Basic Quantum Mechanics

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Module No. # 08
Angular Momentum - II
Lecture No. # 4
The Larmor Precession and NMR Spherical Harmonics using Operator Algebra

In my previous lecture, we had discussed the consequences of the electron having a magnetic moment which is proportional to the spin angular momentum of the electron. Spin angular momentum is the intrinsic angular momentum of the electron. We will continue our discussion from there, but first we will write down the expression for the magnetic moment for the electron.

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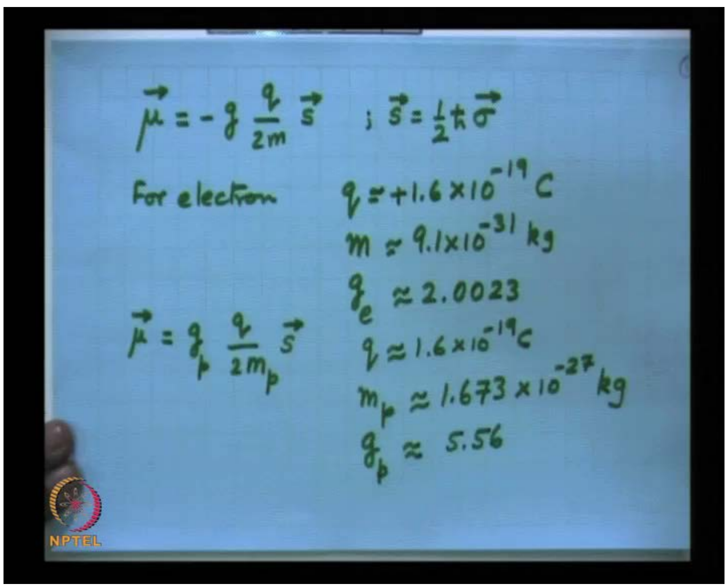

We have said that this was equal to minus g q by 2 m into s. This is usually a lower case and this s vector is equal to the spin angular momentum vector which is denoted by half h cross sigma. Now, for the electron, for electron the q is with for all particles, q is the magnitude of the charge of the electron. So, 1.6, about 1.6 into 10 to the power of minus 19 coulombs. The mass of the electron as we know is about 9.1 into 10 to the power of minus 31 kilograms and the land g factor for the electron is about 2. Actually, in most calculations, we assumed to be 2.

Now, exactly in a similar way, there is a magnetic moment of a proton and for the proton. So, we write mu is equal to plus g t q over 2 m p, where m p is the mass of the proton times the s vector. So, this also corresponds to half spin angular momentum vector and q is again the same. Q is again 1.6 into 10 to the power of minus 19 coulombs and mass of the proton is about 1.673 into 10 to the power of minus 27 kg, but the land g factor for the proton is about 5.56.

So, the proton also behaves as a tiny magnet having a magnetic moment which is much smaller than that of the electron because the mass of the proton is much smaller, but nevertheless, it does have a magnetic moment and if you make a measurement of mu z, you will get two quantized values. Similarly, the neutrons also has although neutron is electrically neutral particle, it does not have a charge, but we can assume that qualitatively speaking that there is a distribution of charge which results in a magnetic moment of the neutron.

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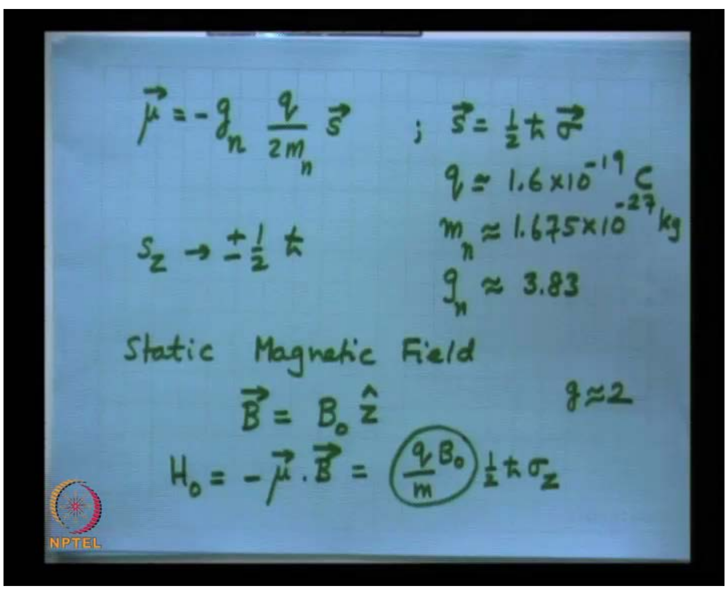

The neutron magnetic moment is actually negative and it is given by the g of the neutron q over 2m n multiplied by the same s. So, s is once again half h cross sigma q is again the same. The magnitude of the electronic charge minus 19 coulombs but mass of the neutron is roughly the same as that of the proton but slightly higher 1.675 into 10 to the power of minus 27 kg and the lande g factor for the neutron is about 3.83. What I wanted to emphasize is that even the neutron which is electrically neutral has a magnetic moment and if you make a measurement of this magnetic moment, it has two possible values corresponding to x z having two eigen values plus half h cross and minus half h cross.

Now, we will briefly discuss that if such a magnet is put in a static magnetic field or any neutral silver atom which has the magnetic moment of the electron is put in a static magnetic field. In a static magnetic field what will be the Hamiltonian and what will be the solution of the Schrodinger equation? So, static magnetic field let us suppose you have a strong magnetic field in the z direction. Therefore, the Hamiltonian h 0 as I had mentioned was equal to mu dot b. So, this will be equal to q by m. I am assuming g to be about 2 for the electron q by this is q by 2 m into g and 2 cancels out into b 0 times half h cross sigma z. So, this is my Hamiltonian. This quantity we denote by omega 0.

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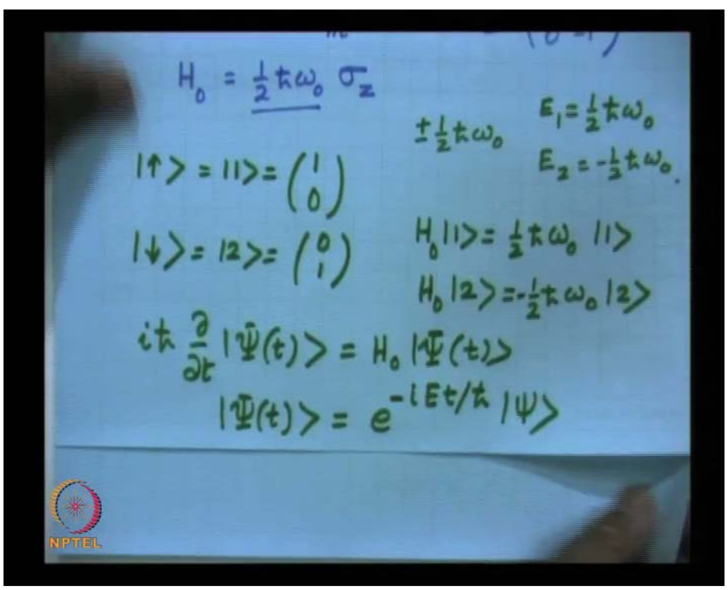

So, omega 0 is defined to be equal to q b 0 m. Therefore, the Hamiltonian is equal to half h cross omega naught sigma z. This is constant. So, this is constant. The eigen values of sigma z as you know sigma z is 1 0 0 minus 1 plus 1 and minus 1. So, the eigen values of h naught are plus half h cross and minus half h cross omega naught. So, my E 1, the two energy state is E 1 is equal to half h cross omega naught and E 2 is equal to minus half h cross omega naught and the two Eigen kets are the spin up state. The z up state which is denote by 1 and that in terms of a column matrix. It is given by 1 0 or the spin down state which I denote by ket 2, which is equal to 0 1. These are the Eigen kets of the Hamiltonian. So, h naught ket 1 is equal to half h cross omega naught ket 1h naught ket 2 is equal to minus, sorry minus half h cross omega naught ket 2.

Now, the time dependent Schrodinger equation is given by i h cross delta by delta t psi of t h naught psi of t. I can use the separate method of separation of variables and write psi of t is equal to e to the power of minus i E t by h cross psi and we will obtain if I substitute this here because h naught is independent of time, we will obtain if I substitute this solution in this equation.

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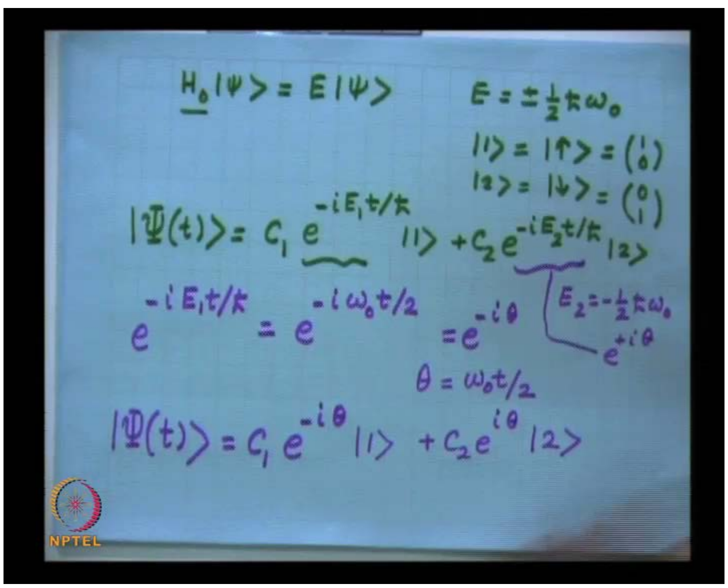

We will obtain H naught psi is equal to E psi. We already know the Eigen values and Eigen functions. Eigen kets of h naught and the two Eigen values are e equal to plus minus half h cross omega naught and the two Eigen kets are ket 1, which is the spin up state and ket 2, which is the spin down state and this is denoted by 1 0 and this is denoted by 0 1. This is the general solution. It is obvious that the general solution of this equation will be psi of t is equal to c 1 e to the power of minus i E 1t by h cross ket 1plus c 2 e to the power of minus i E 2 t by h cross ket 2.

Now, this quantity is equal to say e to the power of minus i e 1 t by h cross E 1 is equal to half h cross omega naught. So, if h cross h cross cancels out, so this becomes minus omega naught t by 2. This I represent by e to the power of minus i theta. That theta is equal to omega naught t by 2 and e 2 as we know is just minus half h cross omega naught e 2 is equal to minus of e 1. So, instead of e to the power of minus i theta, this quantity will become e to the power of plus i theta. Therefore, we will have psi of t will be equal to c 1e to the power of minus i theta ket 1plus c 2 e to the power of i theta ket 2.

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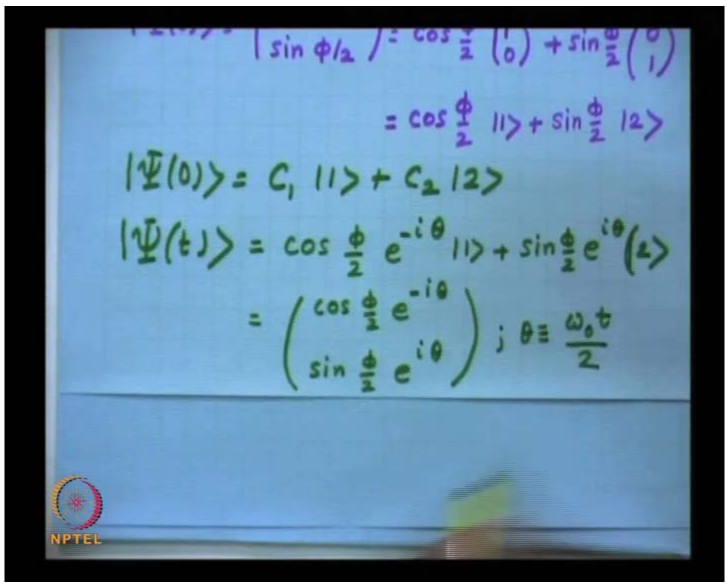

Now, let us suppose that at t equal to 0, this system is in the state given by this ket psi of 0 is equal to cos phi by 2 sin phi by 2. This is equal to, it is in the superpose state cos phi by 2 ket 1. Let me write it down carefully 1 0 plus sin phi by 2 0 1. This is ket 1 and this is ket 2. So, this is equal to cos phi by 2 ket 1 plus sin phi by 2 ket 2. So, we had seen that c 1 at t equal to 0. This factor is 1 and this factor is 1. So, psi of t let me write this down that psi of t equal to 0 was c 1ket 1plus c 2 ket 2.

So, therefore, c 1 is equal to cos phi by 2 and c 2 is equal to (refer time: 14:00) sin phi by 2 and therefore, psi of t. If i write this down that c 1 is equal to cos phi by 2 cos phi by 2 e to the power of minus i theta ket 1 plus sin phi by 2 e to the power of i theta ket 2, I am sorry ket 2. So, therefore, this becomes cos phi by 2 e to the power of minus i theta sin phi by 2 e to the power of plus i theta, where theta as I had mentioned was defined to be equal to omega 0 t by 2. Now, this wave, this ket, therefore describes the entire information about the system. Let us suppose we want to find out what is the expectation value of s x or s y or s z. So, I have here psi of t.

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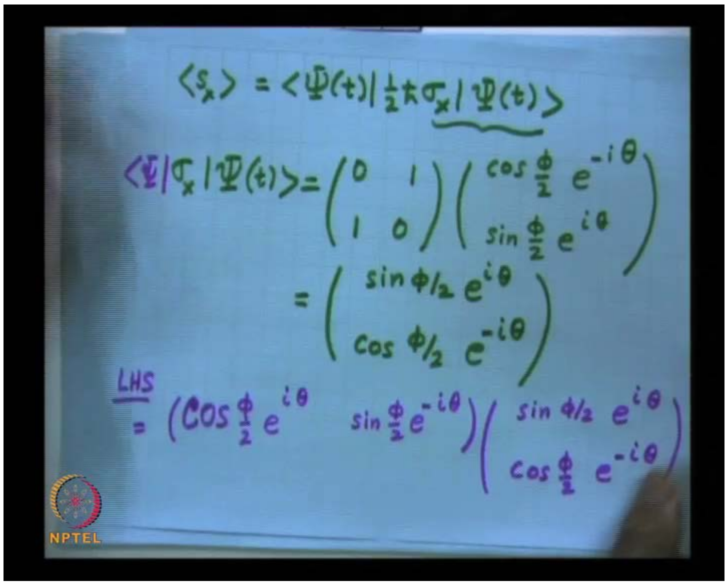

So, I want to make a measurement of s x. So, the s x, the average value of s x will be s x is psi of t. Just we have to do it carefully s x is half h cross sigma x psi of t. So, let me just do this part. Sigma x psi of t will be equal to sigma x is 01 1 0 operating on cos phi by 2 e to the power of minus i theta sin phi by 2 e to the power of i theta. The algebra is trivial. So, this becomes this goes sin phi by 2 e to the power of i theta cos phi by 2 e to the power of minus i theta.

Now, I have to multiply by bra psi t, so the left hand side will become equal to the bra will be psi t. So, this will be cos phi by 2 e to the power of plus i theta sin phi by 2 e to the power of minus i theta conjugate imaginary of this multiplied by sigma x psi t. So, that is equal to sin phi by 2e to the power of i theta cos phi by 2 e to the power of minus i theta. So, if you work this out, so this becomes as you would readily see cos phi by 2 sin phi by 2.

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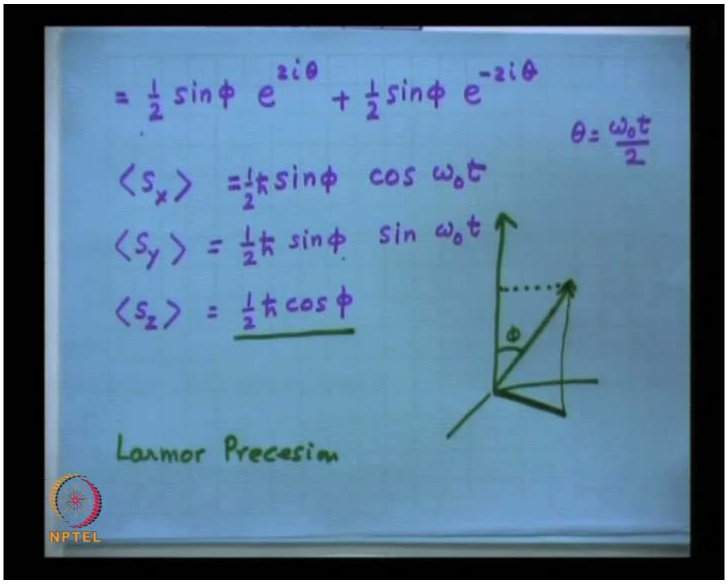

So, that is equal to half sin phi because you have sin 2 a is equal to 2 sin a cos a. So, cos phi by 2 sin phi by 2 is half sin phi, then e to the power of 2 i theta plus sin phi by 2 cos phi by 2 is again half sin phi e to the power of minus 2 i theta. So, we obtain finally that the expectation value of s x is if I take sin theta outside, so this would be e to the power of 2 i theta plus e to the power of minus 2 i theta divided by 2. So, that is cos 2 theta and theta was equal to omega 0 t by 2. So, 2 theta will be cos of omega 0 t.

So, this is the expectation value of sigma x. So, sorry this was the expectation value s x. So, this was the, I have taken out half h cross. I have only calculated the expectation value of sigma x. So, if I multiply by half h cross, so this will be half h cross times this. If I similarly leave it as an exercise for you to calculate expectation value of half s y half h cross sin phi sin omega 0 t and this changes with time but s z, you will have half h cross cos phi. Notice that whereas s x and s y vary with time, s z does not vary with time. This gives us a classical module for the time variation of the angular momentum vector. If I denote this by the angular momentum vector and this as my z axis, then the z component is constant.

So, if this angle is phi and if this vector processes about the magnetic field processes about the z axis, then this precession is known as the Larmor precession. So, you can see that as it processes over along the z axis, its z component remains constant. So, you have this is my s vector. The z component is half h cross sin phi cos phi and if I project it on the x axis, then the projection on the x y plane will be half h cross sin phi which will rotate in the x y plane with angular frequency omega 0.

So, this is a classical model. You explain this three equations and this precession is known as the Larmor precession. In fact, classically also one obtains the same result. We conclude this part mentioning that in addition to this, we have this constant magnetic field.

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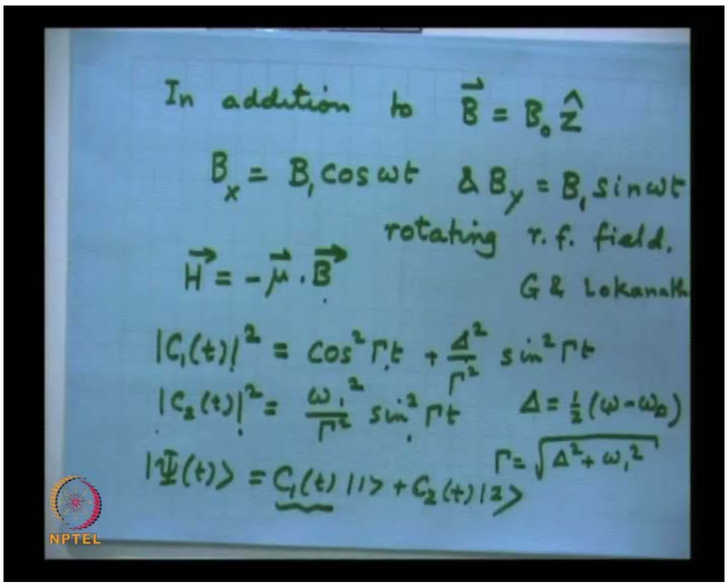

So, in addition to the constant magnetic field, that is b equal to b 0 which is in the z direction, let us suppose we have in the x y plane, a rotating magnetic plane b 1 cos omega t and b y is equal to b 1 sin omega t. So, this is known as a rotating radio frequency field, rotating r. f. field, radio frequency field but when we can similarly in the presence of times, we can again write down the hamiltonian minus mu dot b, but in this case, the Hamiltonian itself will be time dependent but it is a two state problem.

It is really a very straight forward problem. It is solved in our book by myself and Professor Lokanathan and I leave it as an exercise. It is a very nice exercise. I would like you to show that the c 1 t mod square comes out to be cos square gamma t the coefficient. You see we write the solution as psi of t is equal to c 1 of t. Now, here you will have the coefficients will change with time because the Hamiltonian itself is a time dependent quantity.

So, c 1 of t ket 1 plus c 2 of t ket 2, so then you will find the mod c 1 square t cos square gamma t plus delta square gamma square multiplied by sin square gamma t and the final result is c 2 of t mod square is equal to omega 1 square by gamma square sin square gamma t, where delta is equal to half into omega minus omega 0 half into omega minus omega 0. Just let me check this.

So, this is b 1 and gamma is equal to delta square plus omega 1 square. So, then delta is 0, capital delta is 0 omega is equal to omega 0. So, this becomes 0 and this becomes 1 and therefore, c 1 t mod square becomes cos square gamma t and c 2 mod t square becomes sin square gamma t. There is a periodic flipping of spin and each time the spin flips, it absorbs a certain amount of energy from the radio frequency field and this field tripping of the spin allows us to determine either the magnetic moment of the substance of the proton or of the molecule or the strength of the magnetic field.

This principle is used not only in nuclear magnetic resonance techniques and also in the electrons spin resonance techniques. So, in this experiment what we have is a constant static magnetic field in the z direction and a rotating magnetic field in the x y plane. We then write down the Hamiltonian and solve it, but here it is one of those unique cases in which the Hamiltonian itself depends on time. So, therefore, we write down the solution as c 1 of t ket 1 plus c 2 of 2 t ket 2 and substitute it in the Schrödinger equation and obtain a set of coupled equations between c 1 and c 2. It is possible to solve this equation rigorously.

One obtains rigorously correct solution for this two state problem and one finds that a particular frequency when omega is equal to omega naught, the spin flips for certainty and when that happens, it absorbs a certain amount of energy. If the radio frequency feel which can be observed, which can be very accurately observed, this allows us to find out the magnetic moment of the proton or the molecule that we are trying to measure. Hence, one can determine the concentration of the type of particles or the proton distribution that we are looking even inside the human body and this phenomenon is known as the nuclear magnetic resonance technique and has found extremely white spread application in medicine and also other areas.

So, therefore, we have considered two simple cases. One in which there is a static magnetic field as we demonstrated the Larmor precession and then, we just outlined the solution and gave it as a problem when we have in addition to a static magnetic field, a magnetic field in a rotating radio frequency field in the x y plate. Now, we will conclude our analysis of operator algebra by deriving the expressions for the spherical harmonics.

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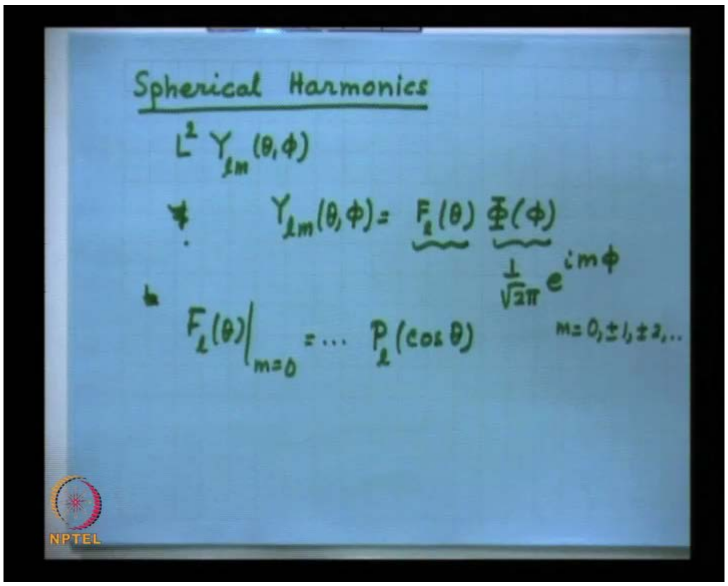

Now, you see we had while discussing the, let me write it down. We had not really considered the theory of angular momentum. We have not derived expressions for spherical harmonics. What we did find out was that the one square operator, the eigen function we denoted by y l m theta phi and we showed that for l equal to for m equal to 0 that y. First of all we showed that y l m theta phi could be written as f l of theta phi of phi and this we had shown was equal to 1 over root 2 pie to the power of i m phi and for the Eigen function to be, well to be single valued m can take the value 0 plus minus 1 plus minus 2 etcetera.

Then, we had shown, also we did not derive the expression of f l of theta, but we did say that for m equal to 0 that the m equal to 0, the function l of theta for m equal to 0 was some multiple of p l, the Legendre polynomials.

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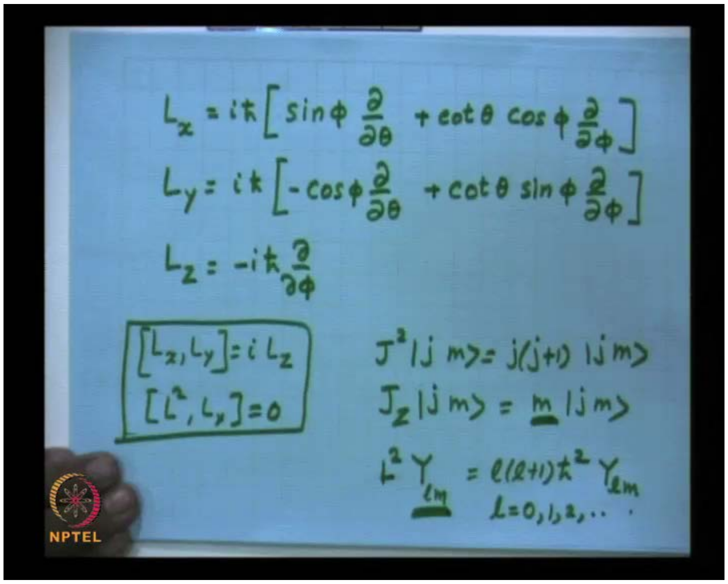

Now, as you may recollect that we have derived the operators, the differential operators for l x y and l z l x was equal to i h cross sin phi delta by delta theta plus cos theta cos phi delta by delta phi l y is equal to i h cross minus cos phi delta by delta theta plus cos theta sin phi delta by delta phi. We have from first principle also derived that l z, the differential operator, differentiator representation was delta by delta phi.

Now, these obviously satisfy the same commutation relations, that is l x, l y is equal to i l z and then, l x square, l x is 0. So, using these just these commutation relations, we had found that j square, the Eigen values are j into j plus 1 h cross square j m and j z. These are simultaneous Eigen kets of j square and j z. So, therefore, since in this particular case, since m takes integer values, so the eigen values of l square must be and we denote the eigen functions by y l m theta phi must be equal to l into l plus 1 h cross square because m now we have showed that m takes only integer values of 0. So, the Eigen values of l square must be equal to this times y l m theta, sorry, where l now takes 0 1 2 3 etcetera.

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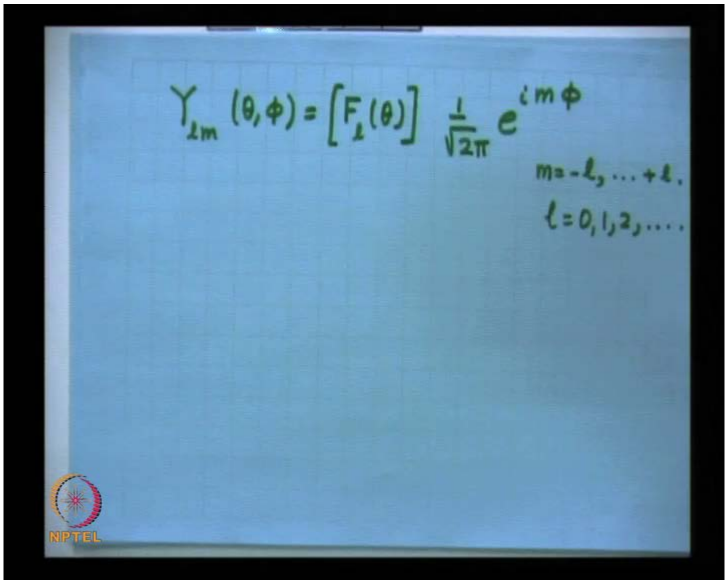

Now, what are the expressions for y l m? For that we proceed as follows. First of all we note that y l m theta phi, the phi dependence f l of theta, this is the theta dependence and the phi dependence is 1 over root 2 pie to the power of i m phi and m goes from minus l to plus l and l can take values 0 1 2 3 etcetera.

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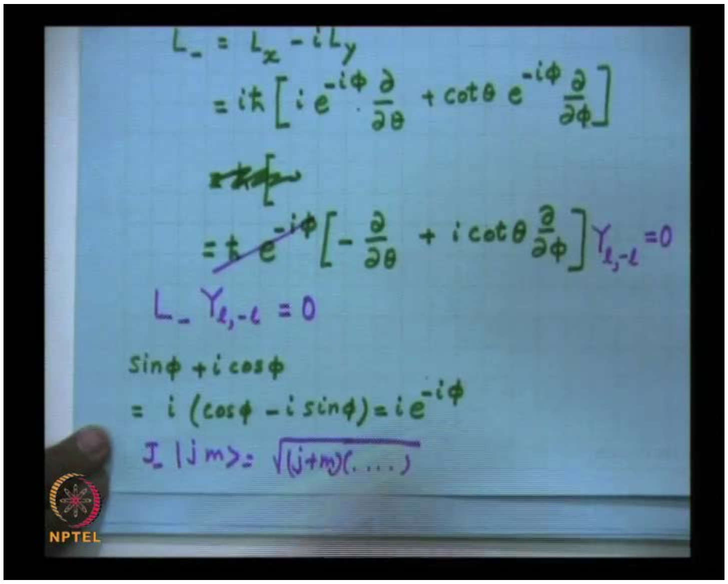

Now, from this equation we can write down, sorry l plus and l minus, that is I can write down plus which is l x plus i l i l y or l minus, let me write down l minus. I leave it as an exercise for you to find out l x plus i l y. So, l minus let us do it very carefully. l x minus i l y minus i l y. So, this will be sin phi minus minus plus i cos phi. So, I will have, let me write it down below that sin phi plus i cos phi. So, if I take outside I, so I get cos phi minus i sin phi.

So, this will be i times e to the power of minus i phi. So, you will have, so if I take i outside, sorry so l minus will be l x minus i l y. So, you will have i h cross i e to the power of minus i phi delta by delta theta plus cos theta cos phi minus i sin phi. So, cos theta e to the power of minus i phi delta by delta phi, if I take i inside, so this will be equal to h cross. If I can assume a system of units, where h cross is 1, so we will have h cross e to the, I am sorry I can take e to the power of minus i phi also outside. So, h cross e to the power of minus i phi if I take outside, so i times i is minus 1 minus delta by delta theta plus i times cos theta delta by delta phi, ok.

Now, l minus by l when m is minus l, we had this relation that j minus j m is under root of j plus m. So, m cannot be less than minus l and also from this relation l j plus m will be l minus l that should be 0. So, this operating on this will be 0. So, l minus operating on y l minus l must be 0. So, this is just a factor. So, I can neglect this and you must remember that.

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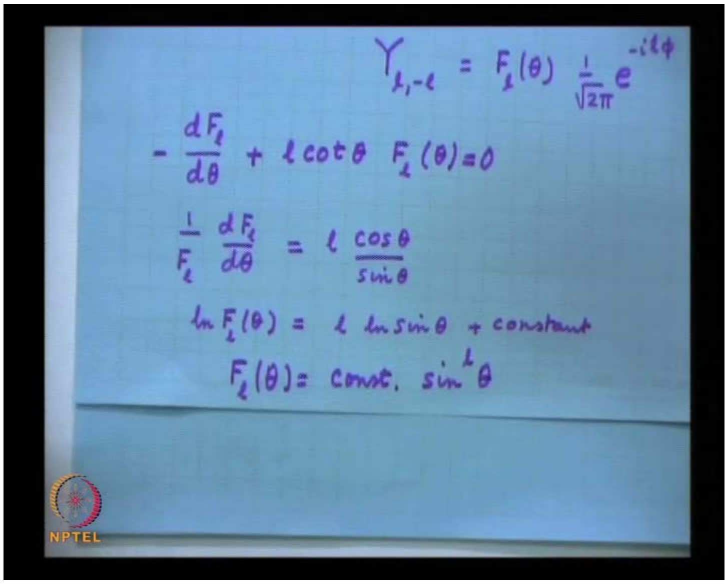

So, let me write it down again. So, therefore, minus delta by delta theta of y l, minus l plus i cot theta delta by delta phi y l minus l is equal to 0, but we know that l, minus l is equal to f l theta 1 over under root of 2 pi e to the power of i m phi m is minus l, so minus i l phi. So, this tells us that if I differentiate this with respect to theta, so you get d f l by d theta with a minus sign and if I differentiate this, so this would be i times minus i plus 1. So, plus 1 l times cot theta f l of theta is equal to 0 and this root 2 pi and this factor cancels from both the term. I hope this is clear.

So, I take these to the other side. So, the minus sign minus sign goes off. So, 1 over f l theta d f l by d theta is equal to l cos theta by sin theta. So, if I integrate this log of f l theta is equal to l of log of sin theta plus constant, so this is log sin theta to the power of l.

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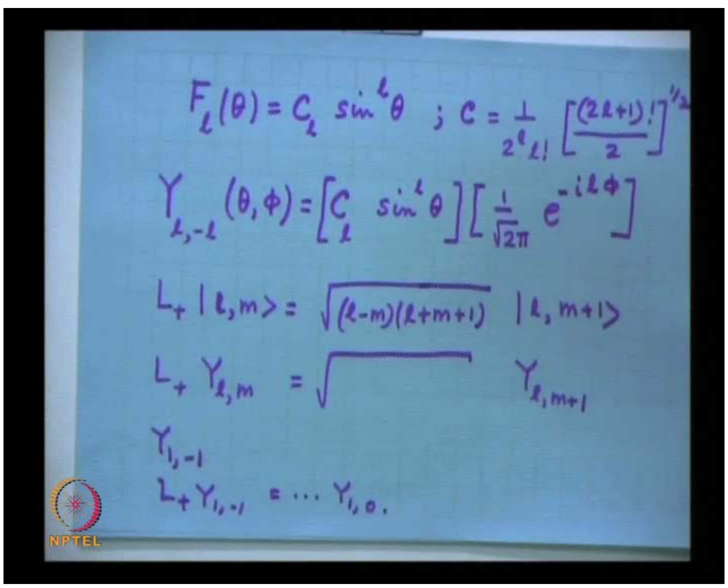

So, therefore we get the relation that f l theta is equal to constant times sin to the power of l theta. You can find out the normalization constant and the normalization constant is that the f l theta is some constant times sin to the power of l theta, where the determination of normalization constant is slightly tedious, but let me give you the final result, 2 to the power of l l factorial 2 l plus 1 factorial by 2 raise to the power of half. So, my Y l, so let me put it as c l, minus l theta phi is equal to c l sin l theta multiplied by 1 by root 2 pi e to the power of minus i l phi. We have therefore, the complete expression for Y l, minus l.

Now, I can now use ladder operator to find out higher other spherical harmonics. For example, l plus as you know j m j l plus l, m as you know is equal to l minus m l plus m plus 1 l, m plus 1. So, l plus Y l, m is equal to under root of this thing Y l m and it will be automatically normalized. So, therefore if I know for example Y 1, minus 1 that I can using this relation l plus y, minus 1 will give me something like y, 1, 0. So, I hope I have been able to tell you a very nice method for obtaining without solving any differential equation. You tell me the value of l, you tell me the value of m and just by doing series of operator algebra, I can find out different spherical harmonics.

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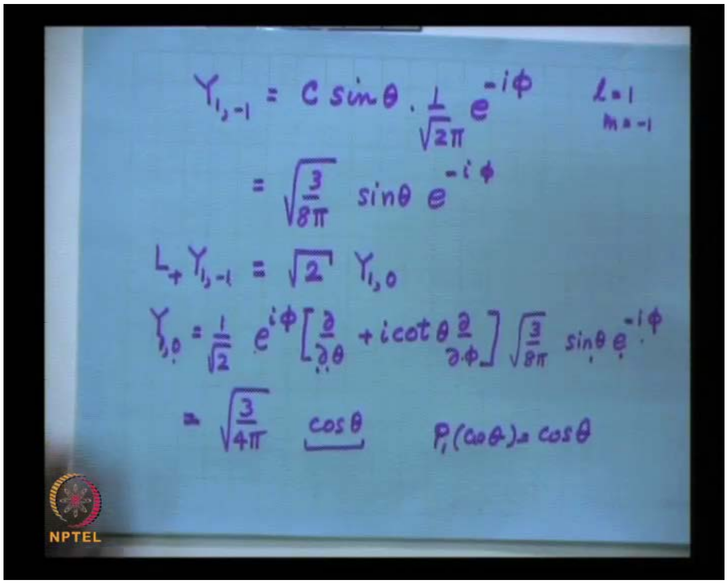

As a simple example you get Y 1 minus 1 that is l is 1, l is 1 and m is minus 1. So, you have here, so you have here. So, c times sin to the power of l theta, that is sin theta 1 over root 2 pi e to the power of minus i phi. You can immediately calculate the value of c and the final result is 3 by 8 pi sin theta e to the power of minus i phi. Now, you have l plus y 1 minus 1. So, this will be equal to j minus l. So, this will be equal to under root of 2 substitute the values and this will be equal to Y, 1 0 because you have j minus m will be 1 minus 1, that is 2 and j plus m plus that will be just 1. So, Y 1, 0 will be 1 over root 2 l plus and the l plus operator is e to the power if I assume e to the power of i phi delta by delta theta plus i cot theta delta by delta phi operating on y 1 minus 1.

So, this is under root of 3 by 8 pi sin theta e to the power of i phi. So, I differentiate this with respect to theta, I differentiate this with respect to phi. It is a very simple algebra and once you do that, you will get under root of 3 by 4 pi cos theta and you can see that e to the power of minus i phi here and e to the power of i phi. So, there will be no phi dependence and this you should have expected because m is 0. So, you will have just this and this is proportional to p f of cos theta, p 1 of cos theta, p 1 of cos theta is just cos theta.

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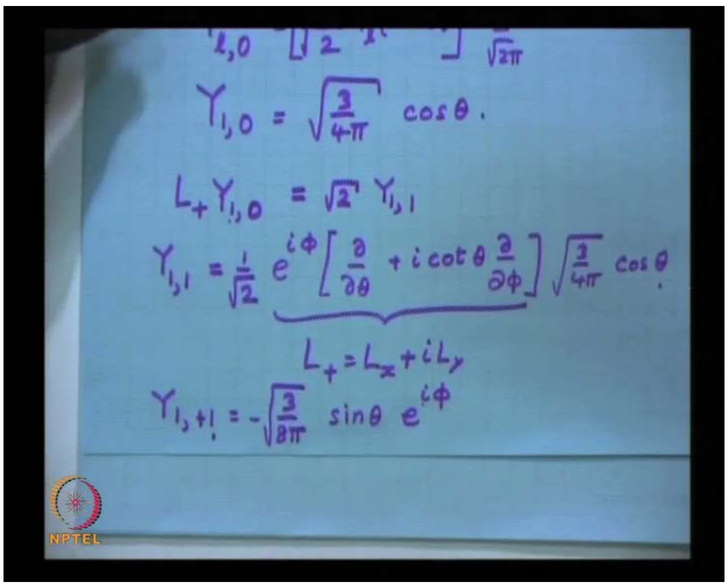

What I am trying to tell you is that in fact, there is another formula Y l, 0 which we have said that this was constant times p l of cos theta multiplied by I think under this factor is 2 l plus 1 by 2 into 1 over root 2 pi to the power of m is 0. So, Y 1, 0 is equal to 3 by 4 pi, 3 by 4 pi. So, 2 l plus 1 is 3 into 2 into 2 is 4 pi and that is cos theta. Now, I can immediately write down what is Y, 1. So, l plus Y 1, 1 sorry 1, 0 is under root of j minus m. So, that is 1 and that is square root of 2 again and then, J plus m plus 1 that is, so this is equal to y 1. The m value has to increase by 1.

So, Y 1, 1 is equal to 1 over root 2 1 over root 2 and l plus operator is e to the power of i phi delta by delta theta plus i cot theta delta by delta phi. This is the 1 plus operator. That is l plus which is equal to l x plus i l Y. This is operating on Y 1 0. So, this is 3 by 4r pi cos theta. So, this is independent of phi. So, this term is 0. There is no phi dependent term here and cos theta will become minus sin minus sin theta and there will be e to the power of i phi. So, this will give me an expression for Y 1 1 and the final expression will be, just let me look at the book and this will be just one second. So, y 1 1 will be equal to minus under root of 3 by 8 pi sin theta into e to the power of i phi. Notice that. So, plus 1, so since m is 1, so you will get this sin e to the power of i phi and there is a minus sign now.

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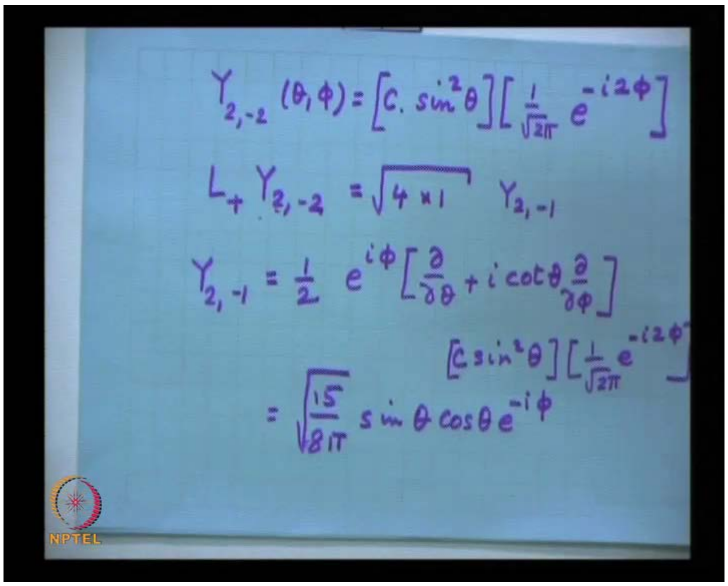

Similarly, you can start with Y 2 minus 2 theta phi. I can write down immediately this thing. This will be some constant sin of a theta that is sin square of theta. This is the theta dependence and then, the phi dependence is 1 over root 2 pi e to the power of minus i 2 phi. So, now you operate this by l plus operator l plus Y 2 minus 2 will be equal to Y 2 minus 2 is l is equal to 2 and m is equal to minus 2 l minus m is l minus will be 4 and then, l plus m plus 1. So, that is 1 this into y 2 minus 1.

So, y 2 minus 1 will be equal to 1 over 2 square root of 4 is l plus l plus is e to the power of i phi delta by delta theta plus i times cot theta delta by delta phi and then, the whole thing sin square theta c times. You have to find out the constant into 1 over root 2 pi e to the power of minus i 2 phi and so on. So, just differentiate this. You get expressions for y 2 minus 1 comes out to be, let me write down. Then, answer 15 by 8 pi sin theta cos theta e to the power of minus i phi.

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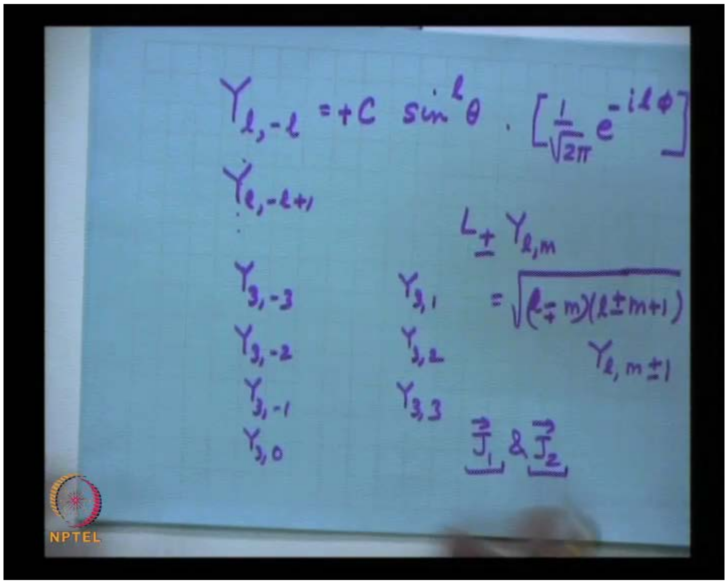

So, I have told you the recipe. You start with for any given value of l. You start with y l, minus l. So, this will be c sin to the power of l theta and then, the phi dependence will be 1 over root 2 pi e to the power of minus i l phi and then, keep on applying ladder operator. This is the convention. This always has a plus sign and then, you do this as the number of one value becoming larger. Of course, it becomes more cumbersome, but in principle it is very straight forward. We use continuously the ladder operators and find out Y l l minus l plus 1 and so on. So, if you start with Y 3minus 3, then you land up with Y 3 minus 2. Then, you land up with Y 3 minus 1 and then, Y 3 0 and then, Y 3 1 and Y 3 2 and 2 and Y 3 3.

So, you will be able to get rigorously correct expressions with the appropriate sign appropriately normalized. All that you have to continuously used are these equations that l plus minus Y Y l m is equal to under root of l minus plus m l plus minus m plus 1 Y l m plus 1 for the upper sign minus 1 for the upper. In our next lecture, what we will do is we will add angular momentum. We will consider two angular momentum vector j 1 and j 2. J 2 we will assume that the components of j 1 commute with component of j 2 and then, we will add these two angular momentum vectors.

For example, if you add two spin half particles, if you have neutron and electron and a proton or the neutron and the proton, both are spin half particles. So, these two forms the total angular momentum which is 1 or 0. The first one is called the triplet state and the second one is called a single state. So, we will develop the theory of addition of angular momentum and which lead to the concept of the Clebsch-Gordan coefficients. Thank you.