Basic Quantum Mechanics
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Module No. # 08
Angular Momentum – II
Lecture No. # 03
Pauli Spin Matrices and The Stern Gerlach Experiment

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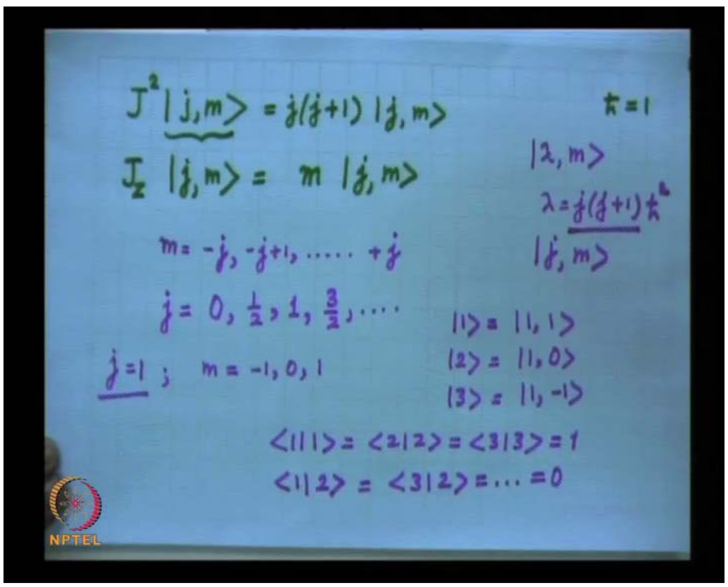

In our last lecture, we had discussed the simultaneous eigen functions for the operator J square and J z and we had obtained that we had represented ket j m as the simultaneous eigen vectors of the operator J square and the corresponding eigen values were j into j plus 1. We are actually assuming a system of units in which h cross is equal to 1.

So, this is j, m and these are simultaneous eigen kets of J square and J z and the eigen value of J z is m h cross. We will suppress the h cross j m. If you recollect that initially we had represented the eigen kets as lamda, m where lamda was the eigen value of the operator J square, but then lamda is equal to j into j plus 1. So, we replaced lamda by just 1 symbol instead of j into j plus 1 by the symbol j, m with the understanding that here the eigen value of J square is lamda, but here the eigen value of J square is not j. It is j into j plus 1 multiplied by h cross square.

Then, we showed that the the values of m goes from m takes the value for a given value of j from minus j to minus j plus 1 so on to plus j and the values of j can take are 0, half, 1, 3 by 2 etcetera. Now, let me take a simple example. Let me take the example j is equal to 1. Then, the corresponding value of m will be three values of m minus 1 0 1. So, corresponding to the eigen value j equal to 1, there will be three eigen kets. Let me denote this by 1 as 1 1. This is the value of j and the second is the value of m 2 is 1, 0 and the three independent eigen kets is 1 minus 1.

We are assumed to form an orthonormal set of kets, that is bra 1 ket 1 is equal to bra 2, ket 2 is equal to bra 3, ket 3. These are all orthonormal. These are all normalized. On the other hand, any dot product of two vectors 1, 2 or 3, 2, they are all 0.

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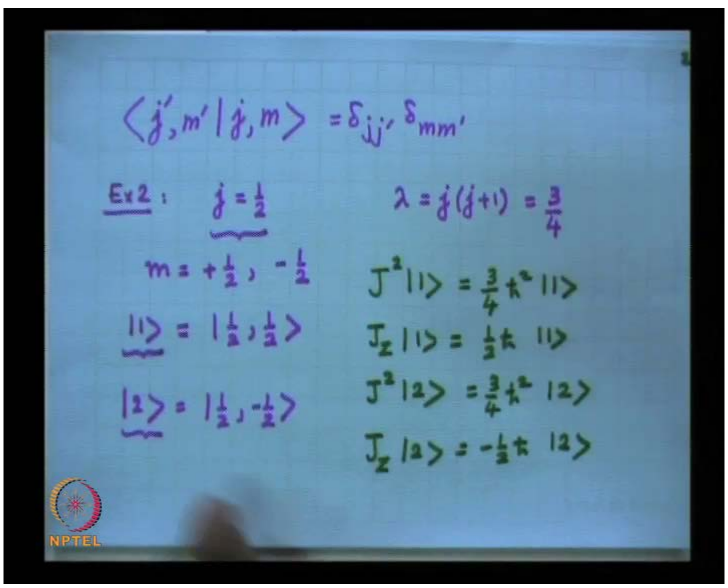

So, therefore, we have the orthonormality relation that j prime m prime j, m, this is equal to delta j j prime delta m m prime. That is this is equal to 0. If j is equal, not equal to j prime or m is not equal to m prime, but if j is equal to j prime and m is equal to m prime, then the value is 1.

Now, let me take the second example. Example two corresponding to the angular momentum is equal to half. So, j is equal to half. So, the eigen value lamda which is equal to j into j plus 1. So, this is the eigen value of the operator J square and the two eigen kets. So, the 2 m values are m is equal to plus half and minus half. So, its span has two orthonormal vectors. One we represented by ket 1 which is equal to half half and the second is half minus half. Now, this and this are simultaneous eigen vectors of the operator J square and j z.

So, therefore, J square ket 1 is equal to half into half plus 1, that is 3 by 2, that is 3 by 4. Actually, h cross square ket 1 if I assume a system of unit in which h cross is equal to, then this is unity. Similarly, J z ket 1 is equal to half h cross ket 1 and J square ket 2. The eigen value of the j value remains the same. So, that is equal to 3 by 4 h cross square ket 2 and j z ket 2 is of course minus half h cross ket 2. So, this allows as to write the matrices for j for any of these operators.

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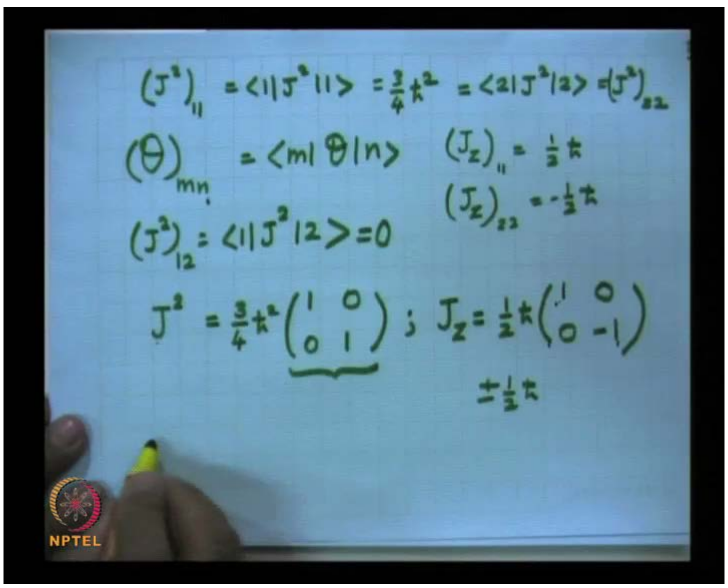

You can see that that J square 1 1 is equal to 1 J square 1. So, if I take this here, so this will be just 3 by 4 h cross square. Similarly, since j square 2 is so much, so this is also equal to 2 j square 2. So, this is j square 2 2 any operator. If I write m n, it is bra m o n. This is the m n-th matrix element of the operator 0 in a system, where ket n are the base vectors are the unite vectors. Similarly, J z ket 1 is proportional to ket 1. So, you have J z 1 1 will be equal to half h cross and j z 2 2 is equal to minus half h cross.

Now, in this case, if I write J square 1 2, so this will be 1 J square ket 2. So, J square ket 2 is proportional to ket 2 and the bra 1, this will be 0. So, we get the following representation, following matrix representation of the operator j square. So, this will be 3 by 4 h cross square 1 0 0 1 and the corresponding operator representation for J z is equal to half h cross. This is half h cross and minus half h cross. So, 1 0 0 minus 1.

Notice that the eigen values of this matrix is just 1. So, with the eigen values of this matrix is just of this operator is just 3 by 4 h cross square. The eigen values of J z is 1 minus 1 as of this matrix is 1 and minus 1. So, therefore the eigen values of J z are plus half h cross and minus half h cross. Similarly, we had obtained for J x also.

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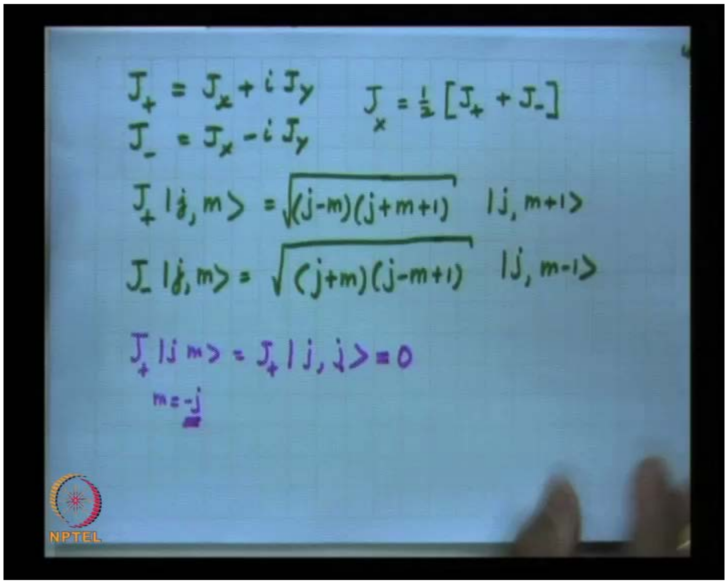

For example, we had defined the two operators. If you recollect j plus was equal to j x plus i j y and j minus was equal to j x minus i j y, if I add the 2, I will get j x is equal to half j plus plus j minus. So, now we know that j plus ket j m we had derived. This is equal to under root of j minus m j plus m plus 1 ket j m plus 1 and then, j minus ket j m was equal to this. You must remember j plus m j minus m plus 1 j m minus 1. The easiest way to remember is that if m, the maximum value of m is j, so therefore j plus j m is equal to j. If I assume that m is equal to j, then it will become proportional to j plus 1 which is impossible. So, this must be a null ket.

So, when m is j, this factor must become 0 and when m is equal to minus j, then since this is the minimum value of m, this factor must be 0, ok. Now, we now use these two relations to obtain j plus ket 1 and j plus j plus ket 2.

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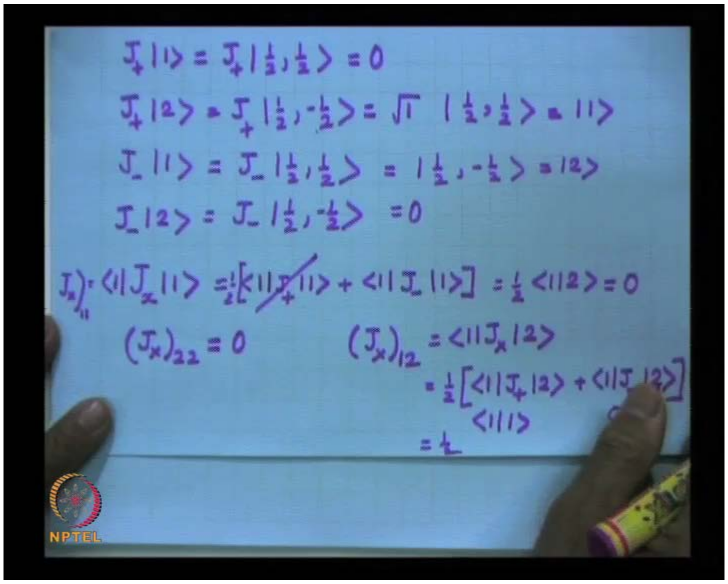

So, let me write it down. So, we have here j plus ket 1 is j plus 1 is 1, 1. So, j value is 1 m value is 1. So, if I apply that, so j minus m is 0. So, this is a null ket, ok. Similarly, j plus ket 2 will be equal to, I am sorry we are considering j is equal to, sorry let me redo this.

So, we have these two equations. So, we consider j is equal to half. So, we have two states, ket 1 is equal to half half and the these are the two base kets half minus half. So, you will have j plus ket 1 will be equal to j plus ket half half. So, j is equal to half m is equal to half. So, from this equation, we get 0. So, this is a null ket. Similarly, j plus ket 2 will be equal to j plus half minus half. So, this will be equal to m is minus half. So, half minus minus half, that is plus half half plus half is 1 and this is 1. So, this is square root of 1 and this will be half m plus 1, that is half half square root of 1 is 1. So, this is ket 1. I hope this is clear.

Similarly, let me look at the other. So, j minus ket 1 will be equal to j minus half half. So, this will be half plus half, that is 1 half minus half plus 1. That is also 1. So, this will be half n minus 1, that is minus half. So, that is ket 2 and j minus ket 2 will be j minus half minus half. So, half minus half, this will be a null ket. So, using these relations, we can immediately obtain the matrix representation for the operator j x. So, j x, say 1 1. So, this is my j x 1 1 1 1 matrix element. So, this will be equal to 1 factor of half outside j plus ket 1 plus 1 j minus ket 1 because we had developed, we had seen the relation that j x was equal to half j plus j minus. So, half j plus.

Now, j plus ket 1 is a null ket, so this is 0 j minus ket 1 is ket 2. So, this becomes equal to half 1 2. These are orthonormal kets. So, this is 0. Similarly, I can obtain, I leave it as an exercise for you. J x 2 2 is also 0, but j x 1 2, this is equal to bra 1 j x ket 2. So, this is equal to half half bra 1 j plus ket 2 plus 1 j minus ket 2 j plus ket 2 is equal to ket 1. So, this is 1 1 which is 1 and j minus ket 2 is a null ket, so this is 0. So, this becomes equal to half .

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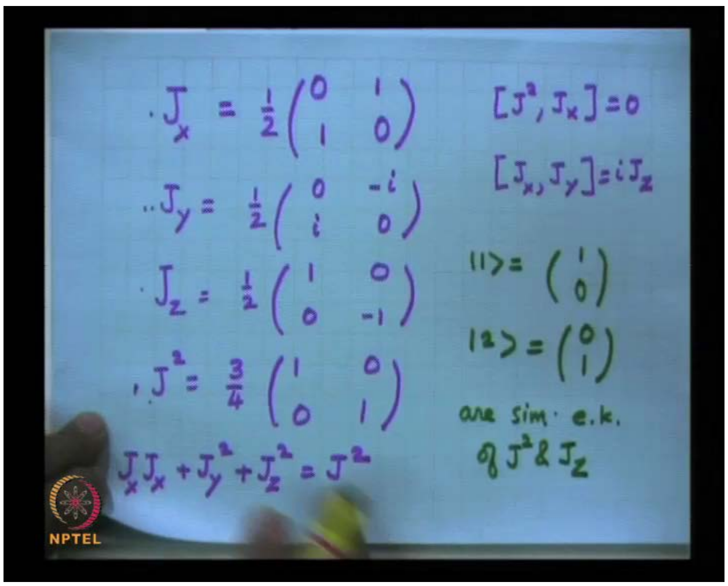

So, we have found out all the matrix element of j x. So, the j x operator is equal to half the diagonal element as 0 and this is 1. Actually, it is multiplied by h cross, but we are assuming h cross to be equal to 1. Similarly, we can. I leave it as an exercise for you to show that j y is equal to half 0 minus i 0 and J z, we have already found out to be half 1 0 0 minus 1 and J square is equal to 3 by 4 1 0 0 1. Having obtained this, I leave it as an exercise for you to show that if you add j x square, that is j x times j x plus j y times j y plus j z times j z and add these 2 by 2 matrices, you will find that this will be j square. J square is proportional to be unit matrix.

So, J square commutes with j x, j square commutes with j y, j square commutes with J z. So, J square, j x is 0, J square, j y is 0, J square, j z is 0, but although j x and j y do not commute with each other, so you can show this that j x j y is equal to i, j z.

Now, listen to this carefully. You can have simultaneous eigen kets of J square and j x, you can have simultaneous eigen kets of J square and j y, you can have simultaneous eigen kets of j square and j z. In fact, the eigen kets that we have been using ket 1 which are eigen kets of j z is ket 1 is 1 0 and ket 2 is 0 1. These two are simultaneous eigen kets, simultaneous eigen kets of J square and j z. These matrices are known as Pauli Spin matrices.

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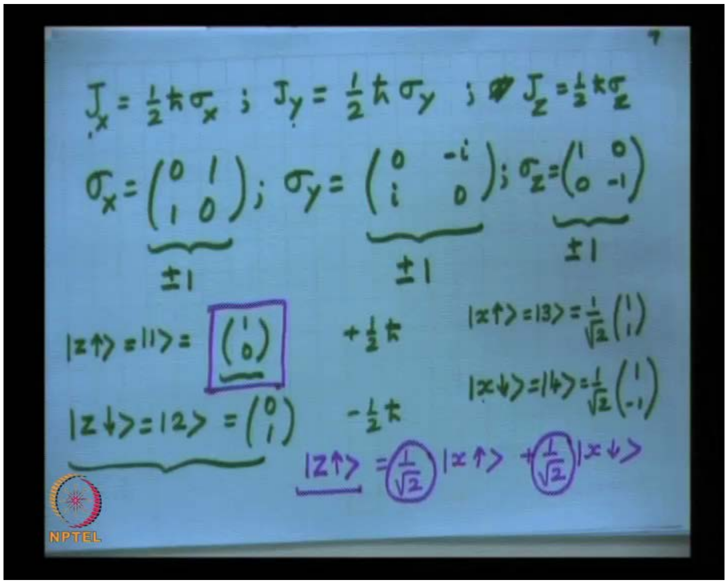

So, let me write these down that we have j x, sorry jx is equal to half h cross. We need not put the h cross sigma x. Let me put this h j y is equal to half h cross sigma y and j z, sorry jz is equal to half h cross sigma z, where sigma x is equal to 0 1 1 0, sigma y is equal to 0 minus 1 1 0 and sigma z is equal to 1 0 0 minus 1.

It is a very simple to show that the eigen values of these are plus minus 1, eigen values of this matrix are also plus minus 1, eigen values of this is a diagonal matrix, the eigen values of plus minus 1. Therefore, if I make a careful measurement of j x, I will get plus half h cross or minus half h cross. If I make a measurement of j y, then I will get plus half h cross and half h cross and if you make measurement of sigma j z, then you will get plus half h cross or minus half h cross.

The eigen kets that we have written 1 1 0, these are known as the z up state or it is sometime written as the spin up state. Spin up means z component of the spin angular momentum is pointing upwards and the eigen value is plus half h cross. These are the simultaneous eigen kets. Similarly, the z down state is the second state and this is denoted by 0 1 and if the system is in this state and if I make a measurement of j z, I will obtain minus half h cross.

Now, therefore, let me ask you this question that the system is in the z up state, that is if I make a measurement of j z, I will get the eigen value half h cross, but if we now make a measurement on the system for j x, what are the eigen values will I get for this? We must express the state as a linear combination of the eigen states of j x. Now, the eigen kets of this of the x up state, let us suppose I denote by ket 3. If you find out the normalized eigen vector of this, it is very easy to show that this is 1 by root 2 1 1. Similarly, the x down state, let us suppose I denote this by ket 4, so this is equal to 1 over root 2 1 minus 1.

Now, my system is in this state and I want to make a measurement of j x. So, I must express this as a linear combination as a superposition of x up and x down state. So, a little algebra will show that the z up state is equal to 1 by root 2 x up state plus 1 by root 2 x down state. If you multiply this by 1 by root 2, I will get half 1 1. If I multiply this one by root 2 half minus 1, if I add these two up, the second row will vanish and you will get 1 0.

So, therefore, this z up state is in a super position of the x up state and the x down state and therefore, this is the beauty of quantum mechanics. This entire concept of super position of states is a quantum phenomena. If I now make a measurement of j x, then there is a half probability of finding it in the x up state and half probability of finding in the x down state.

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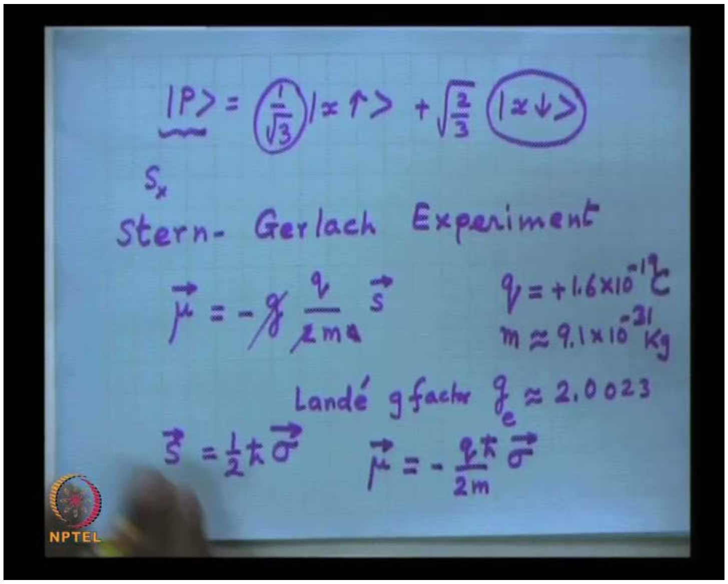

I will illustrate this through an example, but little later. Let me consider another state p. Let us suppose this is 1 over root 3 x up state plus under root of 2 by 3 x down state. Now, this is normalized because 1 over 3 whole square, 1 over root 3 whole square plus root over 2 over 3 whole square is 1. Now, if you ask me the question that if I make a measurement, the system is in a state p and if I make a measurement of the x component of the angular momentum, then will I get x up state or will I get x down state? The answer is I do not know. There is a one-third probability of finding it in the x up state and two-third probability in finding it in the x down state.

Now, I would like to discuss with you a very famous experiment and this experiment is known as the Stern Gerlach experiment. Now, the electron is endowed with an intrinsic angular momentum and this angular momentum is known as is usually referred to as the spin angular momentum. It is not that the electron is rotating about its axis.Tthat is not a correct way to understand the concept of the spin angular momentum. The best way to understand it is to assume that the electron behaves like a tiny magnet and that it has a magnetic moment which is proportional to the spin angular momentum of the electron. This magnetic moment of the electron is given by minus G, which is known as the lande g factor q by 2 m c S vector, ok.

So sorry, there is no c here in the c g a in the m k s system of units. So, q is the magnitude of the charge of the electron. So, this is plus 1.6 into 10 to the power of minus 19 coulombs m is of course the mass of the electron and that as you all know is 9.1. This is 9.1 into 10 to the power of minus 31 kilogram. G is the known as the lande G factor and the value of G for the electron is approximately 2. Actually, it is 2.0023, but we will assume that this to be 2.

So, S this spin angular momentum operator for the electron, this is a small s. Actually, lower case s is equal to half h cross sigma, where sigma x, sigma y and sigma z are the pauli spin matrices that I have just now written. Now, g is equal to 2 if I assume, then this factor cancels out with this factor. So, this becomes the magnetic moment is proportional to minus q by 2 m h cross sigma. Now, let us suppose the electron or actually we consider the experiment cannot be performed, the stern gerlach experiment cannot be performed with an electron because it has an intrinsic charge.

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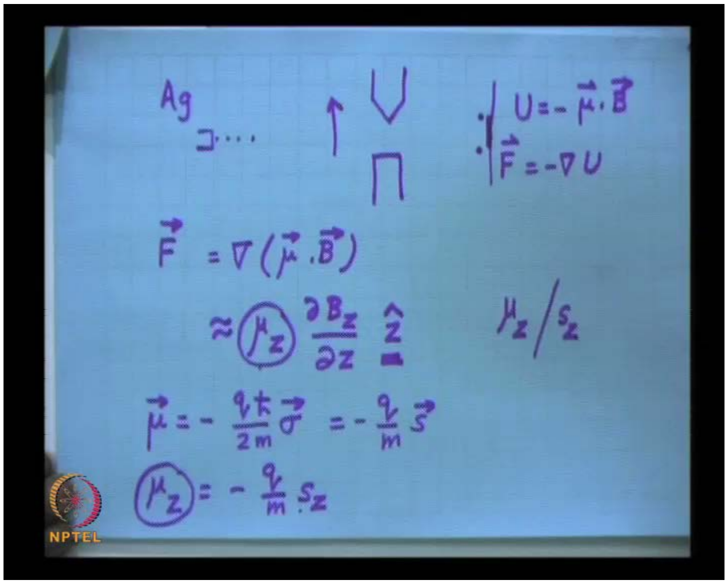

So, the experiment must be performed with neutral silver atoms and because of its wavelength electron, it has the magnetic moment exactly that of an electron. So, I make the neutral silver atoms pass through a very strong inhomogeneous magnetic field in the z direction.

Now, the force that is acting on that because of the magnetic moment of the silver atom, this is equal to gradient of mu times B. Actually, the interaction energy is equal to minus mu dot B and the force is equal to minus gradient u. So, this is equal to gradient of mu dot B and if the magnetic field is predominantly in the z direction, then this is equal to mu z delta B z approximately delta z into z cap. So, predominantly if I apply an inhomogeneous magnetic field in the z direction, the force is predominantly in the z direction and the magnitude of the force is proportional to mu z.

Now, I had just now said, written down that the magnetic moment was equal to minus q h cross by 2 m into sigma or this was actually equal to minus q by m S vector, where s vector is the spin angular momentum vector associated with the electron. So, this is the magnetic moment. So, the z component of this will be equal to minus q by m S z. So, the force that is acting on the silver atom will be proportional to the z component of the magnetic moment or will be proportional to S z. Now, as the silver atoms are coming out from the oven, classically speaking we can assume that the magnets are oriented at random.

So, let us suppose this is a tiny magnet, this is the north pole and the south pole and as it comes out to the magnet, they are oriented at random. Now, the vertical direction is let us suppose the z axis and if the magnet makes an angle theta, then the z component of the magnetic moment will be mu cos theta. Since, the magnets are oriented at random, mu z will continuously vary from plus mu to minus mu as theta goes from 0 to pi.

So, therefore, the force acting on the silver atoms will be proportional to mu z and on the screen, the deflection which will be proportional to the force will have a continuous smear, but when the experiment was carried out by Stern and Gerlach, they had obtained two spots, that is as if the value of mu z or the value of S z was quantized as if you make a measurement of the z component of the spin angular momentum or z component of the magnetic moment. Then, it has two quantized values and this is one of the, considered to be one of the most beautiful and important experiments in quantum theory, this Stern-Garlach experiments.

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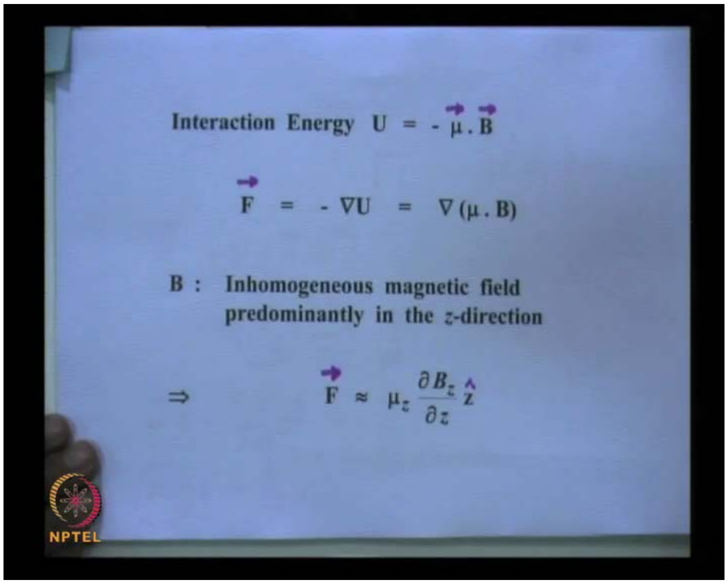

So, let me repeat that the interaction energy for a magnet, all these are vectors, I have denoted by a bolt sign. So, the force, the interaction energy is mu dot e. The force is equal to minus gradient of U. I now apply an inhomogeneous magnetic field predominantly in the Z direction.

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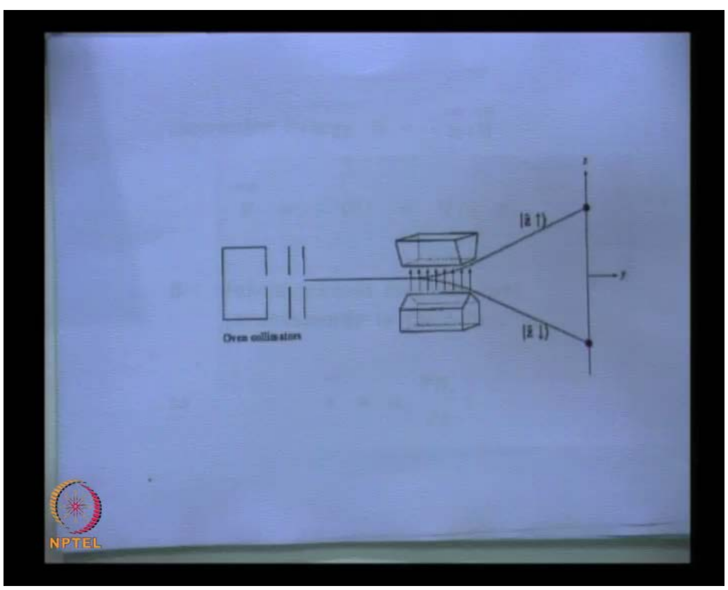

So, therefore, the force that is acting on the silver atom, this is a scalar quantity mu z delta b z by z and this is the unit vector in the z direction. Now, as the silver atoms come out of the (()), the magnets if I consider them as a tiny magnets, they are oriented at random and the deflection which is proportional to the z component of the magnetic field, the z component of the magnetic moment will vary from plus mu 0 to minus mu 0, but instead only two spots were observed. This corresponds to the z up state and this I have exaggerated, the splitting. Actually, the splitting is usually extremely small. So, this corresponds to the z up state and this corresponds to the z down state.

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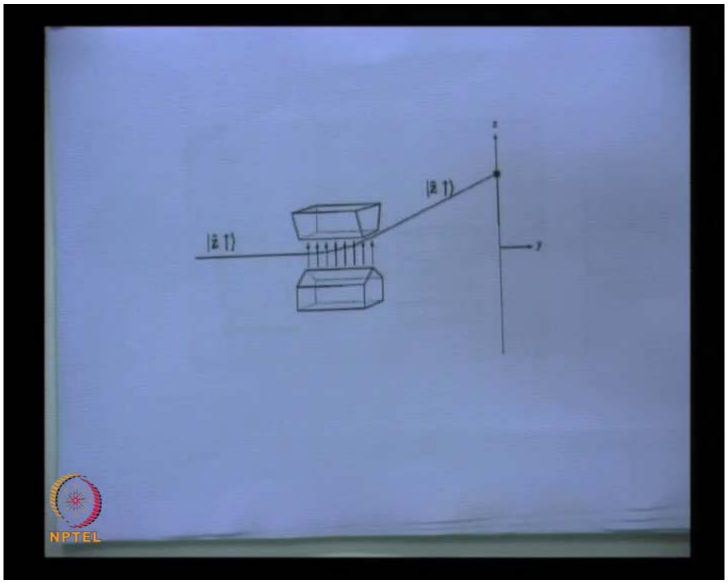

Now, let us suppose I block the beam and I allow the z up state to pass through again a similar inhomogeneous magnetic field, then all the magnets pointing upwards all the magnets are in an eigen state of the operator mu z or s z. Therefore, since I am measuring again mu z, it remains in the same state.

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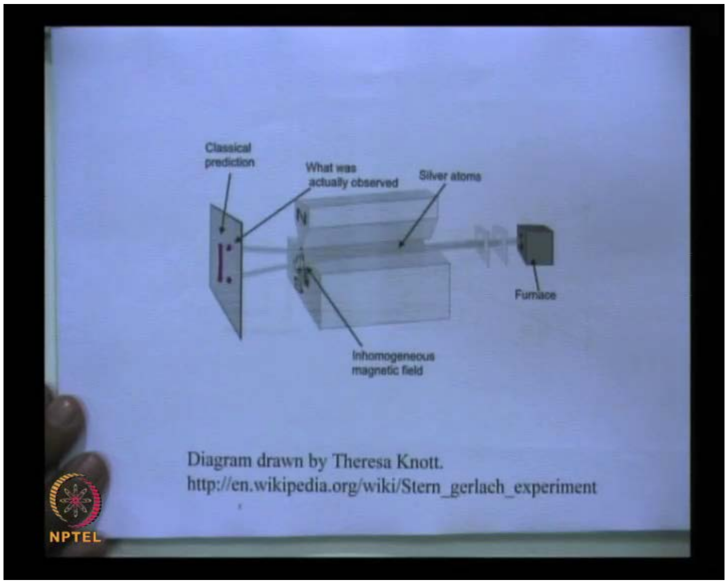

You obtain only one spot on the screen. This is the schematic of the original experiment. I have taken it from the internet from Wikipedia. This is the furnace. This is the silver atoms which come, this is the inhomogeneous magnetic field and as you probably can see that it splits into two spots and the classical prediction is a smear like this, but in the one that is obtained are only two spots. So, you have the experiment the silver atom coming out of the furnace. The magnetic moment of the silver atom does not have any charge, so there is no Lawrence force acting on that. It is passed through a very strong inhomogeneous magnetic field in the z direction. The force acting on the silver atom is proportional to the z component of the magnetic moment and since, the z component of the magnetic moment is quantized, you obtain two spots instead of a smear that would have been classically predicted.

So, once again I have an (()) which sends out silver atoms. Now, the silver atoms I visualize this as tiny magnets. Now, the magnets are oriented at random and therefore, classically the z component of the magnetic moment will be if this is the vertical direction will be mu cos theta and since, theta goes from 0 to pi by pi, the z component of the magnetic moment should have continuously varied from plus mu to minus mu. Therefore, the force that is acting on the magnet would have continuously varied from plus to plus mu 0 to minus mu 0.

You would have the deflection which is proportional to the z component of the magnetic moment, you would have obtained a smear continuous variation, but when the experiment was performed, it was found that there are only two spots showing as if the magnetic moment, the z component of the magnetic moment is quantized. It takes only two values that means the spin angular momentum, the z component of the spin angular momentum vector takes two discrete values and those two discrete values are either plus half h cross or minus half h cross.

Now, the two spots that come out as I mentioned, the upper one corresponds to the z up state and the lower one corresponds to the z down state. So, if I block one of the beams and all them now are pointing upwards. Classically speaking, now I pass through it is in an eigen state of mu z and I pass through and again I am trying to make a measurement of mu z. Then, it will remain like that it remains in the state in the z up state.

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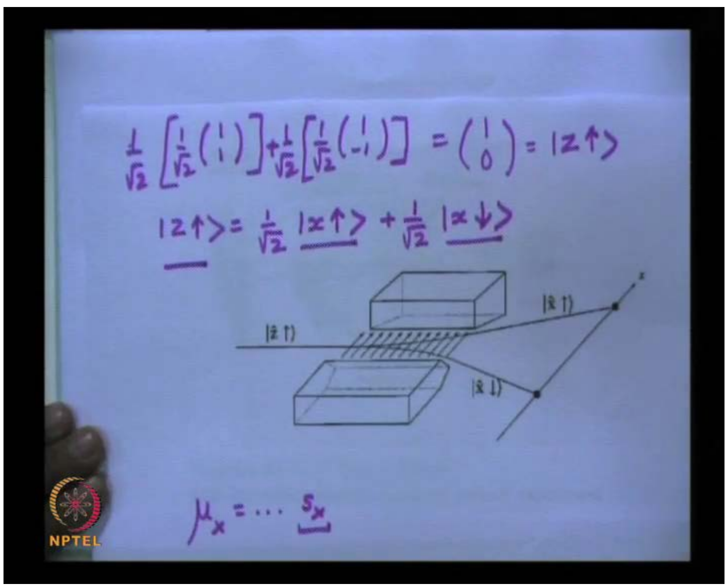

Now, we consider that the magnetic field, the magnets here, the magnets here are now placed horizontally, so that the inhomogeneous magnetic field is now in the x direction. So, the inhomogeneous magnetic field is in the x direction. So, my experiment now try to make a measurement of mu x which is proportional to s x. So, since I am trying to measure s x, I must write z up state as a super position of the eigen kets of the operator s x and of the operator s x, these are two states are 1 1 under root of 2 and 1 minus 1 under root of 2.

So, if I multiply this by 1 over root 2 plus, if I multiply this by 1 over root 2, so this becomes 1 0. Simple calculation. So, this is my z up state. So, my z up state is actually a super position of the x up state and x down state. So, there is a half probability of finding it in the x up state and half probability of finding in the x down state. Where will it go, no one can predict. I can predict only the odds that is if there are 10,000 silver atoms, each in the z up state, then 5000 will go in the x up state, but if do the experiment with one silver atom which is in the z up state, I will not be able to tell for sure which side, which spot will it go to because it is in a superposed state.

This indeterminism which Einstein could never accept is a consequence of quantum mechanics. This concept of super position that a state is a super position of two different states is a consequence of quantum mechanics. Therefore, let us suppose I can have not along the, let us suppose this is the x axis and this is the z axis and the beam is coming from along the y axis. I put an angle and then, I apply a magnetic filed in an angle. Then, I must find out, let us suppose this is sigma, this is x prime axis. Then, I must write this as eigen state of sigma x prime and we can obtain two spots with probability, say one-third and two-third. Then, one spot will be twice as instance as the other. I conclude this lecture by mentioning that.

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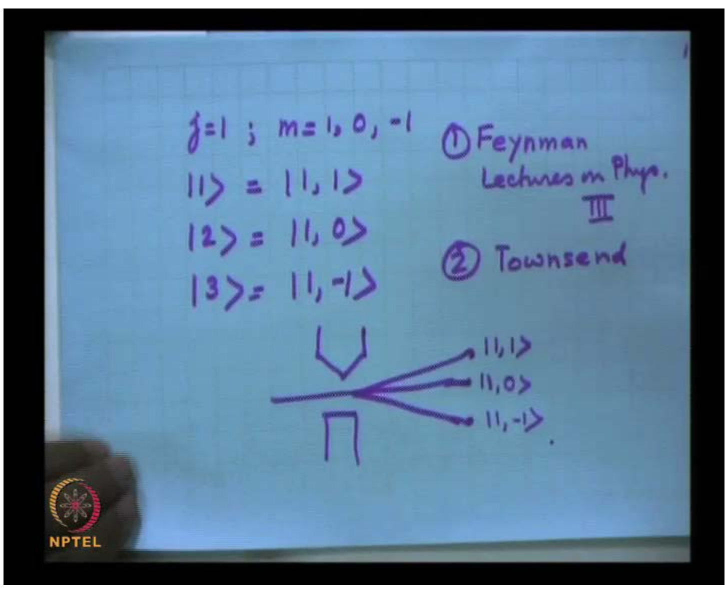

If you have for example a particle system which has angular momentum, then m value as I have told you is 1 0 minus 1. Therefore, there are three states as I had mentioned in the beginning. One is 1, 1 2 is 1, 0 3 is 1, minus 1 and if you have a magnet which corresponds to angular momentum 1 and if it is passed through a Stern-Gerlach experiment apparatus, then it will split into three parts and this experiment, this type of experiment is very nicely discussed at two places, may be many places, but I have the Feynman lectures on physics. Feynman lectures on physics volume 3 very beautifully discussed and also in a book by townsend. In a book by townsend, a modern approach to quantum mechanics the spin half problem is discussed at many places including in our own book with professor, my own book with professor Lokanathan. So, this will correspond to 1 1 state, 1 0 state and 1 minus 1 state, ok.

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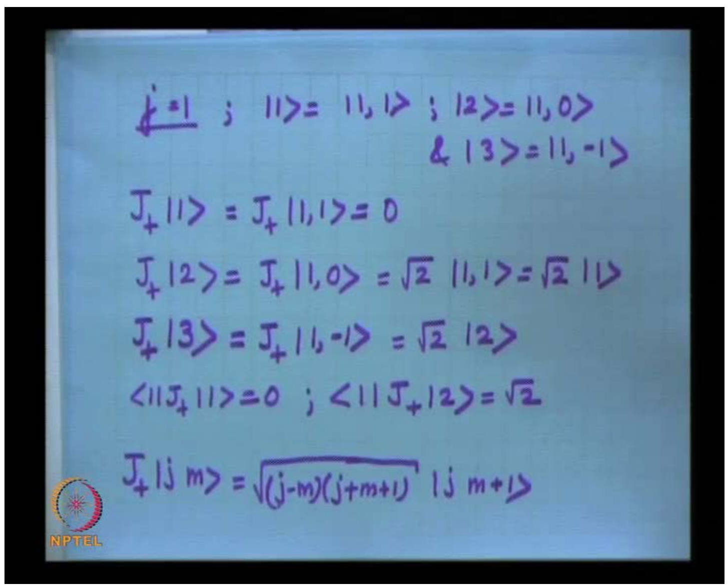

So, let me then also mention that if I have three eigen kets corresponding to j is equal to 1, that is ket 1 is equal to 1, 1, ket 2 is equal to 1, 0 and ket 3 is equal to 1, minus 1, then just as we did for the spin angular momentum half problem, then J plus ket 1. If I remember that J plus is equal to J, let me write down below that. J plus ket j m is equal to this.

All of you must remember j minus m J plus m plus 1 ket j m plus 1. So, J plus ket 1 1. So, this is equal to j is 1 m is 1. So, this is a null ket. J plus ket 2 is equal to J plus ket 1, 0. So, j is 1 m is 0. So, this is 1 1 and this is 1 minus 1 plus 0 plus 1 square root of 2. So, square root of 2 1, minus 1. So, m value has to increase by 1. So, 1, 1, so this will be 1, 1. So, this will be square root of 2 ket 1. Similarly, J plus of ket 3 will be equal to J plus of 1 minus 1. So, m is minus 1. So, 1 minus minus 1 is 1 plus 1, that is 2 1 minus 1 is 0. So, this will become again 0 square root of 2 1, 0, that is ket 2 and similarly, I can write it down for all others.

So, using this I can write down that 1 J plus 1 is of course a null ket, 1 J plus 2 1 J plus 2 is square root of 2 and and so on. So, using this we can write down the angular momentum matrices for the j is equal to 1 ket. So, let me just tell you that for this one second, yes.

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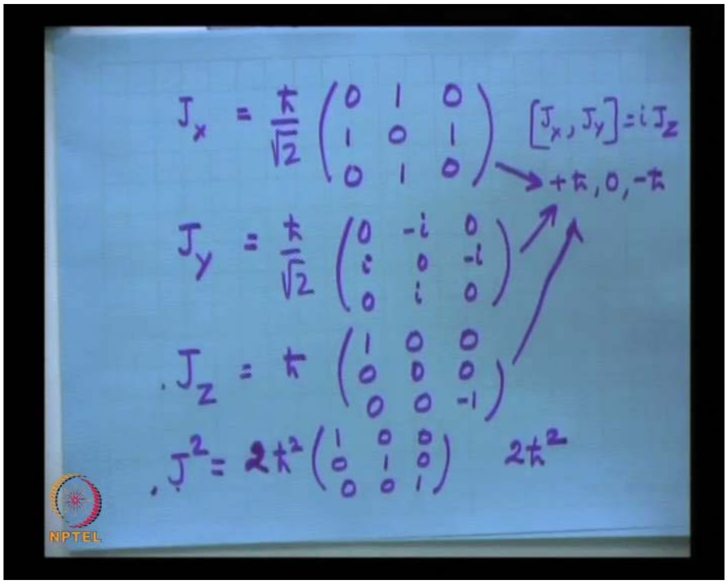

So, we finally obtained for the j x will be equal to h cross by under root of 2 0 1 0 1 0 1 0 1 0. Sll these matrices are hermitian j y becomes equal to h cross by root 2 0 minus 1 minus 1 0 1 0 minus 1 0 1 0 and j z is equal to h cross 1 0 0 0 0 0 0 0 minus 1 and J square will be j x square plus j y square plus j z square. So, this will be 2, sorry j into j plus 1 2 h cross square 1 0 0 0 1 0 0 0 1. So, once again j x will commute with J square, j y will commute with J square, j z will commute with J square, but j x j y j z will not commute with one another and I leave it as an exercise for you to show that j x, j y is equal to i times j z.

The eigen values of j x are plus h cross 0 and minus h cross, the eigen values of j y are also this, eigen values of j z. It is obvious from here that this is so much the eigenvalues of J square. It is a degenerate eigen value j into j plus 1 2 h cross square. So, I can set up again three vectors which are simultaneous eigen kets of J square and j z and do the same kind of analysis with as we had done for j z is equal to for j is equal to half kets.

So, with that we conclude this lecture. In the next lecture, we will use the operator algebra to determine spherical harmonics. Thank you.