch Gordon Coefficient

Today, we will discuss the addition of angular momentum of two angular momentum vectors leading to the calculation of Clebsch Gordon Coefficients. However, before we do so, I would like to spend a few minutes on the calculation of the spherical harmonics that we did in our last lecture.

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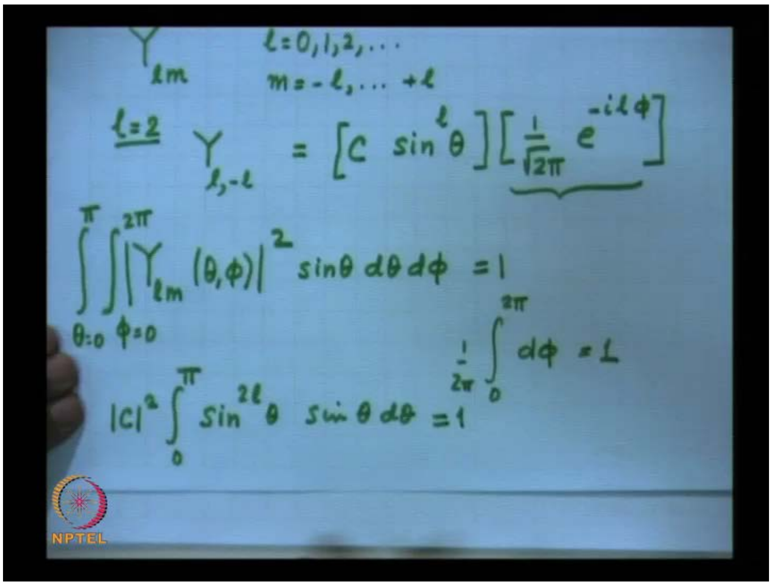

Now, we had said that the Y l m are the spherical harmonics, now as we mention that l takes the values 0, 1, 2, 3 etcetera and for each value of m l m goes from minus l to plus l. So, let me consider the case when l is equal to 2, then we had shown that Y l when m is minus l is proportional to sin to the power of l theta and the phi function was 1 over root 2 pi e to the power of the m value is minus l, so this is minus i l phi. And I thought I will spend a little time on calculating the, on the calculation of the normalization constancy.

So, this part is normalized because the the spherical harmonics are normal, the normalization condition is that Y l m theta phi mod square and integrated over theta and phi.

So, theta is equal to 0 to pi and phi is equal to 0 to 2 pi and then multiplied by sin theta d theta d phi this must be equal to 1. Now, if we take the phi part then modulus square of that is 1over 2 pi so, this square is 1over 2 pi so, 1 over 2 pi and the modulus of this is 1 so, from 0 to 2 pi d phi this is equal to1. So, we are left only with the theta part, and therefore, the normalization condition becomes, becomes mod C square sin 0 2 pi sin of 2 l theta sin theta d theta this should be equal to 1.

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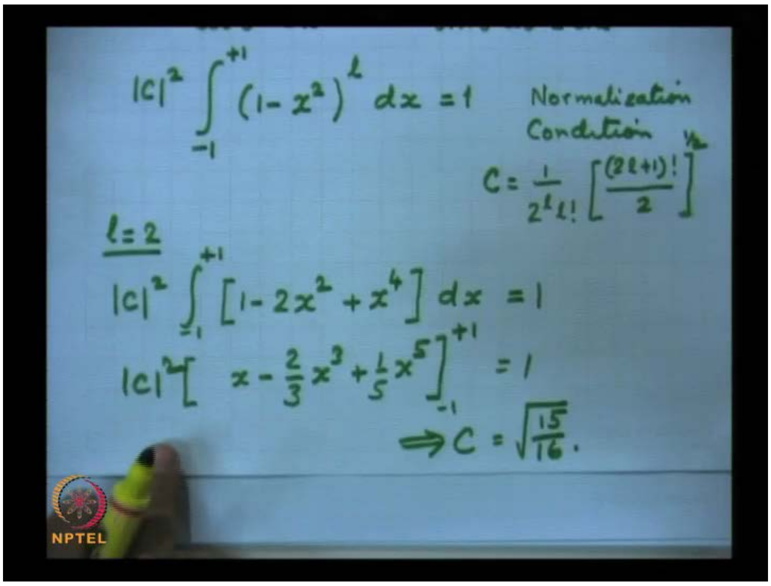

So, therefore, we make the substitution here, that let cos theta be equal to x then minus sin theta d theta will be equal to d x and sin square theta will be 1 minus x square, and when theta is 0 this is 1 and when theta is pi it is minus 1, but there is a minus sign here. So, you get C square I interchange the sign minus 1 to plus 1 sin square theta is 1 minus cos square theta, x square raise to the power of l, and this is Just minus of d x and I have taken care of the minus sign so, this is d x is equal to 1. So, this is the normalization condition and if I carry out the integration this the normalization condition. And if I carry out the integration I will get C is equal to 1 over 2 to the power of l l factorial 2 l plus 1 factorial by 2, it is a little cumbersome formula.

So, what I am going to tell you today is that, if you do not remember C the value of C then you can easily calculate, very easily calculate by brute force integration. So, let me consider the case when l is equal to 2, when l is equal to 2 so you get C square minus 1 to plus 1 1 minus x square whole square so, this is 1 minus 2 x square plus x to the power of 4 times d x is equal to 1. Now, this I can integrate immediately, the first term will be x so, that will be so x minus minus two-third x cubed plus one-fifth x 5 this is really very simple from minus 1 to plus 1 this is equal to 1. And if you calculate this then this will readily give C is equal to under root of 15 by 16. So, therefore, we get the normalization constant as C is equal to under root of 15 by 16.

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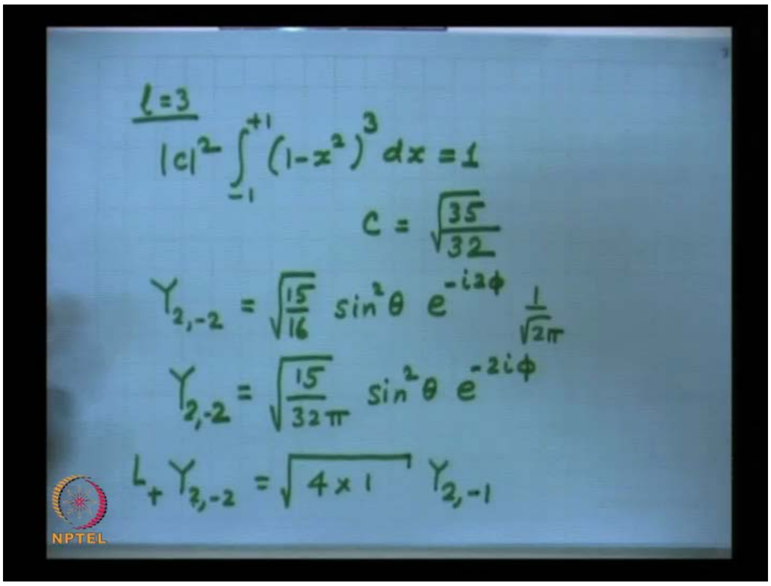

Similarly, for l is equal to 3 mod C square and you will have the now the integration minus 1 to plus 1 1 minus x square raise to the power of 3 d x is equal to 1. So, I can immediately write the 1 minus 3 x square minus 3 x 4 plus x to the so on. I I can immediately expand it and if I if I do this than you will get C is equal to under root of 35 by 32. So, what I wanted to tell you is that in case I you do not remember, the value of the coefficient then by brute force integration I can calculate the normalization constant.

So, therefore, we had the relation that Y 2 minus 2 and this will be under root of 15 by 16 sin square theta e to the power of minus i 2 phi multiplied by under root of 2 pi. So, this will be equal to under root of 15 by 32 pi sin square theta e to the power of minus 2 i phi. So, this then the expression for Y 2 minus 2 and then we had the expression that L plus Y 2 minus 2 will be under root of J minus m that is 2 minus 2, minus minus 2 so that is 4 into J plus m plus 1 that is 1 and this will be Y to minus 1 and this will allow me to calculate Y 2 minus 1.

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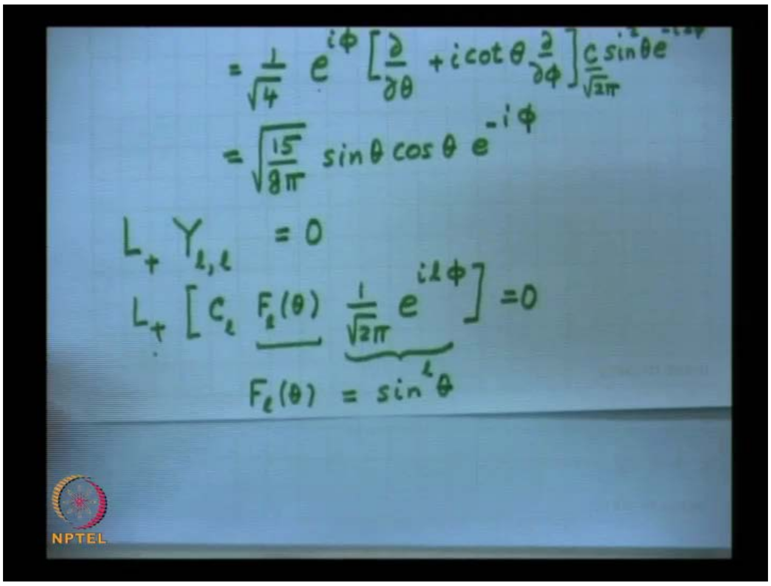

So, Y 2 minus 1 will be 1 over under root of 4 L plus Y 2 minus 2 and I we have derived the expressions for l plus and m minus and we have L plus is equal to 1 over under root of 4 e to the power of i phi delta by delta theta plus i cot theta delta by delta phi. And you substitute the expression for Y 2 minus 2 that is C sin square theta e to the power of minus i 2 phi. And, if you work this out then you will get Y 2 minus this is a very simple problem into 15 by divided by under root of 2 pi. So, the final result will be it is a very simple exercise 15 by 8 pi sin theta cos theta into e to the power of minus i phi, you see because m is minus 1 therefore, it will be e to the power of minus i phi.

There is one more thing that I would like to mention, that we could have equally well started from the fact that L plus y l l. Now, so this is under root of J minus n that is l minus l so this is a null ket this is 0. And therefore, from this we can write down that if I write L plus Y l is c times c l f l of theta and the phi part will be 1 over 2 pi e to the power of i l phi this is 0. Then the phi part will be 1 over root 2 pi e to the power of i l phi and using the operator representation of the l plus operator, which is given by e to the power of i phi so on. One can find out the equation for f l of theta, and if one works out the details I leave that as an exercise then f l of theta will also come out to be sin to the power of l theta.

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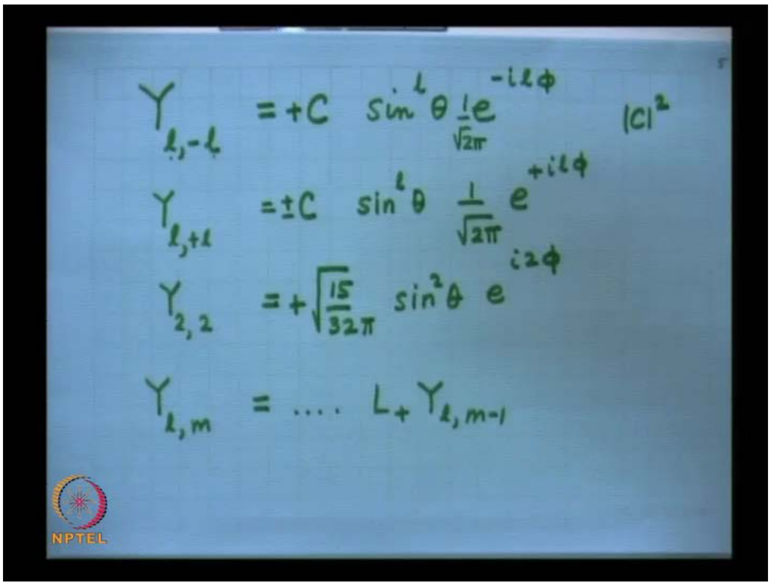

So, please see that Y l comma minus l is C sin to the power of l theta e to the power of minus i l phi multiplied by 1 over root 2 pi, then Y l plus l will be the same constant. Actually, there can be a plus sign here or a minus sign here depending on the value of l this is what I wanted to tell you today and that this multiplied by same sin l theta 1 same constant, but except for a sin 1 over root 2 pi this will be plus i l phi. So, in the case of Y 2 2 that is l m is 2 we will obtain the same expression for c and it will be a plus sign here, 15 by 32 pi sin square theta e to the power of i 2 phi. So, why do I say that? There has to be a plus sign here because we always choose a plus sign for Y l comma minus l.

Because when you calculate the normalization constant you can either choose, because you calculate mod C square so, it is always arbitrary within a face factor. We always choose for Y l comma minus l a plus sign and then we operate this by ladder operators and calculate Y Y l m by operating some multiplied by l plus Y l m minus 1. And then once the sign is fixed you can keen on calculating the spherical harmonics and you will if l is 2 you will obtain a plus sign here.

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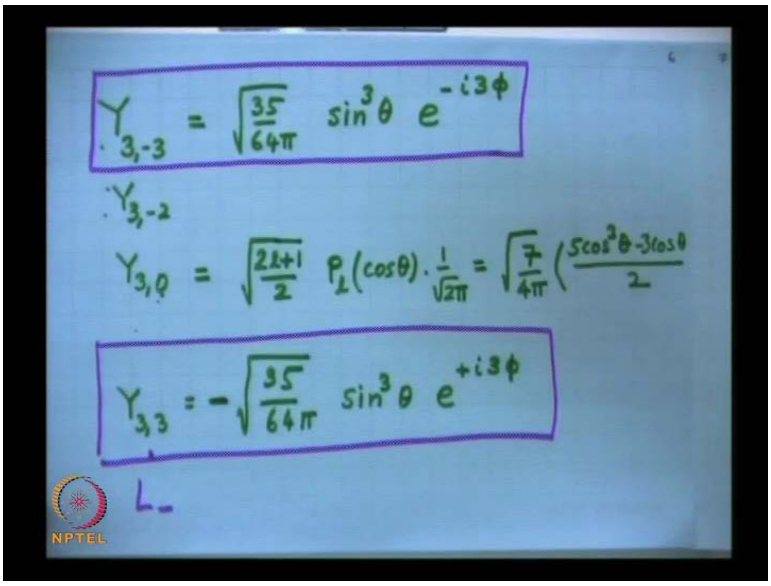

However, if l is 3 then one can show that Y 3 minus 3 is equal to 35 by 64 pi sin cube of theta e to the power of minus i 3 phi. You keep on operating the ladder operators you get Y 3 minus 2 and these expressions are given in any text book Y 3 minus 1 and then Y 3 0 and then Y 3 1, Y 3 2, Y 3 3. So, if you get Y 3 3 from here you will get minus sign here 35 by 64 pi; this will be the same factor, but now it will be a minus sign here sin cube theta but the phi part will now be plus i 3 phi. So, I have told you the recipe for obtaining all these spherical harmonics it Just working out patiently. And the if you use the l plus 1 then what you will get is here, you will get for m is equal to 0 this will be 2 l plus 1 by 2 it will come out p l of cos theta multiplied by 1 over root 2 pi.

And this in this particular case will come out to be under root of l is 3, so 2 into 3 plus 1 is 7 7by 4 pi and p 3 of cos theta is 5 cos cube theta minus 3 cos theta divided by 2. What I would request all of you is? That you start with this expression you start with this expression and use ladder operators to obtain Y 3 0. And then start with this expression for Y 3 0 use again the ladder operators and use obtain this you can do it in any order you can start with this and use l minus operators repeatedly to get Y 3 0 then Y 3 minus 1, Y 3 minus 2 all of them will come out to be consistent. So, therefore, I have given you the recipe that for any value of l or m.

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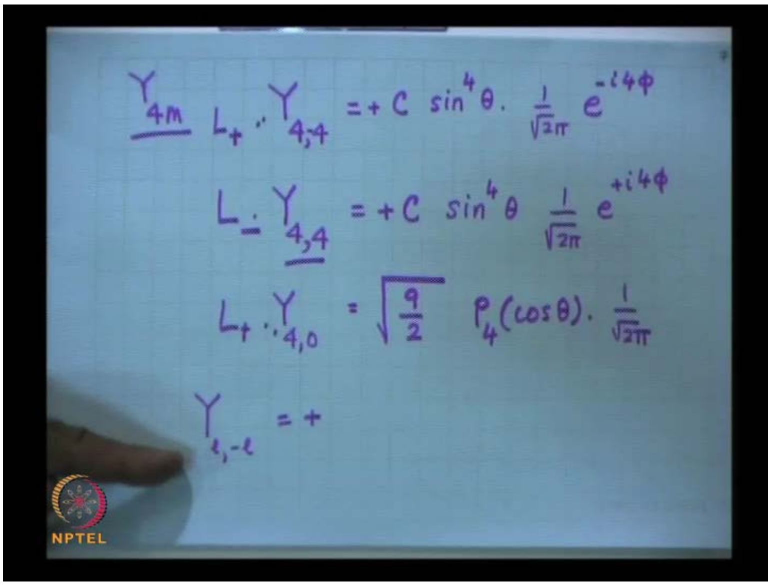

Let us suppose, I want to do Y 4 m I want to I want expressions for Y 4 m so, first I write Y 4 m, Y 4 4 or Y 4 minus 4 will be plus C sin to the power of 4 theta multiplied by 1 over root 2 pi e to the power of minus i 4 phi. I can also, write down Y 4, 4 will be again plus c same constant sin 4 theta 1 over under root of 2 pi e to the power of plus i 4 phi. And I can find out C either by using the expression or by brute force carrying out the integration or I can start with this Y 4 comma 0. So, this is l is 4 so 2 into 4 plus 1 is 9 by 2 p 4 of cos theta multiplied by 1 over root 2 pi of course, I must know the value of C.

I can start with either of these expressions, if I start with this then I must period operate this by l plus and then again by l plus then again by l plus and get all these. If I start with this then I can if I operate this by l plus then I get Y 4 1, Y 4 2, Y 4 3 and so on and if I start with this there I must operate either. So, therefore, in summary I have given you a recipe for at least small values of l, l equal to 0, 1, 2, 3, 4 you start with you. I have given you the recipe for calculating first this spherical harmonic Y l comma l or Y l comma minus l.

You obtain that you write down that normalize it and then use ladder operators to calculate all the other spherical harmonics. And once to start with it is normalized all others will be automatically, normalized. Why did we choose the convention? That Y l minus l has to be positive it is Just like that, we can we could have we could have easily chosen Y l comma minus l can be negative also. But once we have chosen the sign of Y l comma minus l, then the remaining signs are automatically fixed by the use of ladder operators. So, this concludes the the recipe for obtaining the spherical harmonics. We now continue our discussions on the addition of angular momentum, now for example, that we have this electron in the say p state.

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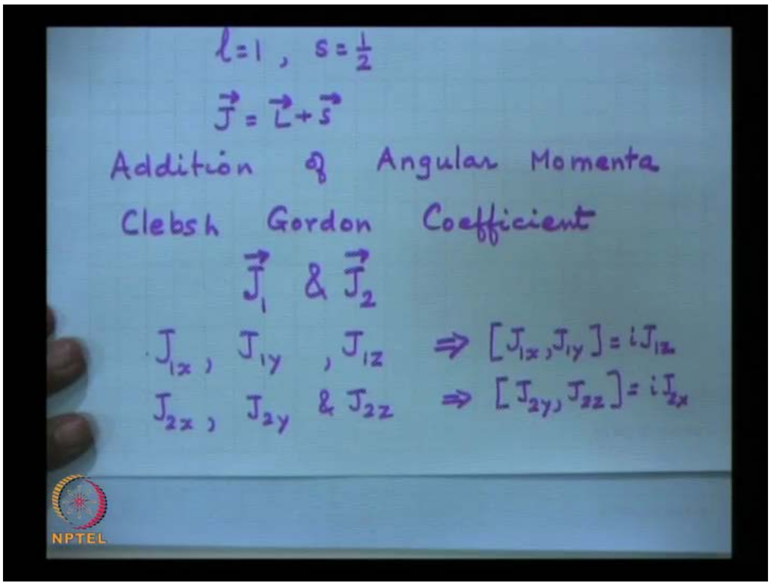

So, electron in the p state has an orbital angular momentum l is equal to 1 and then you have a spin angular momentum, which we discussed which is equal to half. So, there are interaction terms there is an interaction as we will discuss later on between this spin angular momentum and the orbital angular momentum and this leads to an interaction term in the energy. And therefore, we define the total angular momentum J, which is the vector sum of L plus S. So, I will consider a more general case and we start this by saying that we will be considering what is known as addition of angular momenta and we will develop a formal theory for that an addition of angular momenta.

We will consider and which will lead to the calculation of what are known as clebsch gordon coefficients clebsch gordon coefficients. These are extremely important in any any calculations involving atomic and molecular spectroscopy. So, we consider two angular momentum vectors J 1 and J 2 now, by this we mean that we have J 1 x, J 1 y and J 1 z and these satisfy the same commutation relation that we had discussed earlier, that J 1 x comma J 1 y is equal to i h cross J 1 z, but h cross is 1. Similarly, we have J 2 x, J 2 y and J 2 z and they also satisfy same type of commutation relation that is J 2 y comma J 2 z is equal to y z x so, J 2 x and three similarly, other relation .We say that the components of J 1 commute with components of J 2 by that we imply by that.

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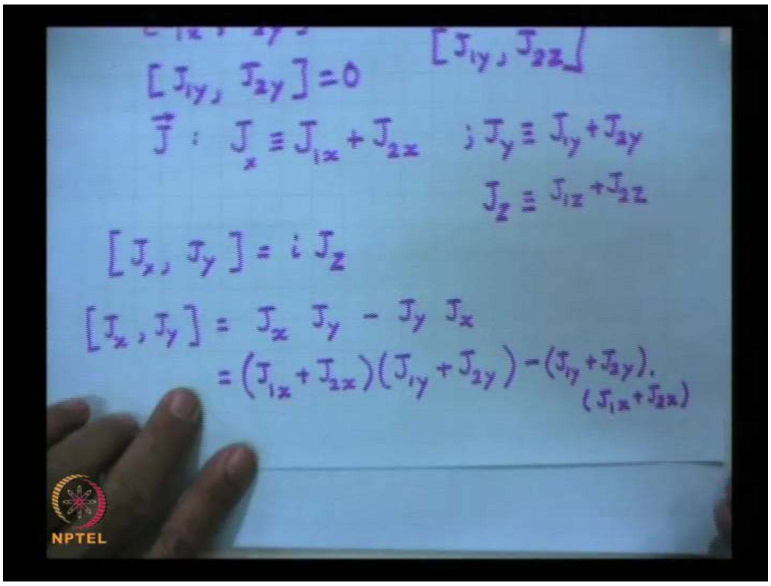

We imply that J 1 x as you can see J 1 x, does not commute with J 1 y but J 1 x commutes with J 2 y we assume, that components of J 1 commute with components of J 2 any two components. So, J 1 x of course, commutes with J 1 y commutes with J 2 y is equal to 0, J 1 y commutes with J 2 z. Any component of J 1 will commute with any of the components of J 2 so we assume that. So, we assume that the components of J 1 commute with components of J 2, but of course, J 1 itself has they satisfy this commutation relation. We then define J x the total angular momentum vector J, such that J x is defined to be equal when I have three equal to signs then it means, define to be equal to this is equal to J 1 x plus J 2 x.

Similarly, J y is defined to be equal to J 1 y plus J 2 y and J z is equal to J 1 z plus J 2 z. So, we define J x, J y and J z by these three relations and we will now show that, we will now prove that J x comma J y is equal to i time J z. The proof is really simple, so we write down so we first write down what is let us do this carefully, J x J y is equal to J x J y minus J y J x. So, you see J x is equal to J 1 I would like to do this little patiently J 1 x plus J 2 x multiplied by J 1 y plus J 2 y minus the same terms, but in the reverse order so, J 1 y plus J 2 y multiplied by J 1 x plus J 2 x.

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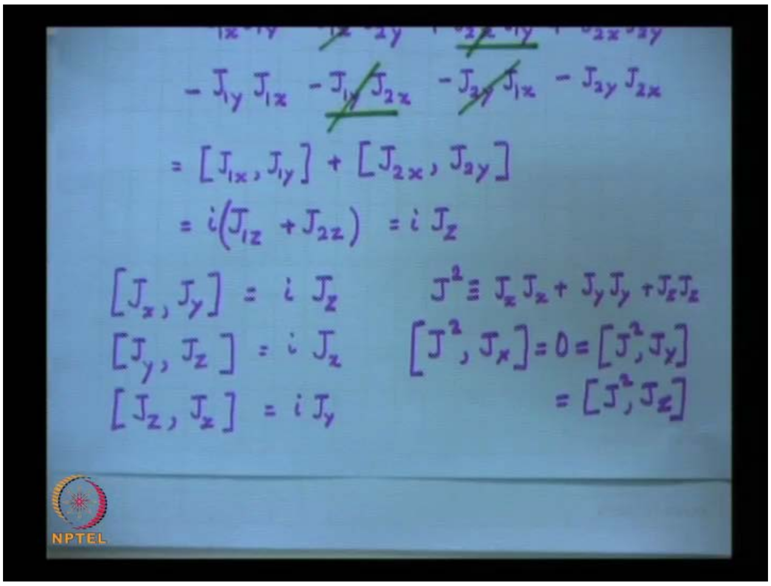

Now, let me Just brute force multiply so, you will have is equal to, please see J 1 x J 1 y plus J 1 x J 2 y plus J 2 x J 1 y plus J 2 x J 2 y and the second term will be minus J 1 y J 1 x J 1 y J 1 x J 1 y minus J 1 y J 2 x minus J 2 y J 1 x minus J 2 y J 2 x so, you see now that J 2 x commutes with J 1 y so this term and this term cancel out. Similarly, J 1 components of J 1 commute with components of J 2. So, this term will cancel out with this term so, I am left with these 4 terms so this J 1 x J J 1 y minus J 1 y this is the commutator of J 1 x comma J 1 y plus the commutator of J 2 x comma J 2 y so, this is i J 1 z plus i J 2 z so this is i J z.

So, I have proved that if the components of J 1 commute with components of J 2, then J x comma J y is equal to i comma J z. Similarly, I can show I leave as an exercise for you to show that J y comma J z is equal to i comma J x and you have J z comma J x is equal to i J y. Therefore, the components of J x the total angular momentum vector also, satisfy the same commutation relations and one can show as we had done earlier. That, the operator J square which is defined as J x square plus J y square plus J z square, this will commute with J x this will commute J square will commute with J y and this will J square will commute with J z. Therefore, J square J x ,J y and J z satisfy the usual commutation relations.

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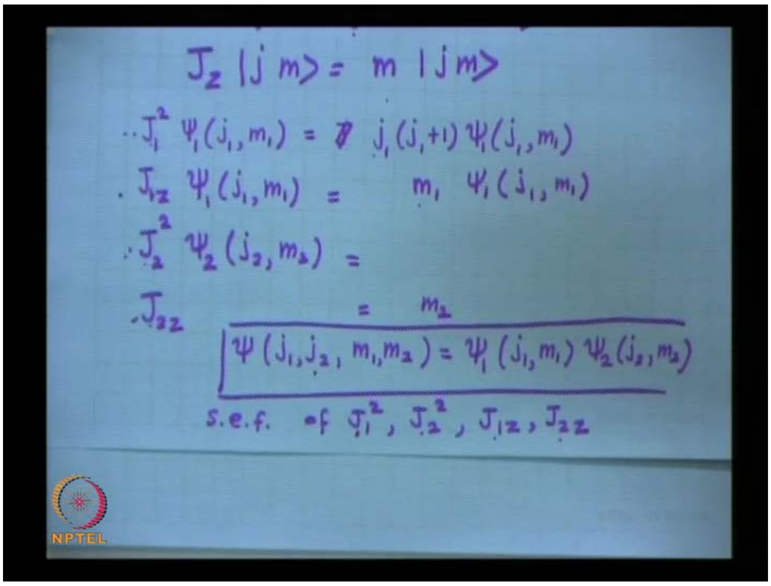

And therefore, we can have simultaneous eigenkets of J square, which I write as J m is equal to J into J plus 1 h cross square, but I assume h cross to be 1 J m and J z J m is equal to m h cross ket J m. I hope I have made myself clear, that the components of the total angular momentum vector satisfy the same commutation relations. And therefore, we can again develop the same kind of algebra introduce operators like J plus and J minus and obtain simultaneous eigenkets of the J square and J z. The question is what are the values of J and m, how they are related to J 1 and J 2? What I am trying to say is that J 1 square and J 2 square I know, so what are the values of what are the eigenvalues of J square and J z this is an extremely important problem.

Now, let me go back to J 1 square let psi 1 J 1 comma m 1 be the simultaneous eigenket of J 1 square and J 1 z, so J 1 square sorry this is equal to J 1 into J 1 plus 1 h cross square psi 1. I will then I give the examples it will become clearer and J 1 z psi 1 is a eigenfunction of J 1 z also, so I write psi 1 J 1 m 1 is equal to m 1 psi 1 J 1 m 1. And similarly, let psi 2 be the eigenket of J 2 square and J J 2 z. So, psi 1 is an eigenket simultaneous eigenket of J 1 square and J 1 z psi 2 is a simultaneous eigenket of J 2 square and J 2 z. Now, please see J 1 square commutes with J 1 z J 1 square commutes with all the components of J 2 so, it commutes with J 2 square and it commutes with J 2 z.

So, these four operators J 1 square J 1 z J 2 square J 2 z they all commute with each other and therefore, we can have simultaneous eigenkets of J 1 square J 1 z J 2 square J 2 z and those wave functions I denote by the product wave functions. So, I write this as psi J 1 J 2 m 1 m 2, which is really a product of the two wave functions psi 1 J 1 m 1 psi 2 J 2 m 2 this product wave function is a simultaneous eigenfunction of J 1 square J 1 z J 2 square and J 2 z. And the corresponding eigenvalues are J 1 into J 1 plus 1 m 1 J 2 into J 2 plus 1 and m 2. So, these are simultaneous eigenfunctions of J 1 square J 2 square J 1 z and J 2 z. So, this represents J 1 represents the eigenvalue of J 1 square; that is J 1 into J 1 plus 1 this represents the eigenvalues of J 2 square J 2 into J 2 plus 1 eigenvalues of J 1 z is m 1 h cross and eigenvalues of J 2 z are m 2.

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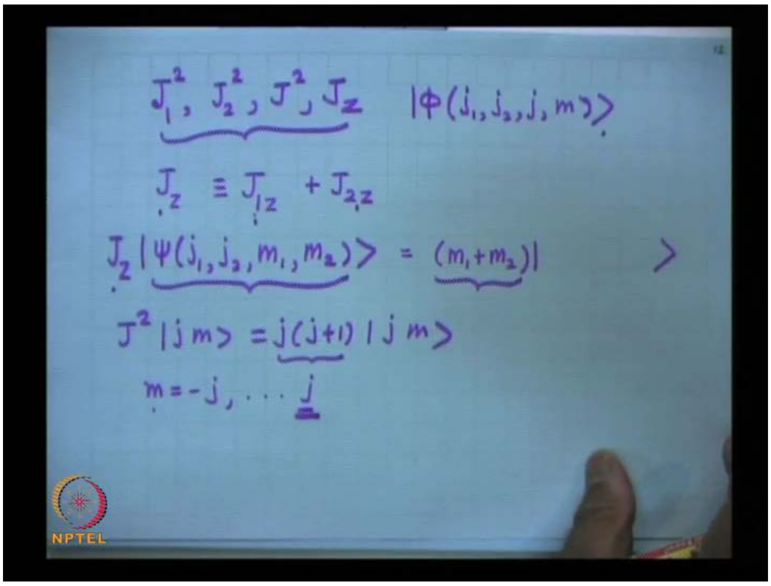

Now, similarly, I leave it as an exercise for you to show is really simple may be I will show slightly later. That the operator J square J 1 square J 2 square J square and J z, they also commute with each other. And therefore, one can form simultaneous eiegnkets which we will denote by phi function so, phi we will put a ket sign here J 1, J 2, J and m. These are simultaneous eiegnkets of the operator J 1 square J 2 square J square and J z, of course, I know J 1 and J 2, they may be one half or something like that, but what are the values of J? That we will find out in a moment we still do not know.

Let me give you an example, but before that let me mention that what is J z so, J z is defined to be equal to J 1 z plus J 2 z. Now, this function psi J 1, J 2, m 1, m 2 is an eigenket of J 1 z and J 2 z and therefore, it is an eigenket of J z. So, these psi functions that I have written may be should write a ket here, so let me write this down that J z operating on psi J 1, J 2, m 1, m 2 is equal to J z is equal to J 1 z plus J 2 z for J 1 z it is m 1 and J 2 z it is m 2 so, this is equal to m 1 plus m 2 times the same ket.

So, this ket is also an eigenket of the operator J z belonging to the eigenvalue m 1 plus m 2, does we now know all the values of the all the values all the eigenvalues of J z once we know the eigenvalues of J z we can find out immediately the eigenvalues of J square. Because I know that J square J m is equal to J into J plus 1 ket J m and therefore and the value of m goes from minus J to plus J. So, if you give me a set of values of m then the maximum value of m will correspond to the value of J. Let me give you an example, let me consider the interaction of neutron and proton or proton and neutron or something like that both have spin both are spin half system.

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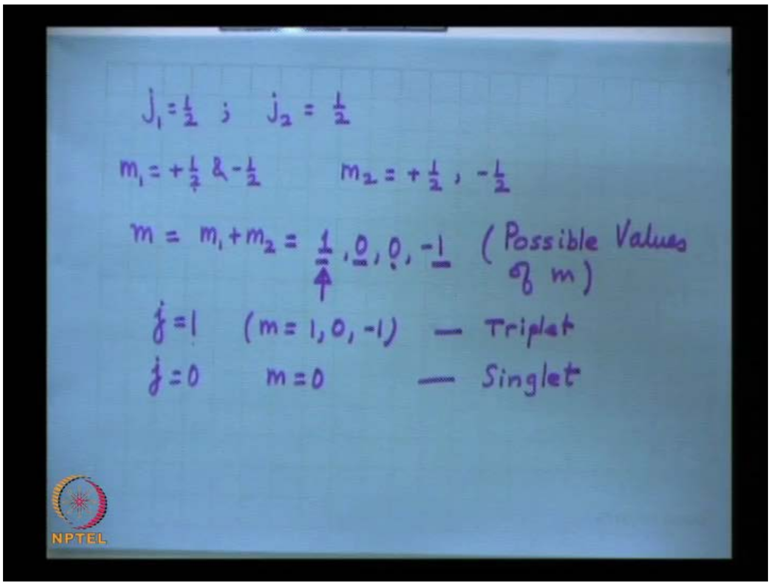

So, J 1 is equal to half and J 2 is equal to half, then we have m 1 can take two values plus half and minus half and m 2 can take also plus half and minus half. So, what are the possible values of m? Which is equal to m 1 plus m 2. So, the possible values of m are half plus half which is 1 half minus half is 0 minus half and half is 0 and minus half and minus half is minus 1. These are the possible value all possible values of m, that is these are the possible eigenvalues of J z. So, the maximum value is 1 so, we must have J is equal to 1, but when J is equal to 1 m can take values 1 0 minus 1 so, this value is taken 1 of this value is taken 1 of this value.

Now, there is 1 0 which is left so, you will have J will be equal to 0 and m is equal to 0, this is known as the triplet state and this is known as the singlet state. Let I will come back to this in a moment, but before that let me give you another example. Let me consider the interaction of the orbital angular momentum with the spin angular momentum so, there is an interaction energy which is known as the l dot s term.

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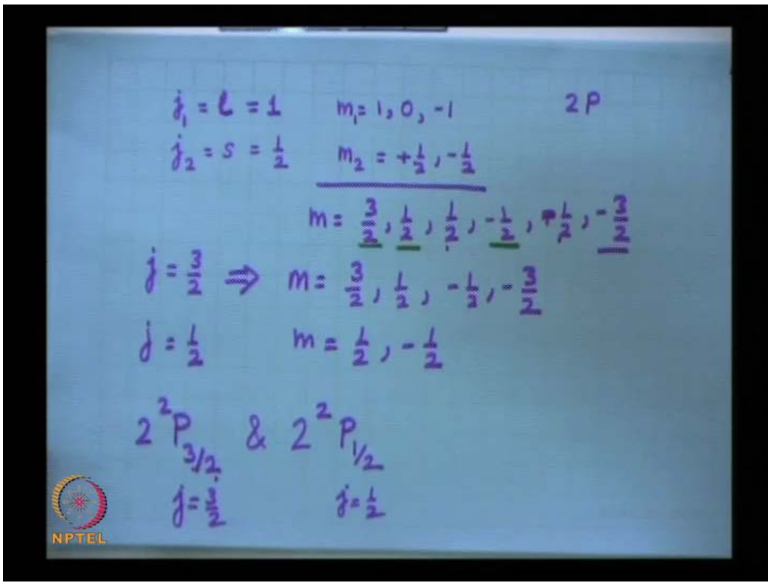

So, we have two angular momentum J 1 which is the l the l sorry the small l and let us suppose, I consider the the 2 P state of the hydrogen atom. So, l is 1 and therefore, m will take 1 0 and minus 1 and J 2 is equal to S is equal to half this is the spin orbit interaction and then this is m 1 and this is m 2 which is equal to plus half and minus half. So, what are the possible values of m? So, please see 1 plus half that is 3 by 2 1 minus that is half 0 plus half is half 0 minus half is minus half, minus 1 plus half is plus half, minus 1 minus half is minus 3 by 2.

So, the maximum value of m is 3 by 2 so, the maximum value of J is 3 by 2, for which we will have m equal to 3 by 2 half minus half and minus 3 by 2. So, this takes care of this one of this one of this sorry 1 has to be minus here and also this. And then there are two remaining half and minus half so, then J must be equal to half so, m is equal to half and minus half. So, we have two states 2 P this is the total angular momentum 3 by 2 and this a doublet, because the S is equal to half so 2 S plus 1 is 2 this is the 2 S plus 1 and this is the l value and 2 doublet P half. So, these are the J values J is equal to 3 by 2 and J is equal to half finally, let me give you I am sure by now you are all familiar with the algebra.

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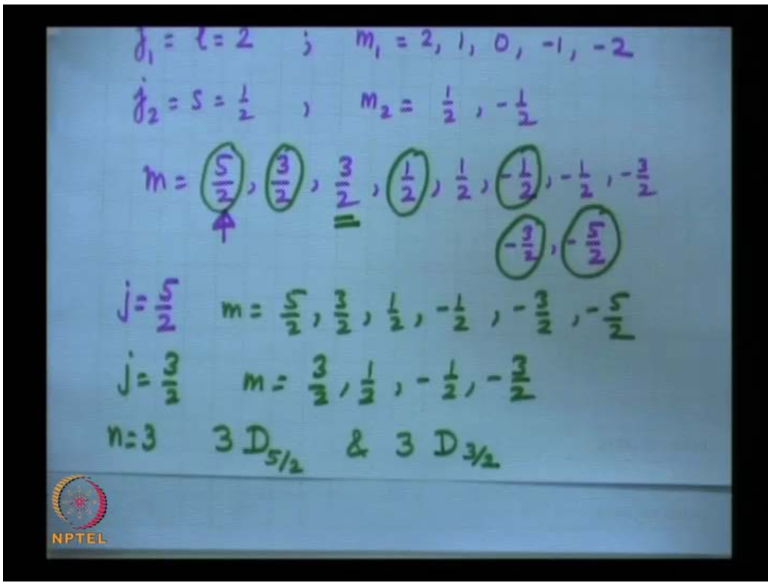

But let me Just give you one more example, say J 1 is equal to l is equal to 2 therefore, m 1 is equal to 2 1 0 minus 1 minus 2, J 2 is equal to S is equal to half so, m 2 the 2 values are plus half and minus half. So, let me write down what are the possible m values 2 plus half is 5 by 2, 2 minus half is 3 by 2, 1 plus half 1 plus half is 3 by 2, 1 minus half is half, 0 plus half is half, 0 minus half is minus half, I hope I do not make any mistake. Now, minus 1 plus half is minus half, minus 1 minus 3 by 2 is minus 3 by 2, minus 2 plus half is again minus 3 by 2 and minus 2 minus half. These are the all possible values of m, once I know the possible values of m I take the maximum value which is 5 by 2.

So, J must be 5 by 2, see if J is 5 by 2 then this is 5 by 2 then m can take 3 by 2 half minus half minus 3 by 2 and minus 5 by 2. So, I will have m is equal to 5 by 2, 3 by 2 half minus half minus 3 by 2 and minus 5 by 2. Then there are four remaining the maximum value is 3 by 2 so J will be 3 by 2 m will be 3 by 2 half minus half and minus 3 by 2. So, thus if n is equal to say 3 n is equal to 3 then an let us suppose, l is 2 so that is the 3 D state you will have J will be 5 by 2 and 3 doublet again D 3 by 2. These are the two states corresponding to l is equal to 2 in the hydrogen atom problem, when we discuss the interaction of the orbital angular momentum and the and the spin angular momentum. Now, let me go back to to this slide let me go back to the case I hope you now understand the algebra.

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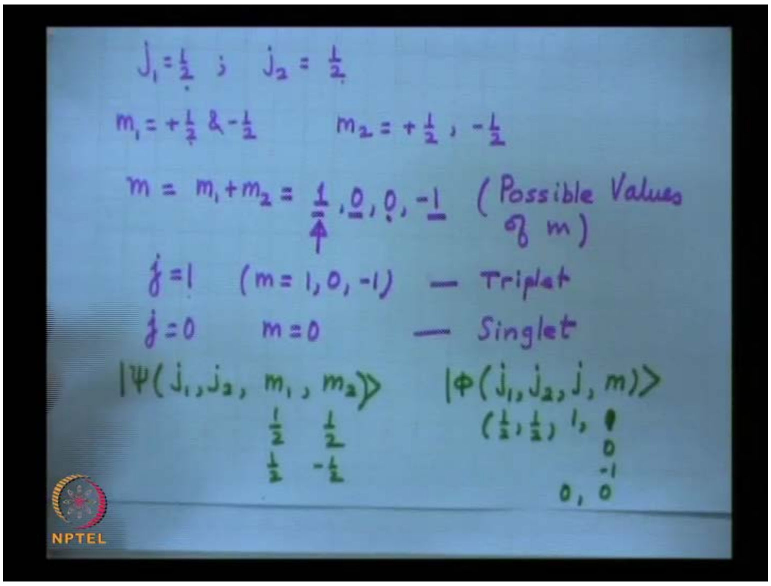

So, we consider two spin half problem, so we have four psi functions please see the psi functions are psi J 1, J 2, m 1, m 2. So, let me write it down so J 1 is half and J 2 is half, now m 1 can take half half, half minus half minus half half minus half. And then there are four phi functions, phi functions are simultaneous eigenkets of J 1 square, J 2 square, J square and J z. so, we have J is equal to 1 so there are 5 4 phi functions so this is half half J 1 and J 2 are half this is 1 and so you have 0 so, m is 1 0 minus 1 and then m is equal to J is equal to 0 m equal to 0 so, let me write it down again.

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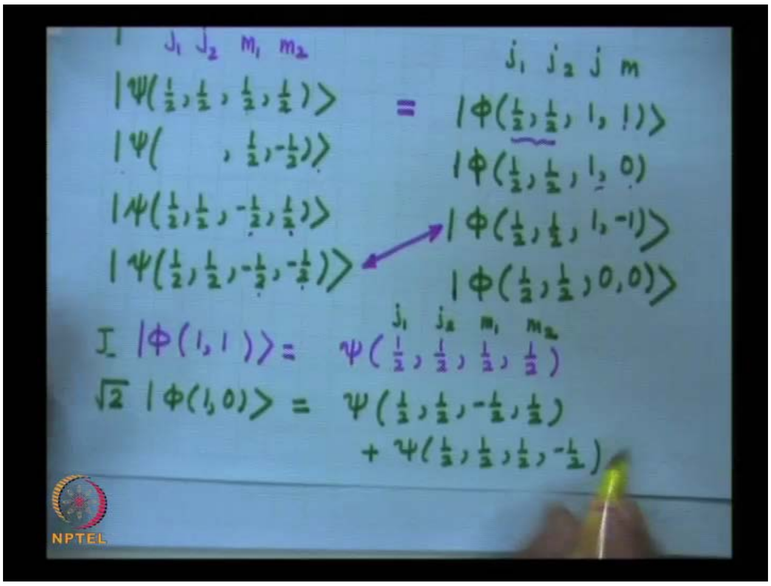

So, you have four psi functions so, let me write here psi half, half, half ,half these are the values of J 1, J 2, m 1 and m 2. J 1, J 2 is the same for all the four functions in fact many people omit writing that. So, there are four such values functions psi half, half and this is half, minus half, psi half ,half, minus half ,plus half and psi half, half minus half, minus half. Now, for the phi functions these are four so, phi functions there are also four J 1 J 2 J m so, these are phi half, half J I said that can take values 1 and 0. So, when J is 1 m can be 1 m can be 0 and m can be minus 1 so this is phi, phi half half, this half half has is repeated everywhere 0 0.

Now, you see when m is equal to 1, m 1 has to be half and m 2 has to be half so, this must be equal to this and when m is equal to minus 1 then m 1 has to be minus half, m 2 has to be minus half because this plus this must be equal to this. So, this must be equal to this and this must be a linear combination of this and this, because m is 0 and here also m is 0 because m 1 plus m 2 is 0. So, this function must be a linear combination of these two and this function must be a linear combination of these two. So, you have for example, let me omit this so phi 1 comma 1 will be equal to psi let me write the half half factor.

I will tell you, why I require that? Now, I want to find out I want to get to phi 1 comma 0. So, what I do is? Operate this by J minus so J minus will be under root of J plus m so, J plus m will be under root of 2 and J minus m plus 1 that is 1. So, this will be phi 1 comma 0 is equal to J minus operator will be J 1 minus plus J 2 minus. So, if I operate this is value of J 1one has to do this very carefully J 1, J 2, m 1, m 2 and you will find that when I operate this by J 1 minus. I have to worry about this so J plus m half plus half is 1 and J minus m plus 1 half minus half plus 1 therefore, this becomes psi half, half, minus half, half and if you operate by J 2 minus you also get this so, psi half, half ,half, minus half.

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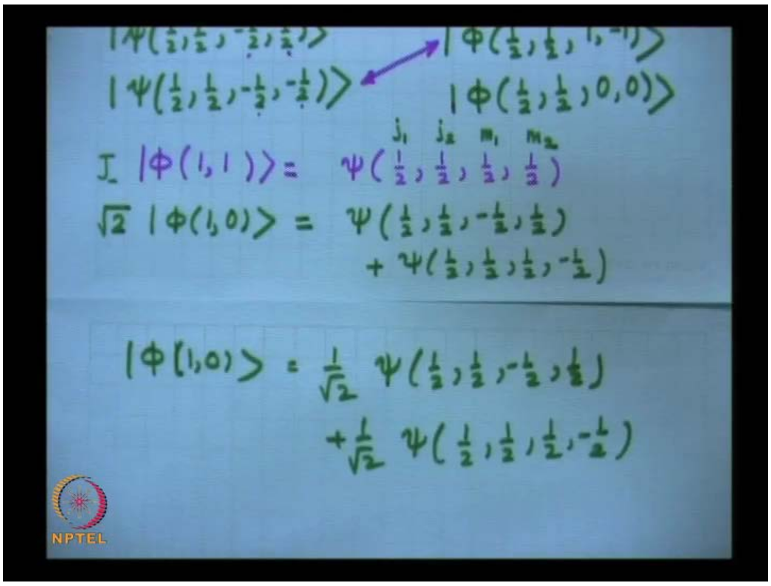

So, I get finally, the result that phi 1 0 is equal to 1 over root 2 psi half, half, minus half, half plus 1 over root 2 we will continue from this point onwards in my next lecture half, half, half. These coefficients are known as the clebsch gordon coefficients. So, we will continue form this point onwards in my next lecture.