Basic Quantum Mechanics
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Module No. # 08
Angular Momentum - II
Lecture No. # 02
Angular Momentum Problem (Contd.)

In our last lecture, we were discussing the operated algebra involving the angular momentum operators. We will continue our discussions on that. We first said that the operators J square and let me just consider the z component of the angular momentum J z. They commute with each other and since, they commute and since, they are observables, we can have a set of simultaneous Eigen vectors of the operator J square and J z.

So, we wrote down J square ket lambda m is equal to so much and J z ket lambda m is equal to m lambda m. Then, we had defined two, we have defined two operators J plus and J minus. J plus was J x plus i J y and J minus was J x minus i J y and now, let me consider the commutation of J plus, J z. So, this is J plus J z minus J z J plus. So, this is equal to J plus is equal to J x plus i J y. So, this will be J x, J z plus i J y, J z. Now, you know that J y J z commutator is just i J x. So, this will be i times i is minus J x and this is in the wrong order. So, J z, J x as you would recall from the previous lecture is i J y. So, J x J z is the minus of that. So, this is minus i J y.

So, if I take the minus sign, so you will get J x plus i J y and so, this is equal to minus J plus. I hope I have done it correctly. So, the left hand side is J plus J z minus J z J plus is equal to so much. What we do is we operate this on ket lambda m, that is we operate these both sides on ket lambda m. So, you will get, you had here. So, let me put a plus sign here, plus sign here and minus sign here and take this to the other side. So, we will have, please see this J z J plus. Let me do this once more.

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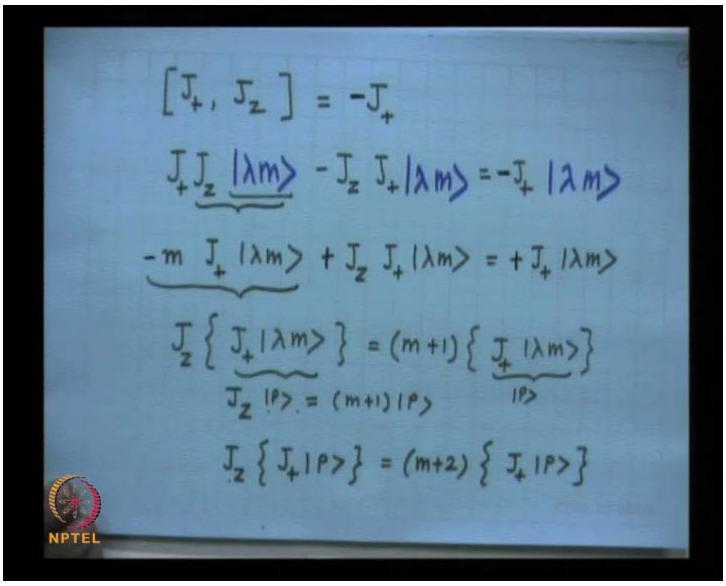

So, we have J plus, J z the commutator this was equal to J plus we derive that so we had J plus J z. Let me leave some space minus J z J plus let me leave some space. This is equal to J plus.

What we do is as we have mentioned operate each term on ket lambda m operate this on ket lambda m operate this on ket lambda m. Then, you can see that J z operating. Did I do it correctly? J plus J z is equal to this was minus J plus minus. So, this is minus, I am sorry, there is a minus sign here. So, J z operating on ket lambda m is m times J plus ket lambda m minus J z J plus ket lambda m minus J plus ket lambda m.

So, I multiply all sides by a minus sign. So, I get a minus sign here, plus sign here and a plus sign here. Now, I take this term on the right hand side. So, I get J z operating on J plus ket lambda m is equal to m plus 1, m plus 1 operating on J plus ket lambda m. Thus, if ket lambda m is an Eigen ket of the operator J z belonging to the Eigen value m, then J plus ket lambda m is also an Eigen ket of the same operator J z, but now belonging to the Eigen value m plus 1.

So, let me denote this file, say ket p. So, J z ket p is equal to m plus 1 ket p. Now, I do the same thing. Instead of operating this by ket lambda m, I operate this on ket p. So, I will obtain J z J plus ket p will now be equal to m plus 2 J plus ket p of course, provided ket p is not a null ket. Similarly, if ket p is not a null ket, then J plus ket p is also an Eigen ket of the operator J z belonging now to the Eigen value m plus 2, provided this is not a null ket.

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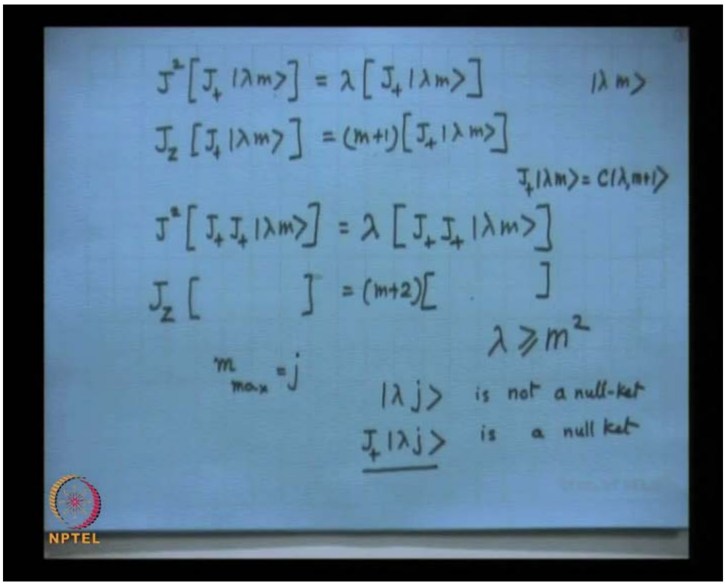

We had seen that J, therefore we had seen that J plus ket lambda m. In the last lecture, we had shown that is also an Eigen ket of the operator J square belonging to the same Eigen value J plus ket lambda and J z J plus ket lambda m is equal to m plus 1 J plus ket lambda m. Therefore, if ket lambda m of course, a non-null ket is a simultaneous Eigen vector of the operator J square and J z. Then, J plus ket lambda m is also a simultaneous Eigen ket of J square and J z belonging to the same Eigen value of J square, but 1 raised Eigen value of J z.

Therefore, J plus ket lambda m must be something like some constant times m plus 1 m increases by 1, but lambda value remains the same. I can go on doing that. Then, we will have J square operating on J plus J plus ket lambda m. That will belong to the same Eigen value of J square J plus J plus ket lambda m, but for this, we say will now be m plus 2. That is why, they are known as ladder operators provided this is not a null ket. So, the value lambda remains the same on m increases, but I have the inequality that lambda must be greater than equal to m square. So, at some m value, we must have a maximum value of m because this process cannot be on indefinitely. Otherwise, it will violate the inequality lambda is greater than or equal to m square and therefore, at some maximum value of m.

So, let there must exist a maximum value of m and let the maximum value of m be equal to J. Then, the max, this Eigen ket is a non null ket, is not a null ket, but J plus ket lambda J is a null ket because J m value with the maximum value of m is J. So, you cannot have m plus 1 and therefore, J plus ket lambda J must be a null ket.

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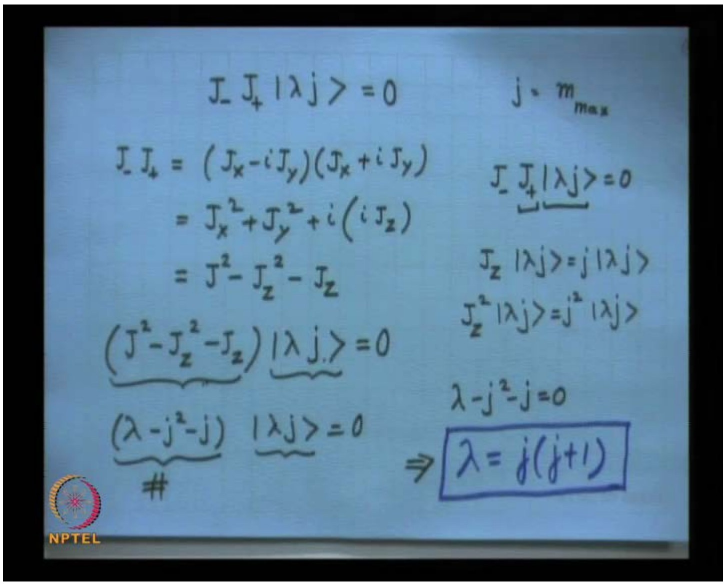

So, therefore, we must have that J plus ket lambda J. That what is J. This is a null ket that J is the maximum value of m. I operate this by J minus and I had shown that you remember that J minus J plus. We had evaluated this. This is J x minus i J y multiplied by J x plus i J y. So, this is J x square plus J y square and this is plus i J x J y minus J y J x. So, that is i J z. So, this is equal to J square minus J z square minus J z. So, I operate J minus J plus this thing.

So, you must understand the equation that J plus ket lambda J is a null ket. This is not a null ket. Of course, this is not a null operator, but operating on a ket gives you a null ket. So, I operate this by J minus. So, I get J square minus J z square minus J z operating on lambda J must be a null ket. This is a non null operator, this is a non null ket, but this operating on this is a null ket.

Now, this is an Eigen ket of J square. So, the J square operating on that is lambda. The Eigen value of J z is J of J z square. If J z ket lambda J is equal to J lambda J, if this is the Eigen value of J z and therefore, if I operate this again by J z, so J z square lambda J will be equal to J operating J z lambda J. So, that will be J square lambda J. So, this will be lambda minus J square minus J operating on lambda J must be a null ket.

Now, here this quantity is a number, this is an operator. It is a non null operator, but this is a non null ket by definition. So, this number must be 0. So, lambda minus J square minus J. This is really beautiful is 0 which gives me the beautiful result that lambda must be equal to J square plus J or J into J plus 1.

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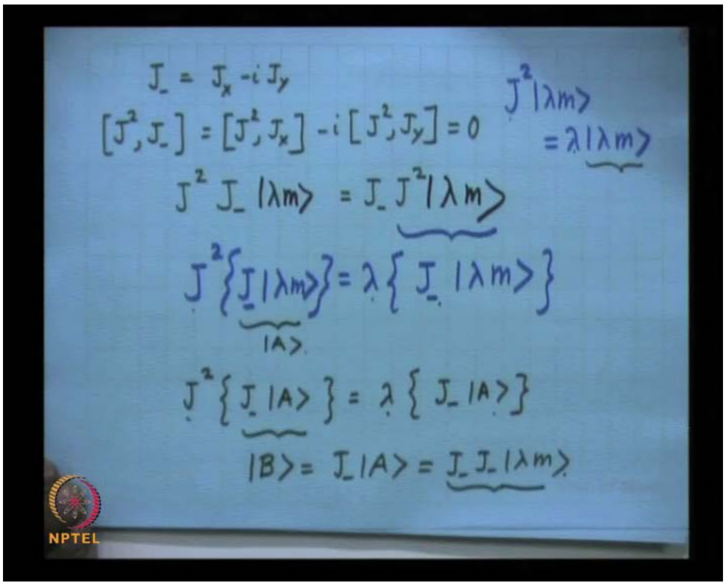

Now, if you have not been able to follow the algebra, let me do it all over again for the J minus operator. So, you have the J minus operator. Let us forget about everything, but keep this result somewhere. So, the J minus operator is equal to J x minus i J y. So, J square commutes with J x, J square commutes with J y. So, J square commutes with J minus. So, J square, J minus is equal to J square, J x minus i times J square, J y. Both are 0, both vanish. So, this is zero.

So, you have J square J minus is equal to J minus J square. I operate both sides by lambda m. So, we have this is equal to lambda. So, this will be lambda J minus ket lambda m. So, you will have J square J minus. Thus, if ket lambda m is an Eigen ket of the operator J square belonging to the Eigen value lambda as expressed by this equation, then J minus ket lambda m is also an Eigen ket J square belonging to the same Eigen value lambda. Similarly, let me denote this by say, ket a. So, then, we can write down J square exactly in a similar way J square J minus ket a will be equal to lambda J minus ket a. So, if I denote this by ket B which is J minus ket A which is equal to J minus J minus ket lambda m, then we say.

Listen, please listen carefully. If ket lambda m is an Eigen ket of J square belonging to the Eigen value lambda, then J minus J minus is also an Eigen ket of this same operator J square belonging to the same Eigen value lambda provided of course, a is not a null ket and J minus a and b is also not a null ket. Therefore, once again we must expect the genre is and this is an Eigen ket, this is an Eigen ket and this is also an Eigen ket. Then, we consider the commutation relation of J z.

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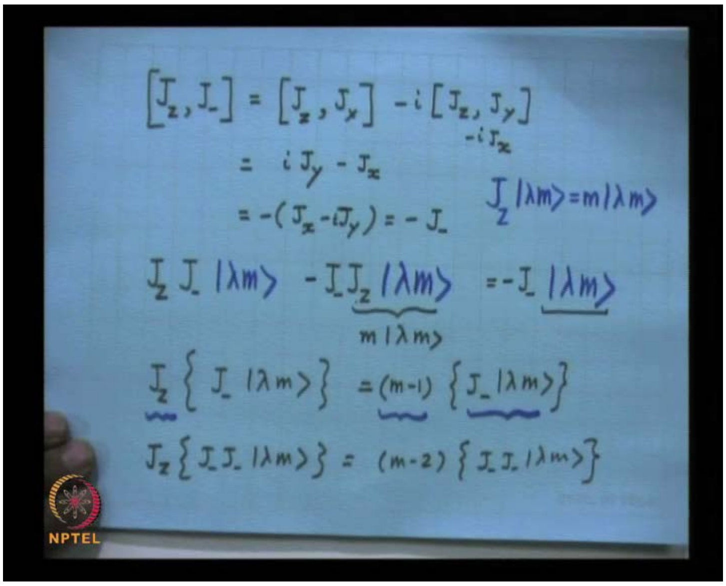

So, let me consider the commutation relation of J z of J minus. This is J z, J x J J minus is J x minus i J y. So, this is J z, J y. So, J z, J x is equal to i J y. I J y and J z J y will be equal to minus. If you look up the first, we wrote down that J z, J y was minus i J x. So, minus minus plus and i square is minus. So, this becomes minus J x. So, if I take minus sign outside. So, I get J x minus i J y, sorry i J y. So, this will be minus J minus. So, I had J z J minus. Please leave some space minus J z J minus J z, leave some space is equal to minus J minus.

Now, what I will do is operate this on lambda m, operate this on lambda m and operate this on lambda m and you have here J z ket lambda m is m lambda m. So, I get this first term becomes J z J minus ket lambda m and if you take it to the other side, then it will become m minus 1 J minus ket lambda m. Therefore, if ket lambda m is an Eigen ket of the operator J z, if ket lambda m is an operator is an Eigen ket of the operator J z belonging to the Eigen value m, then J minus ket lambda m is also an Eigen ket of the same operator J z.

Now, belonging to 1 less Eigen value m minus 1 provided of course, this is not a null ket and I can continue this again, so J z operating on J minus J minus ket lambda m. I hope you can now immediately say that this must be m minus 2 J minus J minus ket lambda m.

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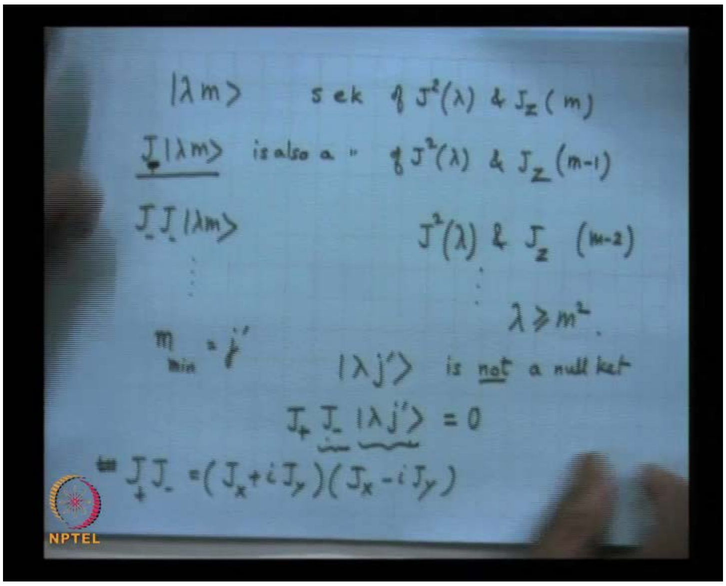

So, therefore, if ket, let me summarize If ket lambda m is a simultaneous Eigen ket of J square belonging to the Eigen value lambda and J z belonging to the Eigen value m, then J plus ket lambda m is also a simultaneous Eigen ket of J square belonging to the same Eigen value lambda and of J s J z. No, I will put minus sign here, but now this will become m minus 1 provided this is not a null ket and then, J minus ket lambda m will also be a simultaneous Eigen ket of J square belonging to the same Eigen value lambda, but of J z belonging to the Eigen value m minus 2 and so on provided we do not hit a null ket, but I have. So, this is an arithmetical preparation m m minus 1 m minus 2 m minus 3. If it goes on indefinitely, then this quantity will become much greater than lambda and we have the inequality that lambda must be greater than m square.

So, there must be a minimum value of m and let that minimum value be equal to J prime. That means that ket lambda J prime is not a null ket. However, J minus ket lambda J prime is a null ket. Even then J prime is not a minimum value. So, once again this is not a null ket, this is not a null operator, but an operator operating on this will give you a null ket. So, let me multiply this by J plus. Then, my left hand side is equal to left hand side. So, this J plus J minus is equal to J x plus i J y J x minus i J y. Let me do it carefully.

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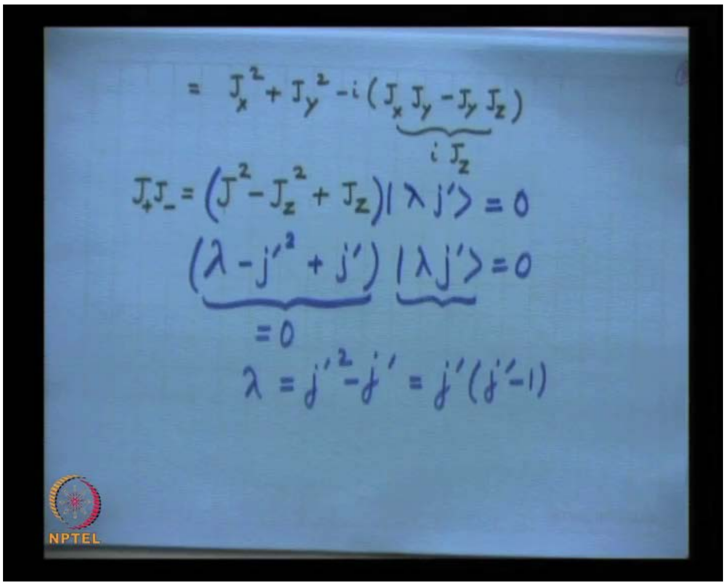

So, this is equal to J x square plus J y square i times i is minus 1 minus minus plus 1 and then, you will have minus i J x J y minus J y J x. So, this is i J z. Actually, i h cross J z. So, this will be i square is minus 1, so plus. So, this will become J square minus J z square plus J z. So, this is my J plus J minus, but we are seeing that J plus J minus operating on lambda J prime. So, this I operate on ket. This whole thing I operate this on lambda J prime. So, this will be a null ket. So, J square operating on this, this is an Eigen ket of J square. So, this will become lambda minus J prime square plus J prime operating on lambda J prime must be 0.

This is not a null ket and this is a number. Now, so this must be 0, so this must be 0 and therefore, lambda must be equal to J prime square minus J prime. So, this will be equal to J prime J prime minus 1.

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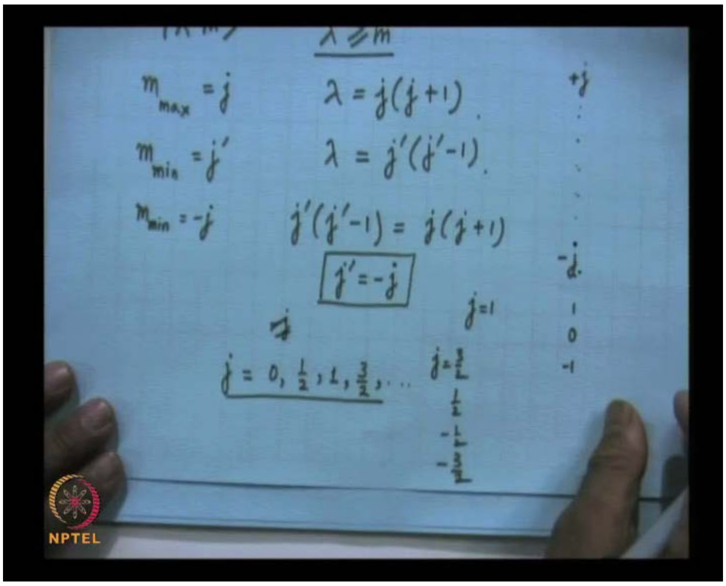

So, please remember two things, ket lambda m is a simultaneous Eigen ket of J square and J z m is the Eigen value of J z and lambda is the Eigen value of J square. First, we derive that lambda must be greater than or equal to m square and we said that what the maximum value is. So, we said that since this, because of this inequality the maximum value of this is J and we found that lambda must be equal to J into J plus 1.

Then, we use ladder operators J minus operators. We found that m must have a minimum value and let that be J prime and we just now found that lambda must be equal to J prime into J prime minus 1. So, J prime into J prime minus 1 is equal to J into J plus 1 and J prime. This is a quadratic equation in J prime, but since J J prime has to be less than J, so the root is J prime is equal to minus J because if I put minus J here, so you will get minus J into minus J minus 1. So, that is J into J plus 1.

So, the solution of this equation is J prime must be equal to minus J. So, the maximum value is J. The minimum value is minus J and now, see that I start with a value J and I go down in steps of 1 J minus 1 J minus 2 J minus 3 J minus 4 J minus 5 J minus 6 and then, I must hit minus J. That is only possible if J itself is equal to 0 half 1 3 by 2 etcetera not for any other number because we have said that if m is an Eigen value, then m plus 1 is also an Eigen value. If m is an Eigen value, m minus 1 is an Eigen value, it can hop in units of 1 and it has to land up on minus J. Otherwise, you will never get a null ket and therefore, as you start with plus J, you hop in units of 1 and you go to minus J. That will only be possible if J takes these values. So, if you have J is equal to 1, then you start with

1 decrease by 0. They are the m values. You start with J is equal to 3 by 2, so you have 3 2 half minus half and minus 3 by 2.

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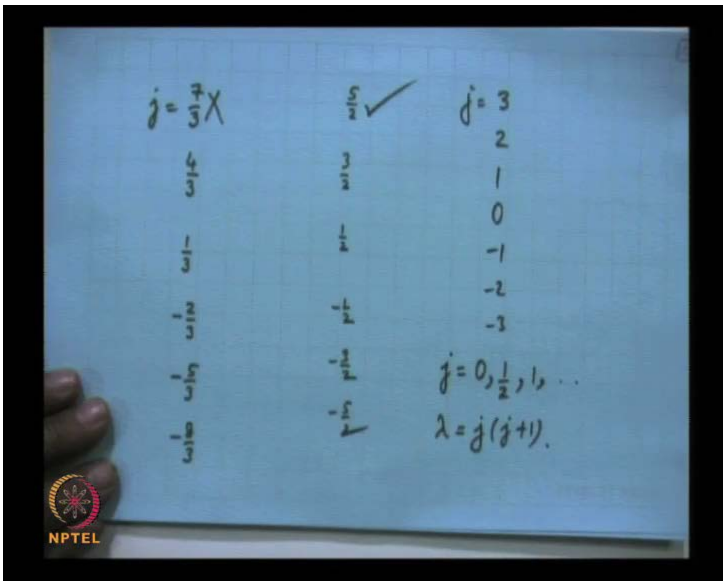

Let me say that I take a, let us suppose I have J is equal to 7 by 3 to 7 by 3 minus 1 is 4 by 3, 4 by 3 minus 1 is 1 by 3, 1 by 3 minus 1 is minus 2 by 3 minus 2 by 3 minus 1 is minus 5 by 3 and then, minus 8 by 3 and so on. So you have not hit minus 7 by 3 to 7 by 3 is not possible, but if you use 5 by, so 5 by 2 minus 1 is 3 by 2, 3 by 2 is half minus, half minus 3 by 2 and minus 5 by 2. So, that is possible. So, J is equal to 3, then 2, 1, 0, minus 1, minus 2, minus 3.

So, we get the result that the value of J can only take 0 to half, 1 and lambda is equal to J into J plus 1. These are the Eigen values of J square operator. Now, you see a remarkable result that we have obtained. Earlier, we had said we had used differential operator representation.

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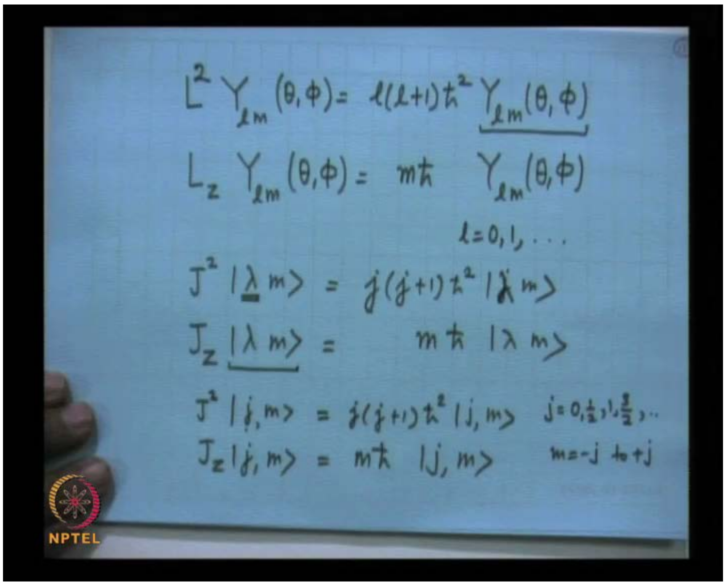

We had found that the Eigen functions are the spherical harmonics and the Eigen values were l square Y l m theta phi were equal to l into l plus 1 h cross square Y l m theta phi. In fact, these were simultaneous Eigen functions of l square and l z, but here l is equal to 0, 1, 2, 3, but here I will get that J square ket J m ket lambda m. This is equal to J into J plus 1 h cross square ket J m ket, sorry lambda m and J z ket lambda m is equal to m h cross, but the difference between the two is that here J can take half integral values.

As we know that the electron and the proton and the neutron are endowed with a intrinsic angular momentum which is half h cross and that is predicted by quantum mechanics that are now derived from quantum mechanics and just by using commutation relations and nothing else. So, we finally obtain actually instead of lambda, the convention is to label the Eigen kets as ket J m. This is not the Eigen value. The Eigen value is really lambda, but the convention is J square ket J m. This is equal to J into J plus 1 h cross square ket J m and J z ket J m is equal to m h cross ket J m I have put. So, these and these values of J that we can take are 0, half, 1, 3 by 2 etcetera and m can take from minus J to plus J.

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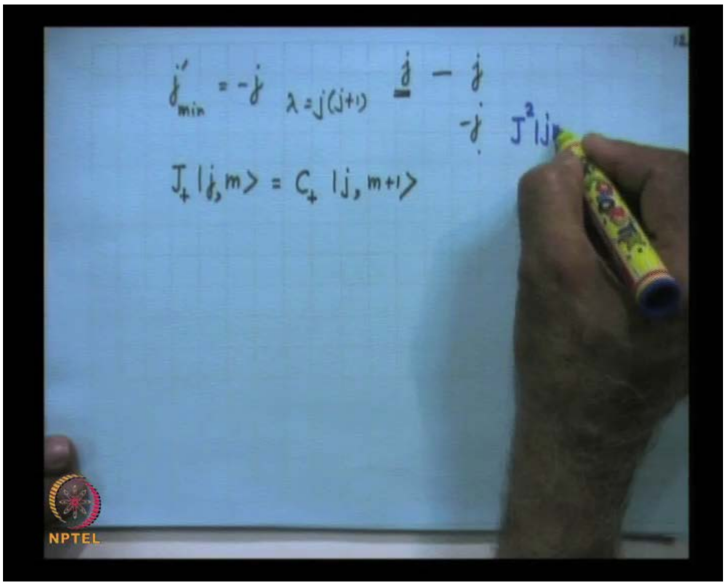

So, I had shown that J prime which is the minus value of which is the minimum value of J was equal to minus J. Therefore, these are the m values for a given value of J. The m values go from J to minus J. I had now proved that just by using commutation relation. So, all the previous results are proved that because the differential operators also satisfy the same commutation relations.

So, therefore, I have proved that for a given value of J, the m values go from J to minus J. Now, let me do now one more algebra and that is we have shown that J plus ket J m. If ket J m is an Eigen ket of J square and J J z, then J plus ket J m is also an Eigen ket, but now belonging to the Eigen, same Eigen value lambda which is J into J plus 1 lambda is equal to J into J plus 1, but m value is raised. So, let me write this down and this must be a multiple of J m plus 1. May be you put a, here. What we must always remember on the right hand side, I may write it down that J square ket J m.

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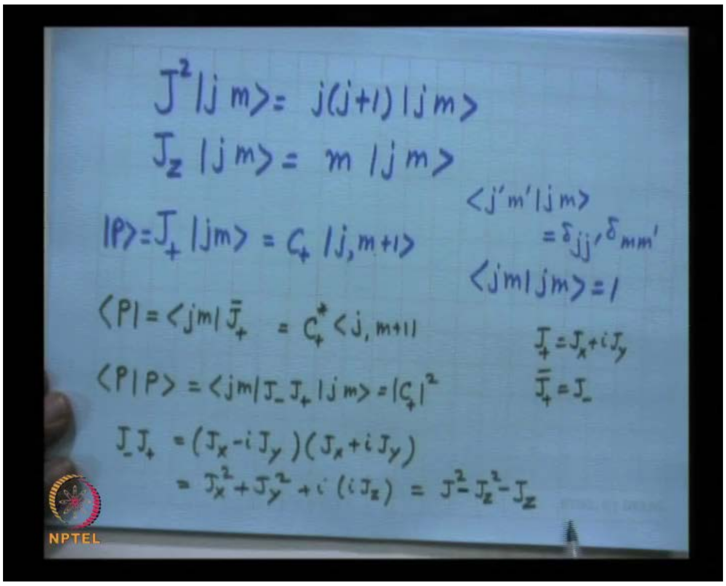

Let me write it on a fresh piece of paper. J square ket J m is equal to J into J plus 1 h cross square is there, but we assume that in a system of units where J h cross is 1 and J z J m is equal to m h cross is 1. We are working in a system of units of this.

Now, we say that J plus J m and these are orthonormal ket. These are simultaneous Eigen ket, but they are orthonormal, that is J prime, m prime, J m here. All delta, J J prime delta, m m prime, they are all orthonormal. Both of them and this is normalized, that is J m J m is 1. We assume that because they belong to (()) operators, so J plus ket J m is therefore a multiple of this ket J m plus 1.

So, let this be c plus, let this be ket p. Then, bra p is equal to bra J m J plus bar. This will be equal to this is just a number, so c plus star bra J m plus 1. So, I now write bra p ket p, this is equal to J m J plus bar is J plus is equal to J x plus i J y. So, J plus bar is J minus because this will be J x minus i J y. So, this is J minus J plus J m is equal to c plus mod square and this is 1 this times this is 1. So, once again J minus J plus is equal to J x minus. This you must remember by now J x plus i J y. So, this is J x square plus J y square plus i times J x J y minus J y J z. So, that is i times J z. So, this is J x square plus J y square is J square minus J z square minus J z. You have derived this earlier.

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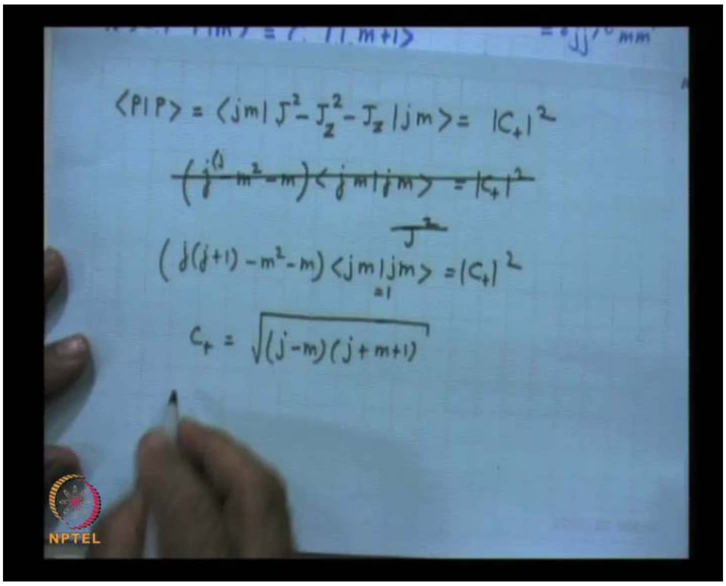

So, we have therefore, bra p ket p is J m J square J minus J plus is J square minus J z square minus J z operating on J m is equal to mod c plus square 2. So, this is operating on this, this is an Eigen ket of J square J z square and J z. So, J square that will be J minus m square minus m J m J m. This is c plus square. I am sorry this will be J into J plus 1. I am sorry J square. I am sorry this is sorry I have to write it again. J square operating on J into this is J into J plus 1 minus m square minus m. This is not the Eigen value. The Eigen value is J into J plus 1 as we had done here J into J plus 1.

So, this will be J into J plus 1 minus m square minus m bra J m ket J m which is 1 because they are orthonormal ket. This is equal to c plus square. So, this is equal to 1 and therefore, we obtain c plus is equal to the under root of that. So, this will be J minus m into J plus m plus 1. Therefore, J plus ket lambda m, I am sorry let me do it again.

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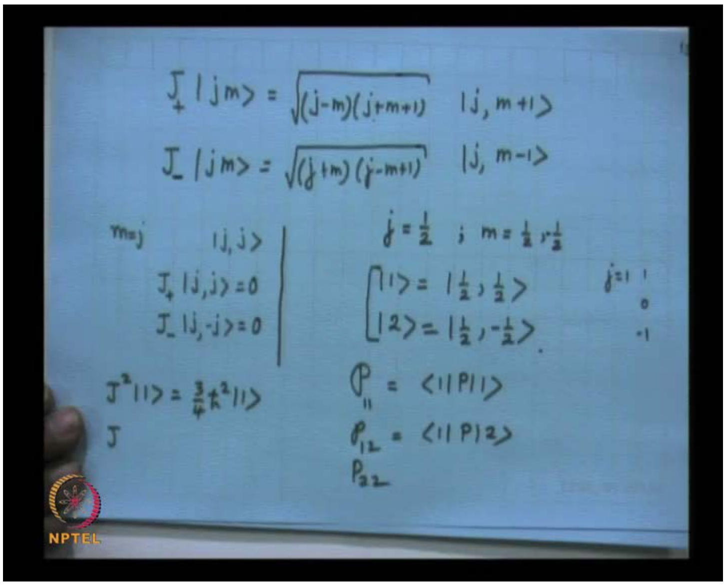

So, therefore, we will have J plus ket J m will be equal to c plus and that is square root of this. You have to remember J minus m J plus m plus 1 into J m plus 1. These are ladder operators. It raises the value of m to m plus 1 and similarly, I leave it as an exercise exactly in a similar way you show that the operator J minus m is proportional to J m minus 1 and this will be equal to J plus m J plus m J minus m plus 1. These two relations we had to remember and we see that when m is equal to J, this is 0. So, J m J is the maximum value of m.

So, J, J is a non null ket. It is a well defined ket but, J plus operating on J J is a null ket. Similarly, J minus J is a non null ket, but J minus operating on that will be a null ket. That comes out automatically from here. Now, let me do an example. Let me consider the case where this spin half J is equal to half. This is an extremely important problem. So, J is equal to half. So, m can be only half and half minus half, sorry m can be half and minus half. So, we can only have two states. State 1 which I denote as half, half. This is the value of J. So, the Eigen value of J square is J into J plus 1 half into half plus 1, that is 3 by 4 h cross square.

So, J square ket 1 will be half into half plus 1, that is 3 by 4 h cross square ket 1 and J square and the other state will be ket 2 half ket minus half. Now, so this I take as the base vectors. I make a representation. I will try to make a representation of the operators J square J x J y J z. Now, when we make a representation, you have to first tell me what your base vectors are like in an I want to describe a vector. If I want to describe a vector and in terms of its component, I will first ask you tell me your x and y axis. So, this is the base vectors. So, in terms of the base vectors 1 and 2, I will write the operators as a matrix. So, for example, since this is a two-dimensional space, there are only two base vectors. We may have 2 by 2 matrices.

If I had J is equal to 1 where m can take 1, 0 and minus 1, so there will be three independent vectors and it will span a three-dimensional space. We will have 3 by 3 matrices. So, of any operator, the matrix representation will be let us suppose of operator p, there will be four numbers p 1 1 which is 1 p 1 2, that is 1 p 2 p 2 2 and p 2 2 1. These are the four numbers which will give you a representation of the operator. So, let me take these as my base vectors.

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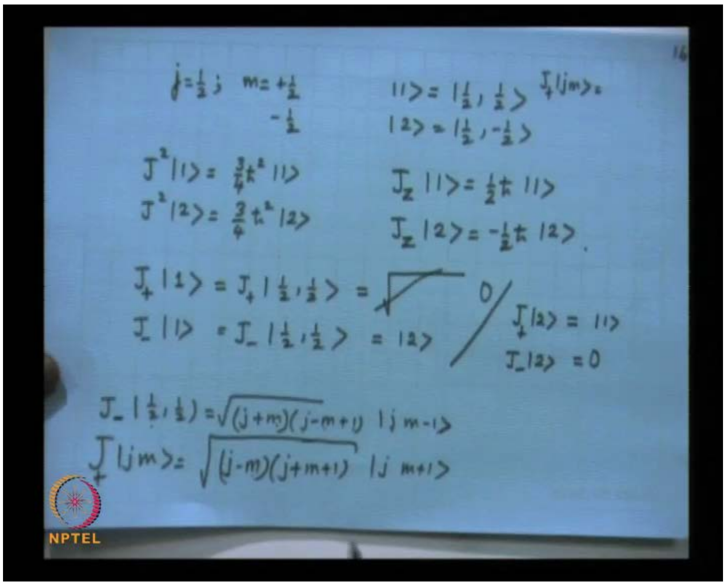

So, I will take ket 1. So, I consider J is equal to half. Then, m is equal to plus half and minus half. So, the base kets are half half and the second is half minus half. These are simultaneous Eigen kets of J square and J x, that is J square ket 1 is equal to half into half plus 1. That is 3 by 4 h cross square ket 1. J square ket 2 is also 3 by 4 h cross square ket 2 and J z ket 1 is equal to this is the Eigen value of J z half h cross ket 1 and J z ket 2 is equal to minus half h cross ket 2.

Now, let me write down what is J plus ket 1. J plus ket 1 will be equal to J plus half half. So, this will be square root of J minus m. So, this is 0. So, remember that J plus ket J m was let me write it below. J plus ket J m was under root of J minus m into J plus m plus 1 J of m plus 1. So this is 0 and J minus ket 1 will be J minus half half. So, this will be please see this J minus half half will be under root of J plus m into J minus m plus half 1 J, m minus 1. So, J is half m is half. So, half plus half is 1 square root of 1 half minus half is 0, 1. So, this will be half minus half. So, that is ket 2.

Now, let me write down what is J plus ket 2. So, here J is half m is minus half. So, here J minus m half minus minus half half plus half 1 half minus half is 0. So, this is equal to 1 and J minus ket 2 J minus ket 2 means m will go down, but that cannot be minus 3 by 2. So, this will be a null ket. Ok, so we all have the relations now.

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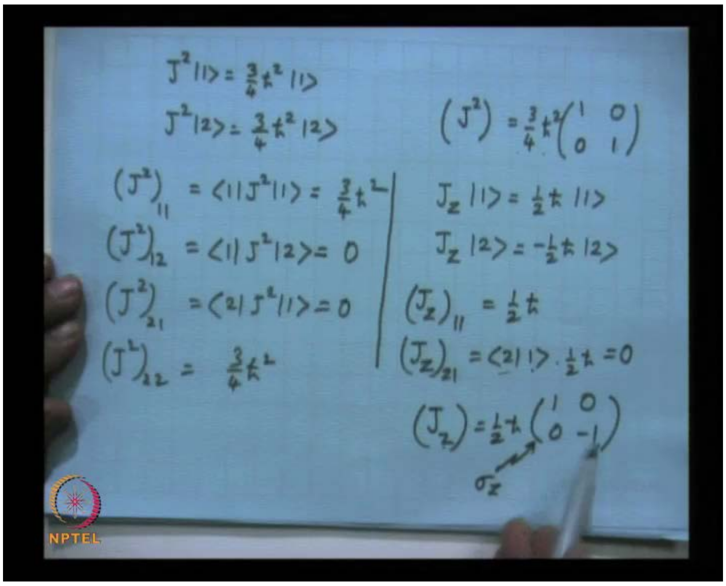

We can write down the matrices. First, let me write down for J square. So, we had first we wrote down J square ket 1 is equal to 3 by 4 h cross square ket 1. J square ket 2 is equal to 3 by 4 h cross square ket 2, so J square. There will be four elements 1 1, that is bra 1 J square ket 1, so bra 1. So, this will be equal to 3 by 4 h cross square. Then, J square 1 2 will be bra 1 J square ket 2.

So, this will be bra 1 ket 2. So, that will be 0. So, J square 2 1 will be 2 J square ket 1. That will be also 0 and J square 2 2 will be equal to, I leave it as an exercise. So, 3 by 4 h cross. So, the matrix representation of the operator J square is 3 by 4 h cross square 1 0 1. Similarly, I have J z ket 1 is equal to half h cross ket 1 because m value is half and J z ket 2 is equal to minus half h cross ket 2. So, again since these are Eigen kets, so if you write J z 1 1, so this will be multiplied by bra 1. So, this is half h cross, but J z 1 2 or 2 1

2 1 will be bra 2 J z. So, this will be multiplied by half h cross. So, this will be 0 and z z 2 2 will be minus half h cross.

So, the operator representation, the matrix representation of J z will be half h cross 1 0 0 minus 1. This is the sigma z, the sigma z, the Pauli matrix. We have derived that what are the Eigen values of J square at this matrix 1 1. Therefore, what are the Eigen values of J square? That is just 3 by 4 h cross square. What are the Eigen values of J z? The Eigen value of these matrix are 1 and minus 1. So, the Eigen values of J z are half h cross and minus half h cross.

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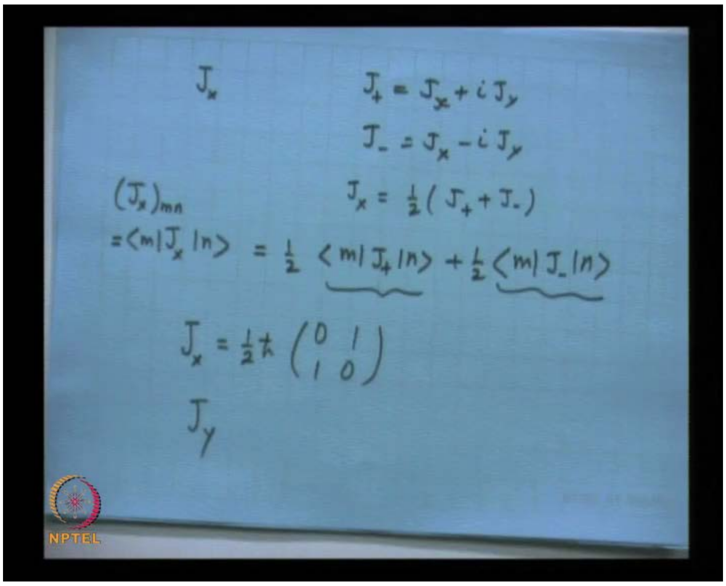

So, I have just given you a hint as to I mean, rest we will do in our next lecture. I want now to make a representation of x J x. So, we have J plus is equal to J x plus i J y and J minus is equal to J x minus i J y. So, I add them up and get J x is equal to half J plus J minus. So, J x bra m ket m, this is the m, mth matrix element is given by this. So, this will be half plus J plus ket n plus half m J minus ket and I leave this as an exercise for you. We will try to do this next time, but if you can work this out, this will be very good and we will, you should find that the matrix representation of J x will come out to be half h cross 0 1 1 0 and similarly, for J y.

So, with this we conclude today’s lecture. What we have achieved is that we have been able to obtain the Eigen value spectrum, simultaneous Eigen kets of the operator, J square and J z. In our next lecture, we will complete the analysis for obtaining the matrix representation of J square, J x, J y and J z. Thank you.

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