Basic Quantum Mechanics

Prof. Ajoy Ghatak
Department of Physics
Indian Institute of Technology, Delhi

Module No. # 01

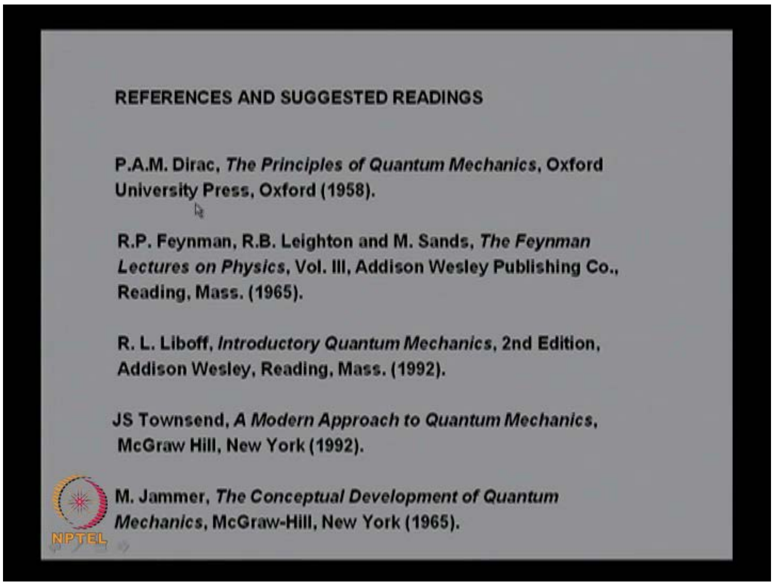
Introduction and Mathematical Preliminary

Lecture No. # 03

Basic Quantum Mechanics : Dirac Delta Function and Fourier Transforms

In the past few lectures 2 lectures, we have discussed the basic phenomenon behind the development of quantum mechanics namely the wave particle duality; and then we continued our discussions to understand the Dirac delta function. We will continue on discussions on the Dirac delta function, and we will introduce Fourier transform of a function.

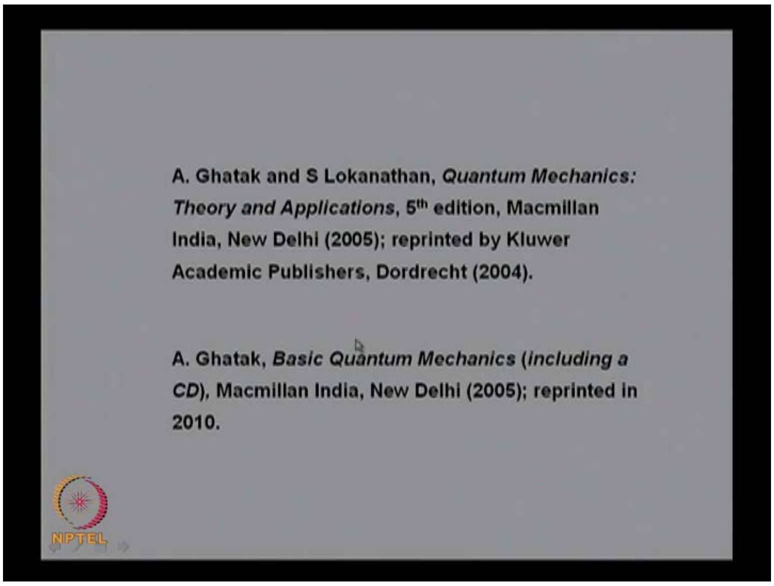
(Refer Slide Time: 01:03)



However, before we do that, I thought I will give you a list of references. There are a large number of books written on quantum mechanics, but some of the extremely good ones are listed below, are listed on this slide. The first one is of course, Professor Dirac book it is a classic on principles of quantum mechanics published by oxford university press. And then we have the famous Feynman lectures on physics volume 3; the first few chapters discuss the wave particle duality and the uncertainty principle in a very beautiful way. I would advise all students, all students to go through the first few chapters of the book.

And then there is a book, a classic a very nicely written text book by Libof, an introductory quantum mechanics and then, there is book by Townsend a modern approach to quantum mechanics published by McGraw hill. And for the history of the development of quantum mechanics, there is a very nice book written by Max Jammer in title the conceptual development of quantum mechanics.

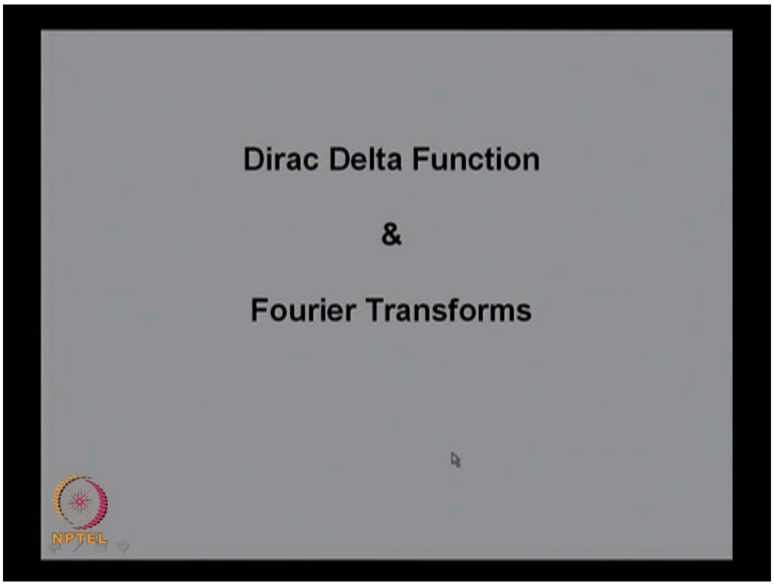
(Refer Slide Time: 02:26)



I will be following my own our own book which is in title quantum mechanics theory and applications, which is currently in its 5th edition and it has been published by Macmillan in India, New Delhi and also the 5th edition has been reprinted by Kluwer academic publishers in Dordrecht.

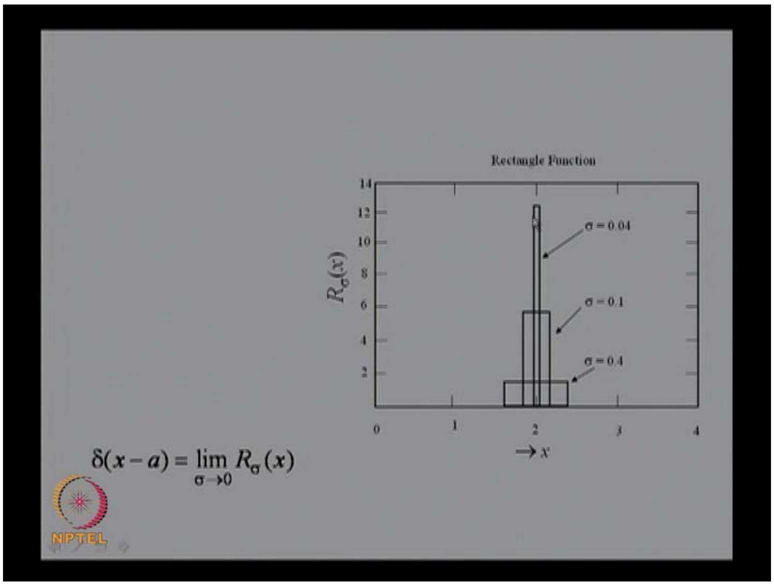
There is another book that I have written a very thin small smaller thin book which is in title basic quantum mechanics and which includes a CD. We have developed a software which will demonstrate in the next few lectures and it has also been published by Macmillan India and the paperback edition now contains the CD.

(Refer Slide Time: 03:20)



So as I mentioned earlier, we will continue our discussion on the Dirac delta function, and we will introduce Fourier transforms, which plays an extremely important role in the development of quantum mechanics.

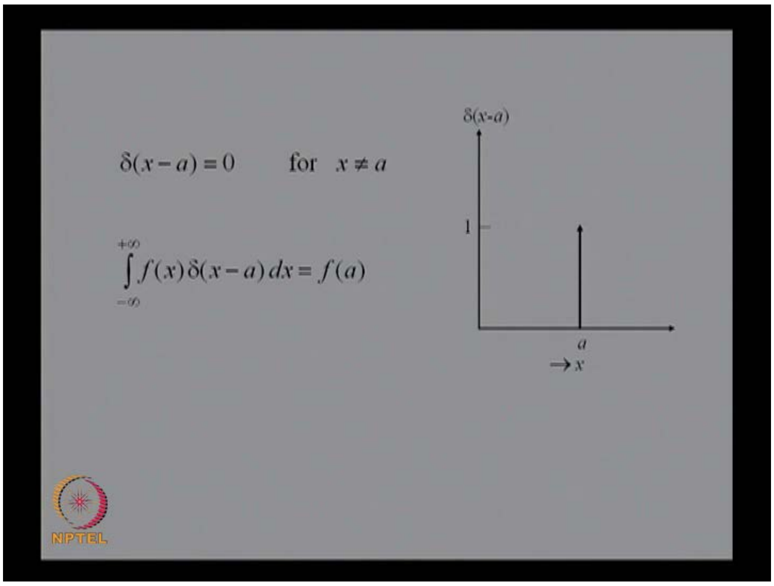
(Refer Slide Time: 03:42)



We had given one of the representation of the Dirac delta function in terms of a rectangle function. Here the rectangle function is such that its value is value is 1 over 2 sigma and then 0 beyond 2 minus sigma and 2 plus sigma, so that as the value of sigma becomes smaller and smaller, the width of the rectangle become smaller and smaller, the height becomes larger. But the area under the rectangle always remains unity.

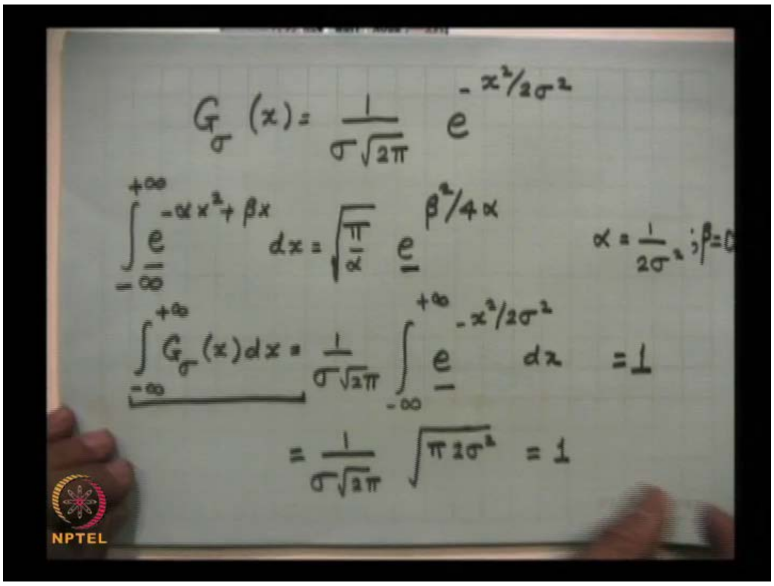
This is perhaps the most straightforward the simplest definition representation of the Dirac delta function and we define and we write this is one of the representation, delta of x minus a as the limit of the rectangle function in the limit of sigma tending to 0. When sigma tends to 0 the width of the rectangle becomes extremely small the height becomes extremely large but the area under the curve remains unity.

(Refer Slide Time: 05:06)



And because of this property the delta function is such that, if I integrate the delta function with multiplied by an arbitrary but well behaved function f of x. Then it picks up its value only at the point x is equal to a and we define the Dirac delta function through these two equations delta of x minus a is 0 for x not equal to a at x equal to a. It is not defined actually it is it has an infinite value but we represent the function by a spike of unit height. This is how a Dirac delta function is represented and it is defined by these two equations.

(Refer Slide Time: 06:13)



We next consider the Gaussian function and the Gaussian function is, we define is such that G sigma of x is equal to 1 over sigma under root of 2 pi e to the power of minus x square by 2 sigma square. Now this integral we will be we had derived last time and we will be using this very frequently e to the power of minus alpha x square plus beta x dx is equal to pi by alpha e to the power of beta square by 4 alpha the limits all limits are from minus infinity to plus infinity.

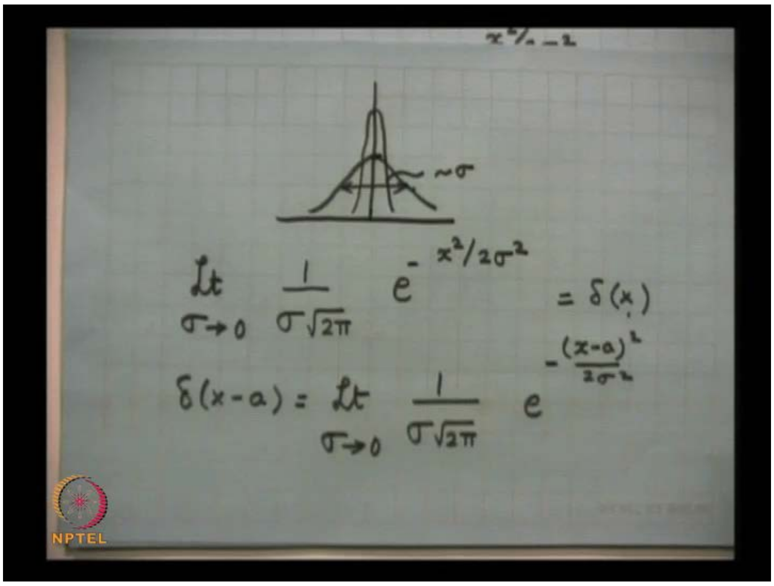
So if I substitute if I now right down the integral of G sigma of x G sigma of x, the Gaussian function G sigma of x dx, then it will be I will have 1 over sigma by root 2 pi outside and then minus infinity to plus infinity e to the power of minus x square by 2 sigma square dx. If I compare the integrant, this integrant with this integrant we find that alpha is equal to 1 over 2 sigma square and beta is 0.

So therefore, this will become the integral will be 1 over sigma under root of 2 pi under root of pi alpha that is pi divided by alpha. So 1 over alpha is 1 over 1 over 2 sigma square so, this is 2 sigma square and then e to the power of beta square beta is 0. So this factor becomes 1, so as you can see the 2 pi cancels out under root of sigma square is sigma so this is equal to 1.

So therefore, this integral G sigma of x dx is equal to 1 is unity the area under the curve is unity for all values of sigma. So, let me plot this function let me plot this function this

G sigma of x that I have written down and of course, sigma is greater than 0 sigma is greater than 0.

(Refer Slide Time: 08:52)

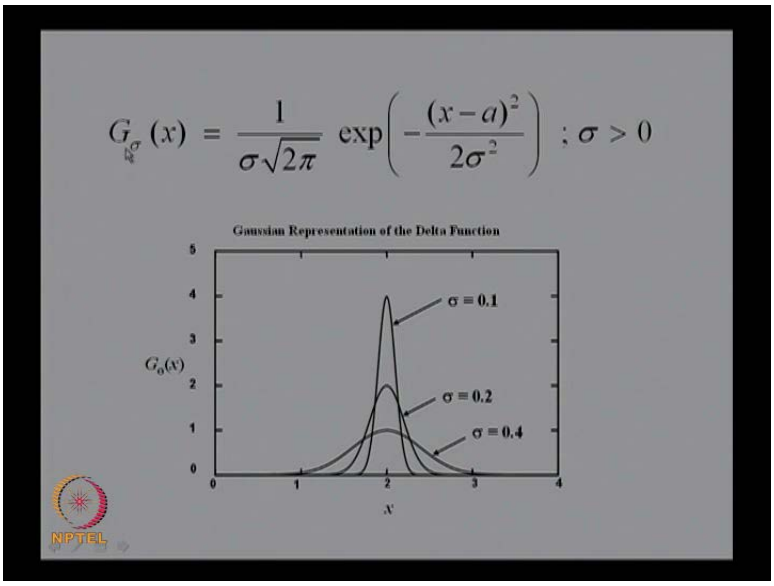


So if you plot this you will find that if I take sigma is equal to 1 then it will have a width something like this. This is a Gaussian function and this width this width is of the order of sigma this is something like the full width at half maximum if this value is 1, if you take this the peak value as 1, then the full width at half maximum is about sigma may be square root of 2 sigma or 1.5 sigma or something like that.

Now, as I make the area under the curve I have just now shown to be unity. So as I make sigma smaller because of the factor of one over sigma the value of the function at x is equal to 0 becomes smaller and smaller, larger and larger and the width becomes smaller and smaller. But the area under the integral area under the curve remains unity. So therefore, it has all the properties of the delta function and therefore, we can write that limit of sigma tending to 0 the G sigma of x that is 1 over sigma under root of 2 pi e to the power of minus x square by 2 sigma square, this is a yet another representation of the Dirac delta function.

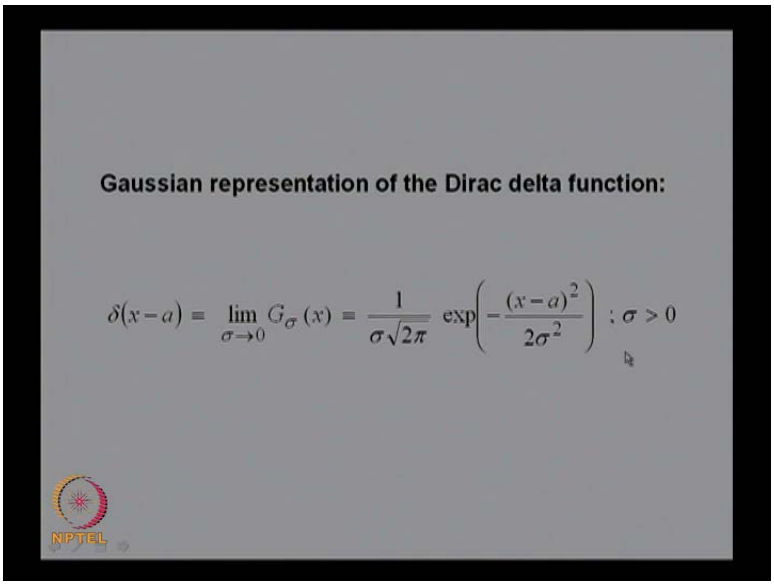
So this is a delta function centered at x is equal to 0. If I want to displace the curve if I want to shift the origin by a. So therefore, we can write delta of x minus a is equal to limit of sigma tending to 0 1 over sigma under root of 2 pi e to the power of minus x minus a whole square by 2 sigma square. This is known as the Gaussian representation of the Dirac delta function.

(Refer Slide Time: 11:04)



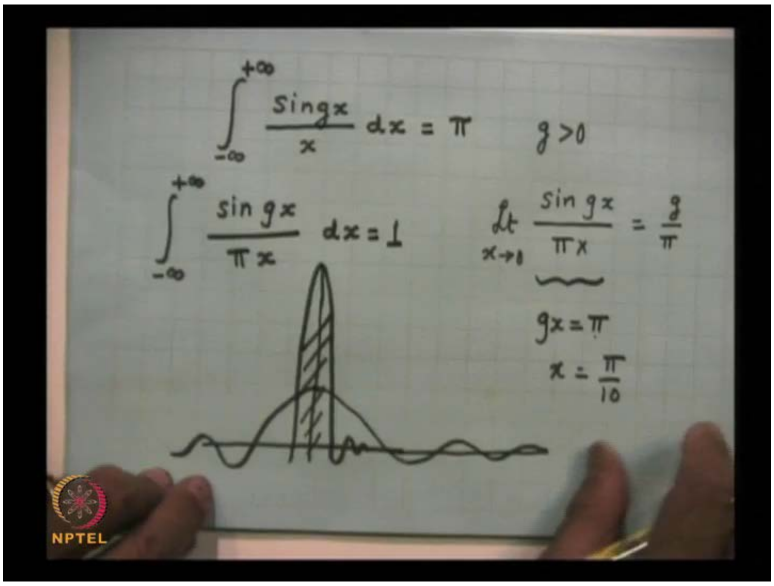
So here on this slide we are shown that the function G sigma of x I define this as 1 over sigma under root of 2 pi and plot. In the figure below we have plotted this function for sigma is equal to 0.1, 0.2 and 0.4. So the width of the function is of the order of sigma and as the value of sigma becomes smaller and smaller, the width becomes smaller and smaller, the localization of the wave function becomes smaller and smaller and the height becomes larger and larger but the area under the curve is always unity.

(Refer Slide Time: 11:46)



So we have what is known as the in Gaussian representation of the Dirac delta function. So this is an important representation of the Dirac delta function. But there are many representations of the Dirac delta function, this is yet one more representation. Now, we will consider yet another representation of the Dirac delta function and that is known as the integral representation of the delta function.

(Refer Slide Time: 12:25)



So let me do it carefully there is a function, there is an integral definite integral which is equal to minus infinity to plus infinity sin g x divided by x the whole thing divided by x.

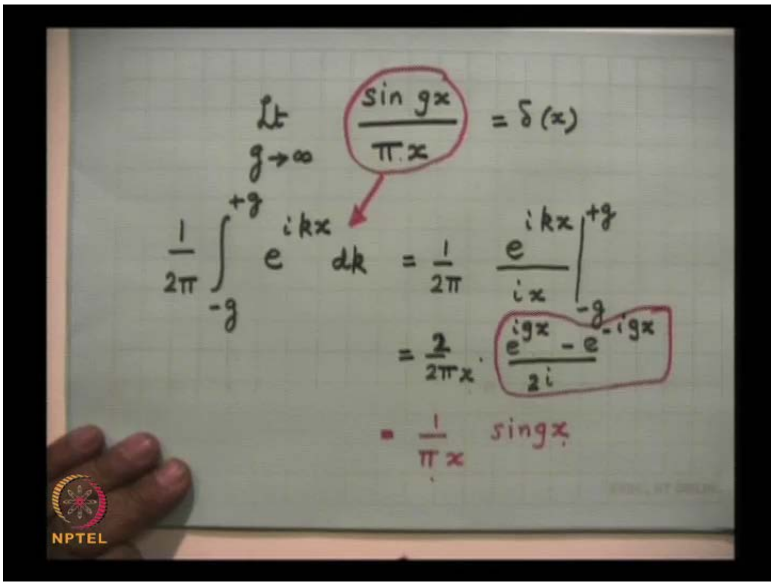
So this is this sink function something like that and integral this is equal to pi for any value of g we assume g to be greater than 0. For any value of g the evaluation of this is extremely straightforward using Laplace transform technique and is given in almost all books of mathematical physics.

But the but the thing that I would like to emphasis is sin g x by x dx is equal to pi. So I write this as so I write this as minus infinity to plus infinity sin g x by pi x dx is equal to 1. If I plot this function you will find you will find that sin g x over pi x in the limit of extending to 0 in the limit of extending to 0 the numerator becomes g x and the denominator becomes pi x so this becomes g by pi g by pi.

And as you can see the first 0 of this function occurs at g x is equal to pi. In fact the 0s of the function appears occurs as g x is equal to pi 2 pi 3 pi 4 pi minus pi minus 2 pi etcetera. So if you plot this function you will find that the value of the function. Let us suppose let us suppose g is 1 so g is 1 so 1 over pi is 0.3 something like this and the first 0 will occur if g is 1 at x is equal to pi something like this. So this is a damp sin curve damp sin curve and it will be symmetric on the other side so, it will be something like this.

If the value of g is increased, if the value of g is increased then that let us suppose g becomes 10. So this value increases by a factor of 10 I cannot show you this but it beco[mes]- and then the first 0 occurs at x is equal to pi by 10 if g is 10. So, therefore, it becomes very shapely peaked and it becomes like this but the area under the curve remains unity.

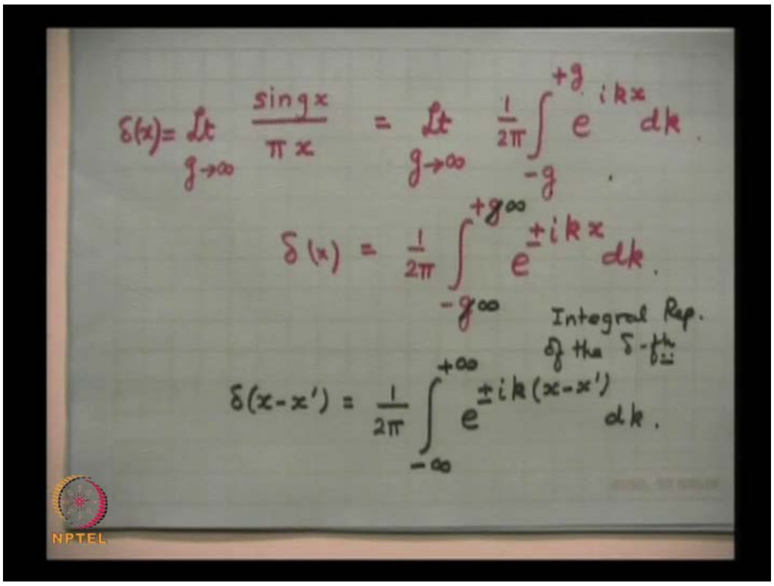
(Refer Slide Time: 15:39)



Therefore, therefore, we have yet another representation of the Dirac delta function, that limit as g tends to infinity sin g x by pi x has therefore, all the properties of the Dirac delta function and I can write this as delta x. So I will use this representation. Now let me write down what is sin g x by pi x, if I integrate a function like this 1 over 2 pi. Let us suppose from minus g consider the definite integral minus g to plus g e to the power of plus i k x or minus i k x it does not matter multiplied by t k. So I carry out the integration over k and the function is e to the power of i k x the integral is trivial its simple 1 over 2 pi and if I integrate this with respect to k, so I get e to the power of i k x divided by i x from minus g to plus g.

So you will have 1 over 2 pi 1 over 2 pi and i x will in the denominator, let me put x outside ~~I~~ here e to the power of i, k becomes g. So I g x minus e to the power of minus i g x I multiply by 2 and divide by 2. So this quantity this quantity e to the power of I g x minus i g x, this quantity is just sin g x as you will wish, so this becomes 1 over pi x sin g x. So therefore, sin g x over pi x this quantity is just this integral is just this integral and therefore, we have a very important relation.

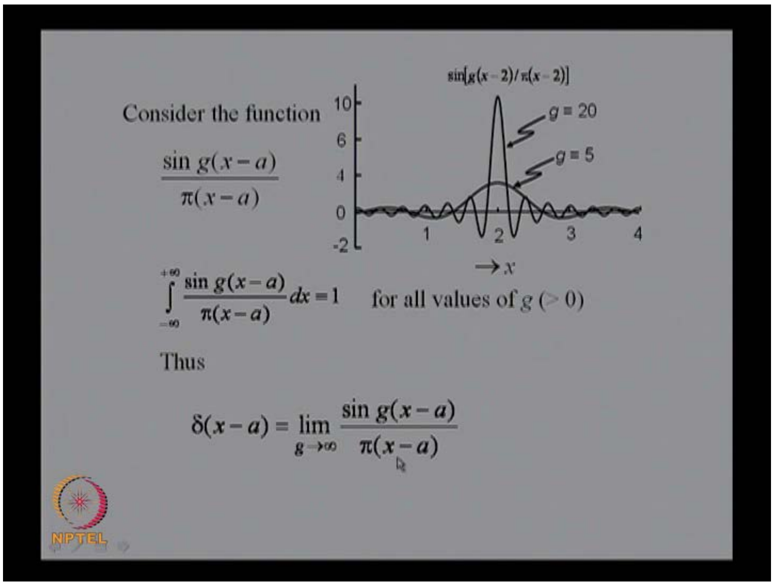
(Refer Slide Time: 18:05)



We have just now shown we have just now shown that that sin g x over pi x, let me leave some space this is equal to 1 over 2 pi 1 over 2 pi minus g to plus g e to the power of i k x dk. So this I have shown just now by evaluating the integral on the right hand side and this follows immediately. But I have just now shown before that that in the limit of g tending to infinity this become the Dirac delta function delta of x. So if I make limit as g tends to infinity so this becomes 1 over 2 pi minus g to plus g actually you can take either plus sign or minus sign it does not really matter plus minus i k x dk. So this is known as the integral representation of the Dirac delta function, this is known as the integral integral representation representation of the Dirac delta function of the delta function.

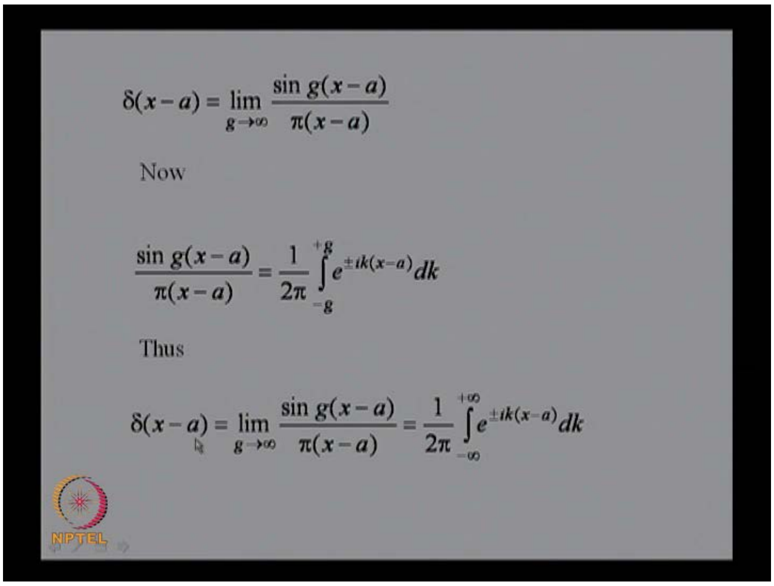
So if I shift the origin if I replace x by x minus x prime, so I get delta of x minus x prime this is equal to 1 over 2 pi sorry this because, I have put g equal to infinity I am sorry this will be minus infinity to plus infinity, so you will have because if I make g tend to infinity this becomes minus infinity to plus infinity. So I should have included that minus infinity to plus infinity e to the power of either a plus sign or a minus sign it does not really matter either 1 e to the power of i k x minus x prime dk. This equation is an extremely important equation, it is known as the integral representation of the Dirac delta function. integral representation of the Dirac delta function.

(Refer Slide Time: 20:38)



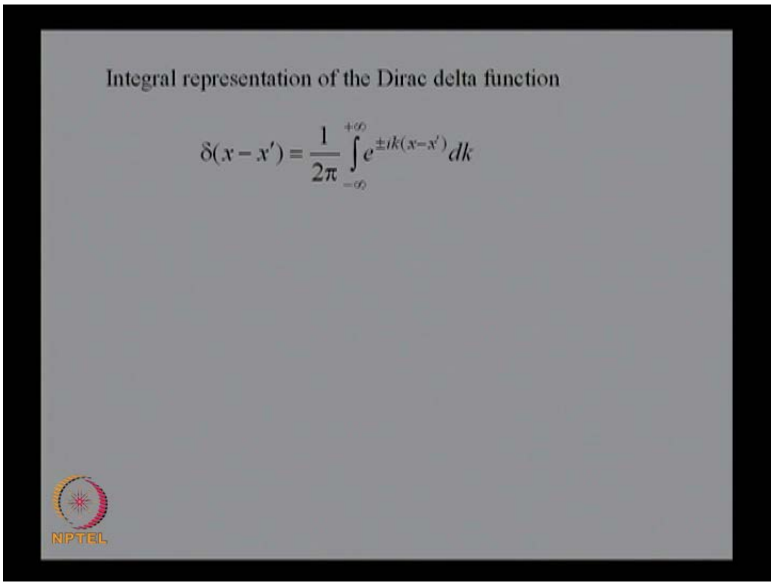
So let me let me show this in the form of slide, that this is for example, the function sin of g x minus 2 divided by pi of x minus 2 for g is equal to ~~phi~~5 the function is spread out. If I increase the value of g it is symmetric around the point x is equal to 2 it becomes sharply peaked and the and and the width becomes extremely small. In the limit of but the area under the curve is always 1 in the limit of g t becoming a very large number this therefore, become this has an infinite value of x is equal to 2. But the area under the curve remains unity and therefore, it has all the properties of the Dirac delta function. So we have so the area under the curve is equal to 1 for all values of g and therefore, we have delta of x minus a is equal to limit as g tends to infinity of sin g x minus a divided by pi of x minus a.

(Refer Slide Time: 21:57)



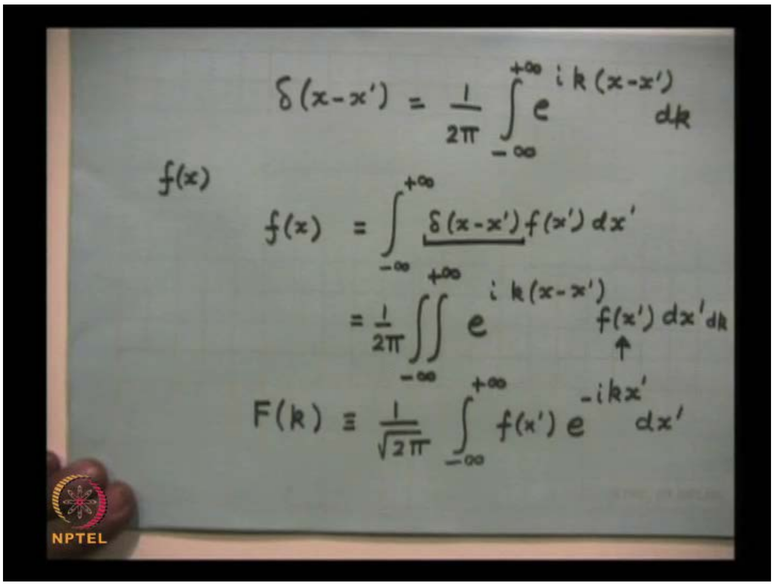
So, we just now showed this that the area under the curve is 1 and it becomes more and more sharply peaked as g becomes large. So, now I have just now shown that if I integrate this then I get the left hand side, if I carry out the integration on the right hand side which is very simple then I will get the left hand side. Therefore, if I make now g tends to infinity so, if I make g tend to infinity so, the limits will become from minus infinity to plus infinity so, I have delta of x minus a is equal to so much.

(Refer Slide Time: 22:43)



So this is known as the integral representation of the Dirac delta function so let me now use this.

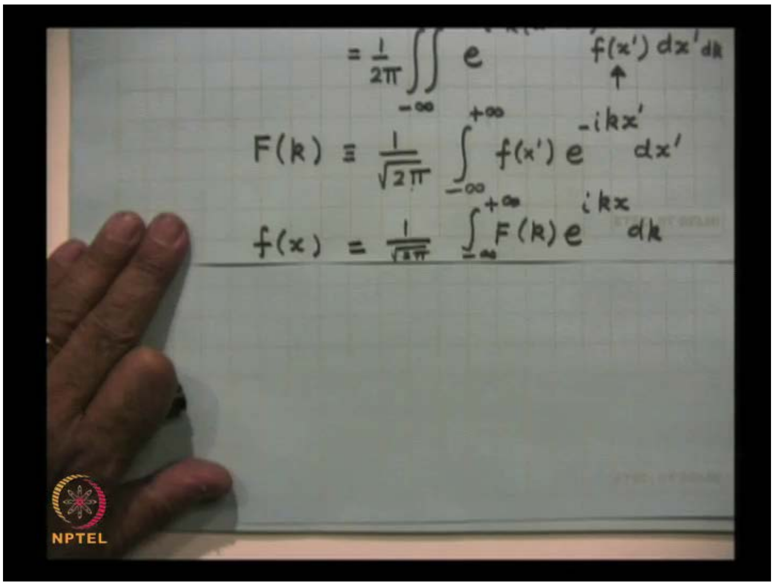
(Refer Slide Time: 22:55)



So I have here e to the power of delta x minus x prime is equal to 1 over 2 pi 1 over 2 pi e to the power of plus or minus so, let me take the plus sign i k x minus x prime dk and the limits are from minus infinity to plus infinity. Now I had told you earlier that I consider a well behaved function f of x, any arbitrary well behaved function f of x and if I integrate this over the delta function that is, if I multiply this f of x by the delta function and integrate over x prime, then I will get the value of the function at f of x. Now for this delta of x minus x prime i substitute this expression so, I write this down as please see this 1 over 2 pi 2 integrals 1 integral is over x prime the other integral is over k so, both limits are from minus infinity to plus infinity e to the power of i k x minus x prime f of x prime dx prime.

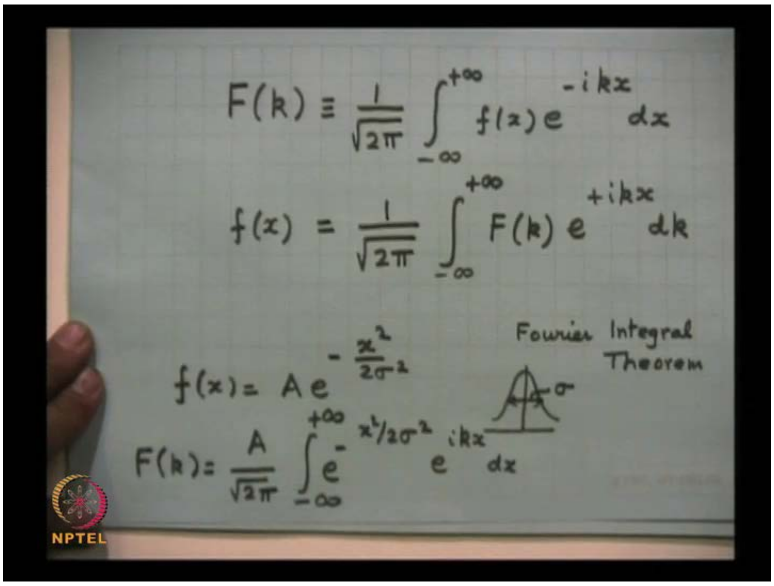
Now I separate the ones which contain x prime so you rem[ember]- so you see this and then dk also dk from this. So, I separate the ones from which contain only x prime and I define a function F of k, I define this function as 1 over root 2 pi, please see this minus infinity to plus infinity f of x prime, which is this e to the power of minus i k x prime which is this and dx prime.

(Refer Slide Time: 22:34)



And then what I do is what I do is that I then write down that f of x f of x will be equal to I have taken 1 over root 2 pi here. So I can write down here 1 over root 2 pi, then F of k e to the power of i k x dk these limits are also from minus infinity to plus infinity. These two equations describe what is known as the Fourier integral theorem, the function F of k is the Fourier transform of f of x and this is the inverse Fourier transform. So in this equation since x prime is a definite integral I can remove the prime on the x.

(Refer Slide Time: 26:37)

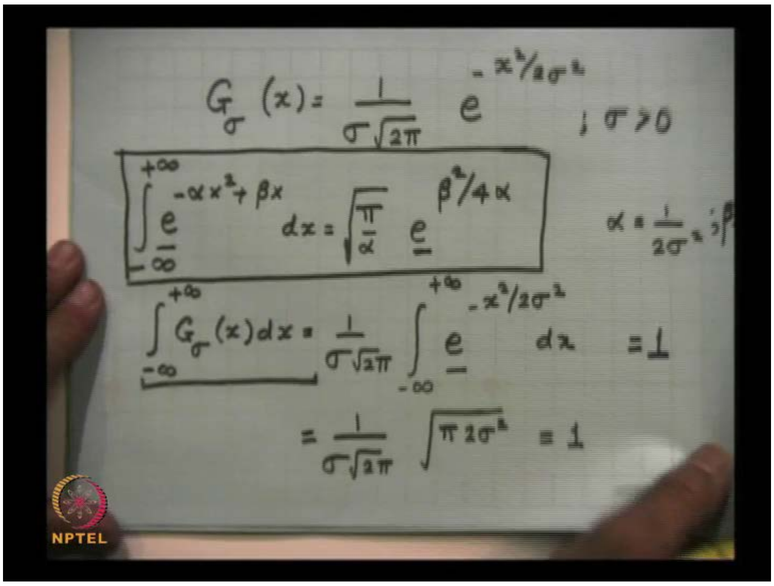


And therefore, I will obtain I can write down you see this I define a function F of k, which is defined to be equal to 1 over root 2 pi from minus infinity to plus infinity f of x e to the power of minus i k x dx. Then f of x is known as the inverse Fourier transform of F of k so 1 over root 2 pi minus infinity to plus infinity F of k e to the power of plus i k x now the integral is over k.

These 2 equations constitute what is known as the Fourier integral theorem Fourier and is a very important concept in mathematical physics. If I have a plus sign here then it will be a minus sign here, if I have a minus sign here I will have a plus sign here, both are extensively used in, so this is one of the very important relationship that I have derived. Now let me take let me give you a simple example so, let me assume that f of x is something like a into e to the power of minus x square by 2 sigma square x square by 2 sigma square. So, it is a Gaussian function a Gaussian function whose width is about sigma something like this is the f of x function this width is of the order of sigma.

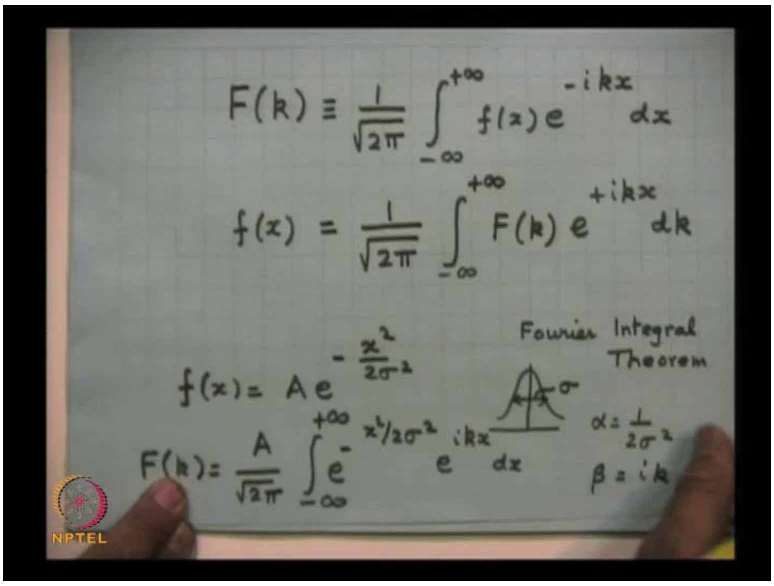
Then its Fourier transform will be F of k will be equal to A over root 2 pi A root 2 pi is should be there. So, then you will have e to the power of minus x square by 2 sigma square e to the power of i k x dx all limits are from minus infinity to plus infinity.

(Refer Slide Time: 29:03)



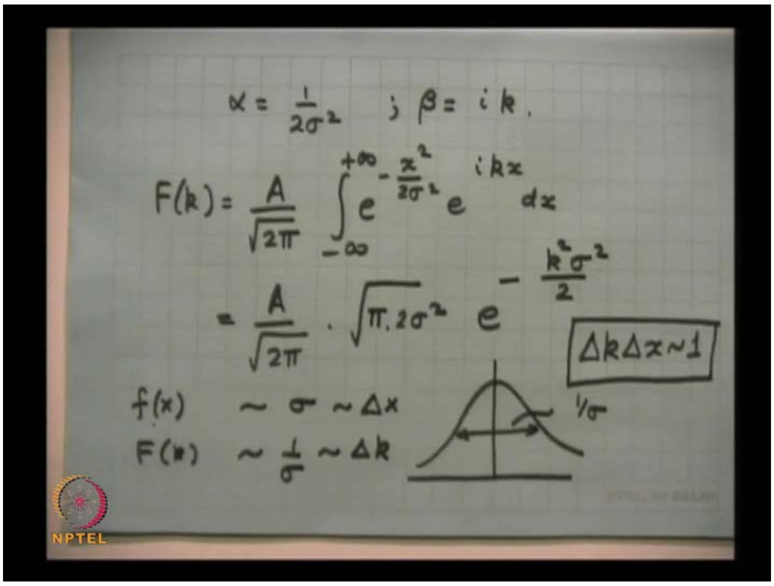
And then we introduce then we have the formula that we had used that e to the power of, this is a very important formula that we will be using quite extensively that, e to the power of minus alpha x square plus beta x is equal to square root of pi by alpha into e to the power of beta square by 4 alpha.

(Refer Slide Time: 29:20)



So here you can see that alpha is equal to 1 over 2 sigma square and beta is equal to i k.

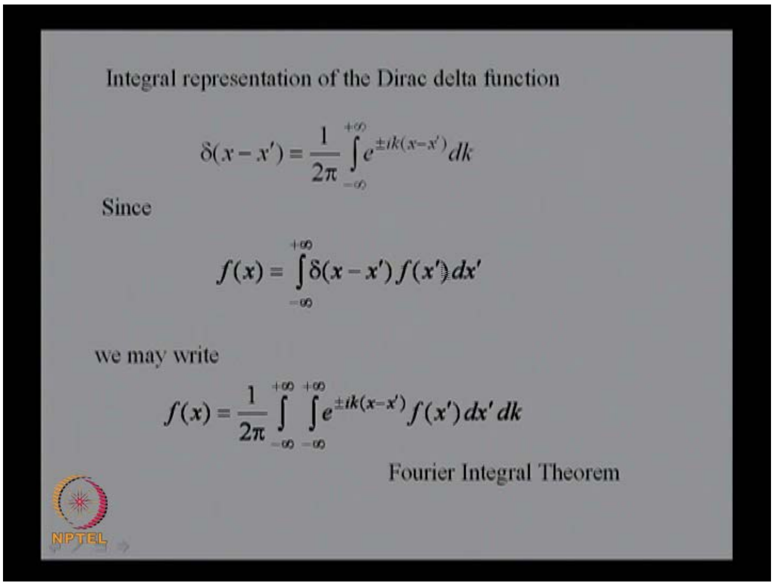
(Refer Slide Time: 29:33)



So you will have so I will have alpha as 1 over 2 sigma square and beta is equal to i k F of k will be equal to as I had written down a moment back A under root of 2 pi e to the power of minus x square by 2 sigma square e to the power of i k x into dx. So, from minus infinity to plus infinity and therefore, you will have A by root 2 pi multiplied by root pi divided by alpha so that root pi times 2 sigma square beta is i k so, square of beta beta square is minus k square minus k square and k square.

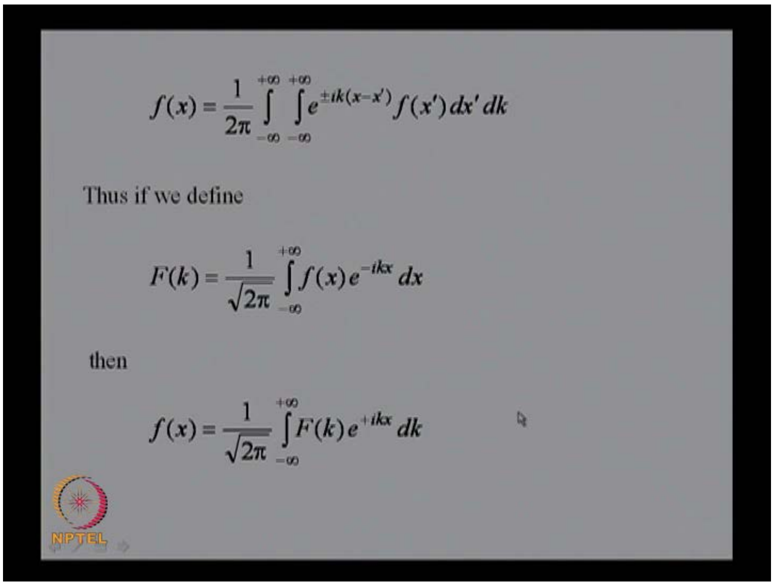
Then 2 sigma square by 4 so, k square sigma square by 2. So, this is just a constant but the main thing that if you plot this function F of k in the k space then that is also a Gaussian e to the power of minus k square by sigma square but the width is now 1 over sigma. Therefore, if a function if a function I had just shown that f of x has a width of the order of sigma then its Fourier transform has a width of the order of 1 by sigma. So the if f of x is localized within a distance which is delta x then, F of k will be localized within a distance in the k space delta k, which is 1 over sigma so that delta k delta x is of the order of 1, this is a very important relationship and from which we will prove the derive the uncertainty principle.

(Refer Slide Time: 31:45)



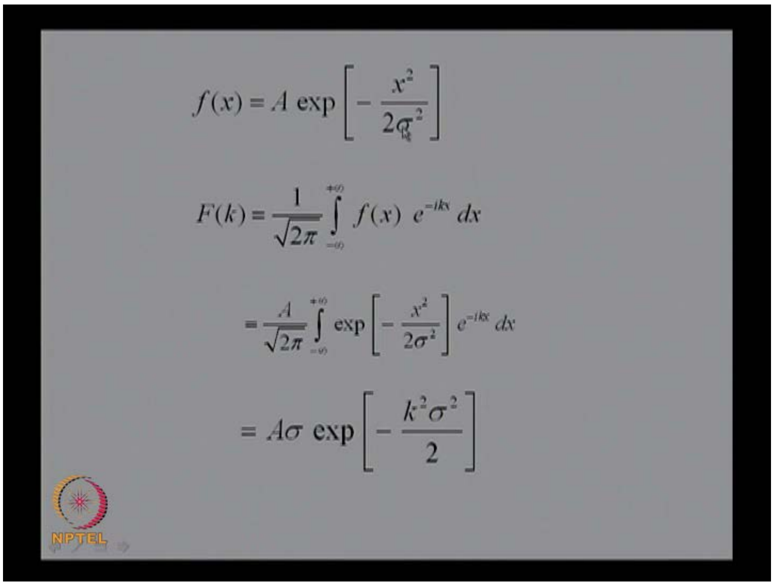
So let me go back to my slides. So, this is my integral representation of the Dirac delta function. So, any arbitrary well behaved function if I integrate by after multiplying by delta of x minus x prime so, I will get f of x. So, then I substitute for delta of x minus x prime from this equation and we obtain this double integral, this integral is known as the Fourier integral theorem. I can have either of this sign I can have a plus sign here or a minus sign here.

(Refer Slide Time: 32:27)



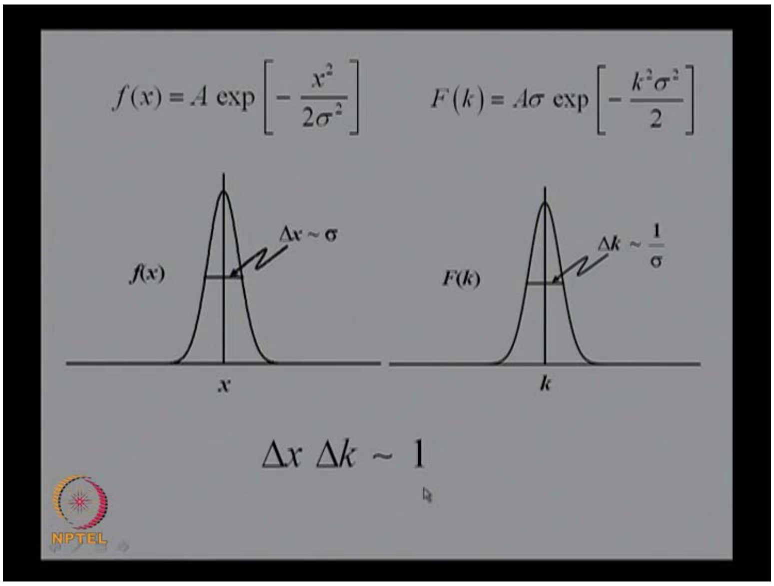
So what I have just now shown is that f of x is equal to 1 over 2 pi e to the power of plus minus i k x minus x prime f of x prime dx prime dk. I now define the Fourier transform of f of x and because of this 2 pi factor here, I spit it into 2, so that the equations becomes symmetric 1 over root 2 pi k. So, f of x e to the power of minus i k x, I have dropped the prime and then I and then I substitute it back then f of x is equal to 1 over root 2 pi F of k e to the power of i k x d k. So this function F of k is known as the Fourier transform of the function f of x and this is the inverse transform relation and these 2 together is often called as the Fourier integral theorem.

(Refer Slide Time: 33:37)



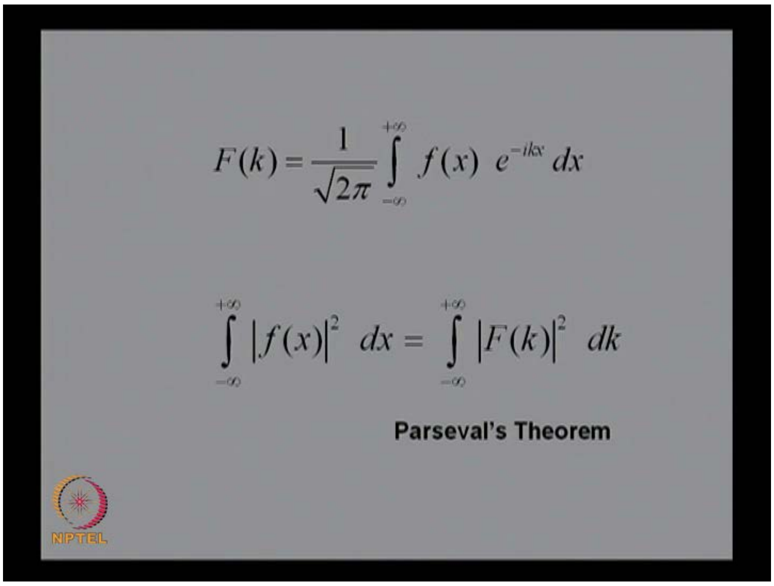
So as I mentioned just now that let me take a Gaussian function f of x is equal to ~~A~~ exponential minus x square by 2 sigma square. So, I substitute it here and calculate its Fourier transform so I substitute it here and here, you again use the same relation with alpha is equal to 1 over 2 sigma square and beta is equal to minus i k so, this square of that is minus k square sigma square by 2. So we say that the Fourier transform of a Gaussian function is a Gaussian function.

(Refer Slide Time: 34:22)



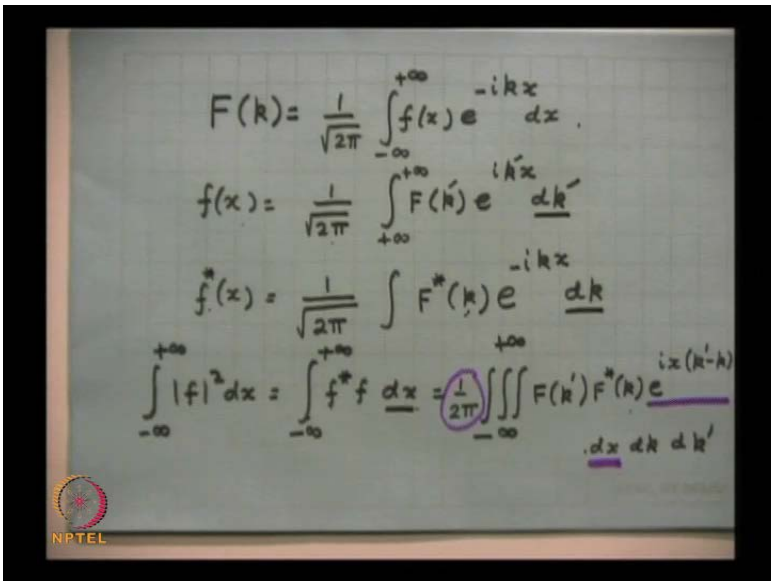
And therefore, this is the function f of x this is the Gaussian function in the x space and its Fourier transform is also a Gaussian function. So, what we have derived is that the Fourier transform of a Gaussian is a Gaussian, the main thing is that its width something like full width at half maximum is of the order of sigma and its half width. If the localization of the function F of k in the k space is of the order of 1 over sigma, so that you have delta x delta k is of the order of 1 delta x delta k is of the order of 1.

(Refer Slide Time: 35:08)



Now let me let me now derive one more relation that I have this Fourier transform and I will show that if f of x is normalized then F of k is also normalized and the proof is very simple and we will again use the Dirac delta function.

(Refer Slide Time: 35:39)



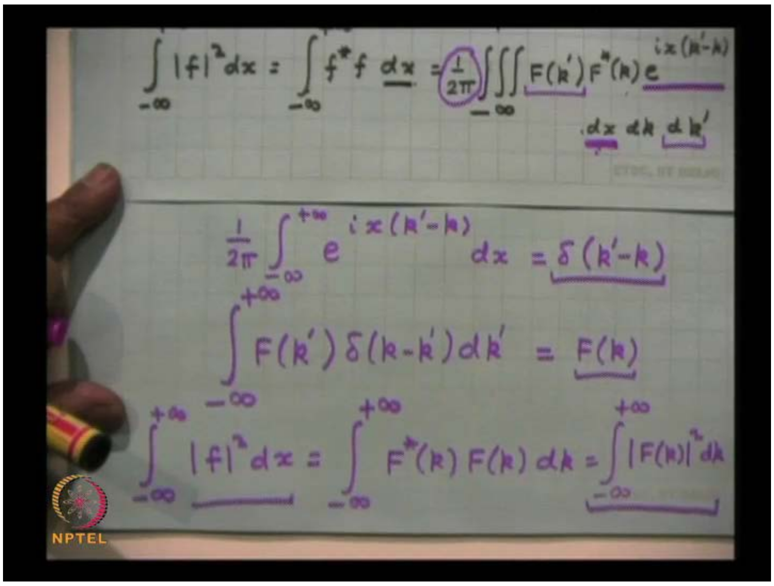
So I have here let us suppose F of k f of k is equal to is equal to 1 over under root of 2 pi, f of x e to the power of minus i k x dx, all limits are from infinity. Therefore, your f of x is equal to 1 over under root of 2 pi, F of k e to the power of i k x dk this is all limits are from minus infinity to plus infinity and if I take the complex conjugate so, f star of x. If the complex conjugate of f of x is equal to the complex conjugate of this equation so, 1 over root 2 pi F star k f star k e to the power of minus i k x d k.

Now what I want to evaluate is integral from minus infinity to plus infinity mod f square x mod f square x dx. So, this is equal to f star times f into dx so, please see this, this this is slightly cumbersome but very straightforward. So what I will like to substitute is f star from here f from here and substitute it and then carryout the integration over x. So, now you must be very careful because this k should not get confused with this k so I will quietly put a prime here k prime here k prime here then everything is all right.

So you will have ~~so~~ if I substitute this f star times this 1 over 2 root 2 pi into 1 over root 2 pi. So, this will become 1 over 2 pi and there are 3 integrals 1 integral here over x 1 over k prime and 1 over k so, all limits are from minus infinity to plus infinity. So you will have first write down F of k prime f star of k e to the power of e to the power of i x, then k prime minus k k prime minus k into dx dk d k prime, dx comes from here dk comes from here and dk co[mes]- prime comes from here.

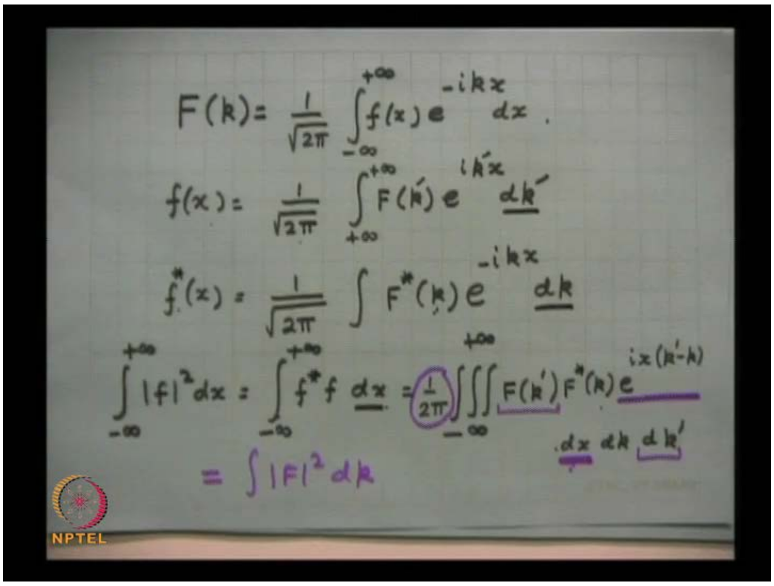
Now let me then collect all those terms which involve x this does not involve x, this does not involve x, this involves x and this involves x. So, I carry out the integration over x so, what I do is I collect this term, I collect this term, and I collect this term and carryout the integration over x so, if I do that if I put it at the top.

(Refer Slide Time: 39:31)



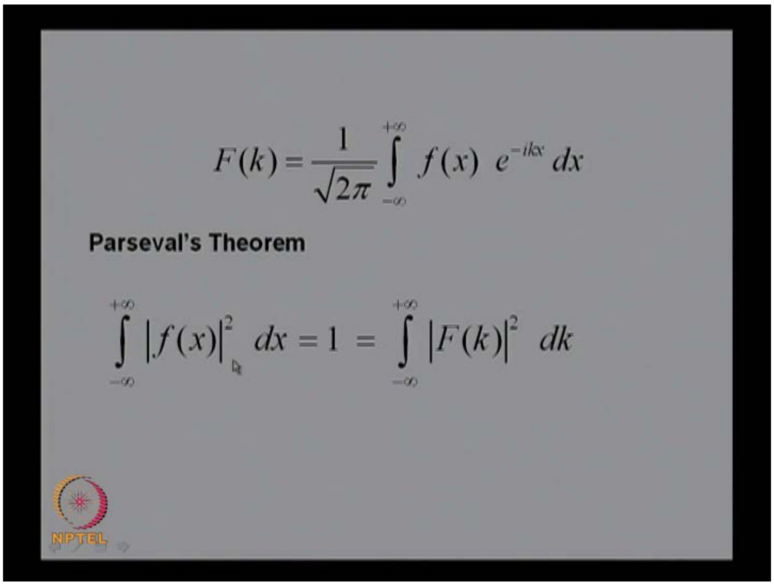
Then what I will find is 1 over 2 pi minus infinity to plus infinity e to the power of i x k prime minus k dx, this is equal to I can write delta k minus k prime or k prime minus k, it does not really matter k minus k prime. So I have carried out 1 integration over x and now what I do is I carry out the integration over k prime. So, I collect all the terms which involve k prime which is F of k prime which is F of k prime dk prime and delta of k minus k prime. So, I write this down so integral F of k prime F of k prime then delta of k minus k prime and dk prime and of course, this limit is also from minus infinity to plus infinity. So, this is just F of k so I substitute it here and I get the relation that minus infinity to plus infinity mod f square dx is equal to minus infinity to plus infinity F star of k multiplied by F of k multiplied by dk. So that we get the very important relation that this is known as parseval's theorem, that minus infinity to plus infinity the mod square of the function and the Fourier transform are equal. So if one of them is normalized one of them is normalized the other will be automatically normalized

(Refer Slide Time: 41:46)



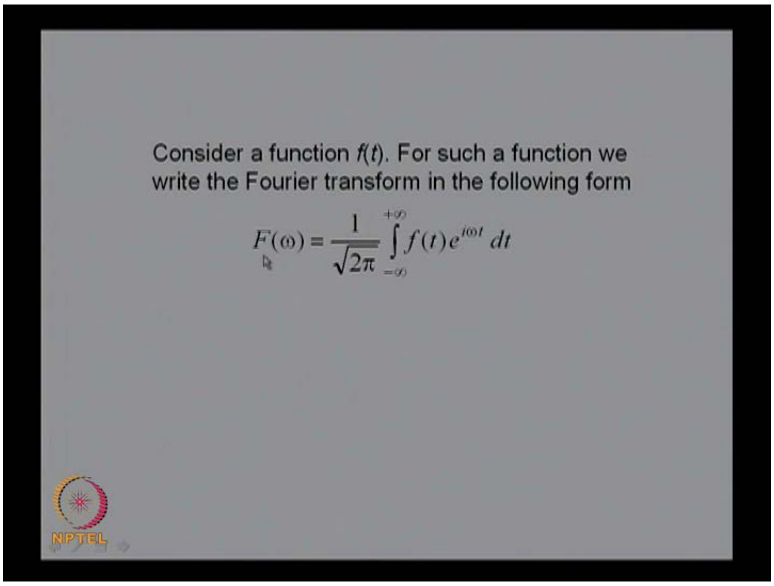
So we obtain the very important relation that that mod f square dx is equal to mod of f square dk. This is known as the parseval's theorem. So this is what I have tried to show here any arbitrary well behaved function, if I square this and integrate over dx in the square of that is equal to 1. Therefore, if this function if a is such that if mod f square dx the curve is 1 then mod f k square the under that function will also be equal to 1.

(Refer Slide Time: 42:24)



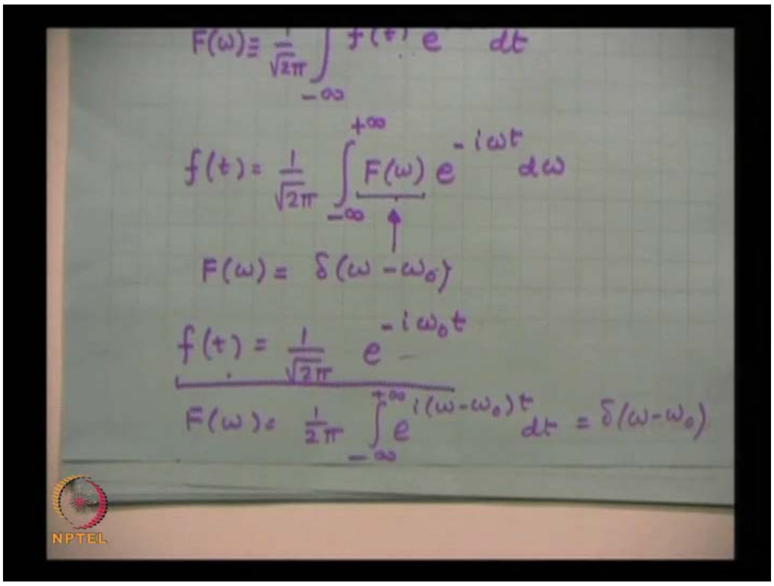
So, ~~if~~ this is what parseval's theorem is, that if 1 of them is equal to 1, then the other one is this is also known as the normalization condition, this is also known as the normalization condition.

(Refer Slide Time: 42:30)



Till now I have done between x and k, there is no reason why we cannot consider a time dependent function, and consider its Fourier transform. So let us consider a time dependent function f of t.

(Refer Slide Time: 42:54)

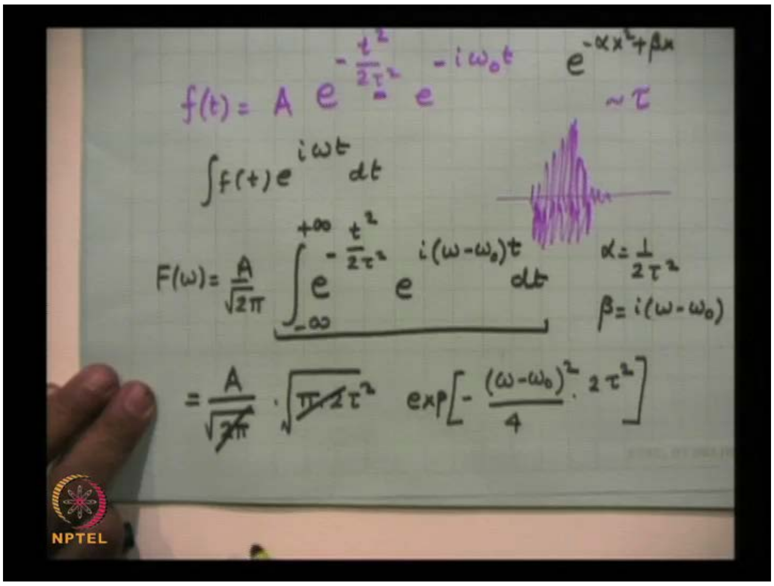


Let us let us consider a time dependent function which I write as f of t and consider its Fourier transform, so the Fourier transform I define as f of omega is defined to be equal to 1 over root 2 p from minus infinity to plus infinity e to the power of i omega t dt. This is known as the frequency spectrum and I will tell you in a moment why this is known as a frequency spectrum. Then the Fourier integral theorem tells us that, if I take the inverse Fourier transform then f of t will become, equal to 1 over root 2 pi 1 over root 2 pi F of omega f of omega e to the power of minus i omega t d omega minus i omega t d omega.

Now please see this what is an integral, an integral is nothing but a sum. So, this is a sum over all the frequencies so, this quantity F of omega is known as the frequency spectrum of the time dependent function f of t. For example, if the time dependent function has only ~~one~~1 frequency then the corresponding F of omega is a Dirac delta function. So if the frequency spectrum is has only one frequency let us suppose delta of omega minus omega 0 then, if I substitute that here carryout the integration then f of t becomes equal to 1 over under root of 2 pi e to the power of minus i omega 0 t.

So this is a monochromatic wave this is a monochromatic wave so, if I substitute this f of t function in in in in here, then I will get F of omega f of omega will be 1 over 2 pi 1 over 2 pi e to the power of e to the power of i of omega minus omega 0 into t into dt. So, this is the definition of the Dirac delta function this is one of the definitions of the Dirac. All limits are from minus infinity to plus infinity so, if the if a time dependent signal has only 1 frequency, then the corresponding frequency spectrum is a Dirac delta function and the corresponding time dependent function is a harmonic wave. So, it is if I take the real path it is Cos omega 0 t from from from t equal to minus infinity to t equal to plus infinity.

(Refer Slide Time: 46:19)

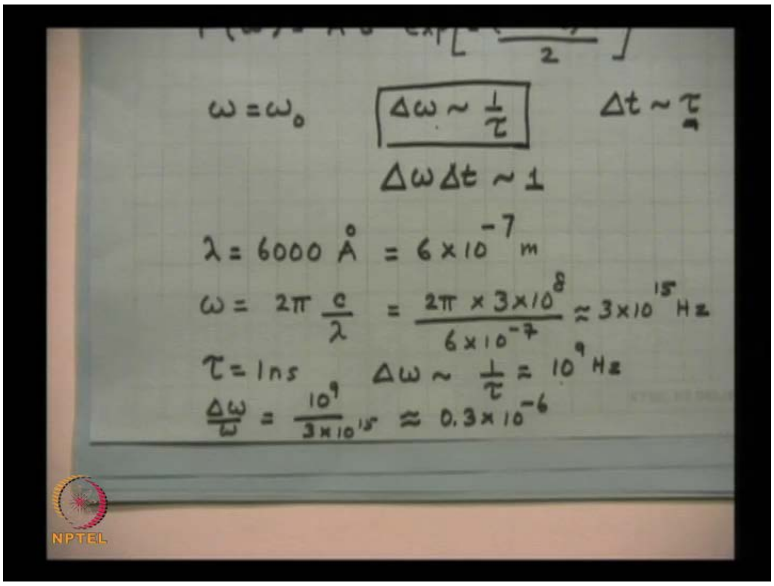


So you have here let me consider a simple function let me consider a simple function e to the power of minus t square by 2 tau square e to the power of minus i omega 0 t. So let us suppose that my time dependent function f of t is something like this, if you take the real path then it will be something like a Gaussian time dependent function multiplied by Cos omega 0 t. So it is a Gaussian pulse of duration tau of duration tau so, it will look something like this. something like this So you will have a Gaussian pulse localized within a time interval of the order of tau.

So let me take calculate its its its Fourier transform, the Fourier transform will be F of omega, F of omega will be equal to 1 over root 2 pi A I can take outside so, integral f of t e to the power of I omega t dt. So I will write it down integral e please do this carefully minus t square by 2 tau square and then e to the power of i so, this will be e to the power of iI omega minus omega 0 multiplied by t times d t.

So here again I again have the relation like e to the power of minus alpha x square plus beta x where x is replaced by t so, my alpha is equal to 1 over 2 tau square and beta is equal to I times omega minus omega naught. So the evaluation of this integral these limits are of course, minus infinity to plus infinity. So this will be please see this carefully 1 over root 2 pi 1 over root 2 pi under root of pi by alpha that is under root of pi times 2 tau square multiplied by beta square, that is exponential beta square beta square by 4 alpha, that is minus omega minus omega 0 whole square by 4 alpha is 2 tau square 1 over 2 tau square. So 2 pi 2 pi cancelled out.

(Refer Slide Time: 49:07)



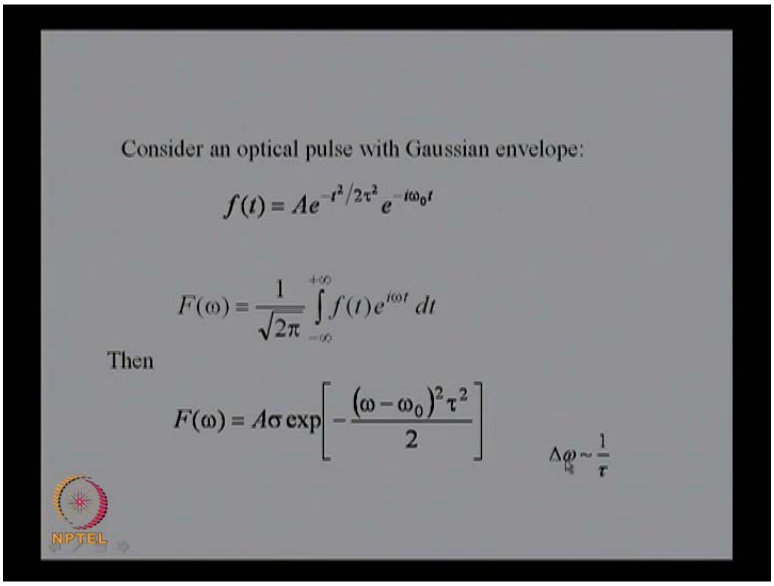
So I will get the frequency spectrum F of omega will come out to be ~~A~~ times tau, this is an unimportant constant factor but once again the Fourier transform of the Gaussian function is a Gaussian function minus omega minus omega 0 whole square by tau square by 2 this is a square here. So this function is picked around omega is equal to omega 0 and it falls by a factor of 2 when omega the del the frequency is spread is of the order of 1 over tau.

So the duration of the pulse is of the order of tau the frequency width of the pulse is of the order of 1 over tau. So I get this uncertainty relation delta omega delta t is of the order of 1, this is known as the mono chromaticity of the pulse smaller the delta variable omega, the greater is the sharpness of the frequency. So t tau is something like the, but we did in optics as the coherence time larger the coherence time smaller is the spectral width and this we have all studied in optics and therefore, if you have a very long coherent pulse then its frequency spread is extremely small.

So let us suppose that I have the wavelength is equal to say 6 000 armstrong in the yellow region 6,000 armstrong. So you will have this is equal to 6 into minus 10 to the power of minus 7 meters and omega will be equal to 2 pi times nu, nu is c by lambda. So this will be 2 pi into 3 into 10 to the power of 8 into 6 into 10 to the power of minus 7, so this will come out to be about 3 into 10 to the power of 15 hertz 3 into 10 to the power of 15 hertz.

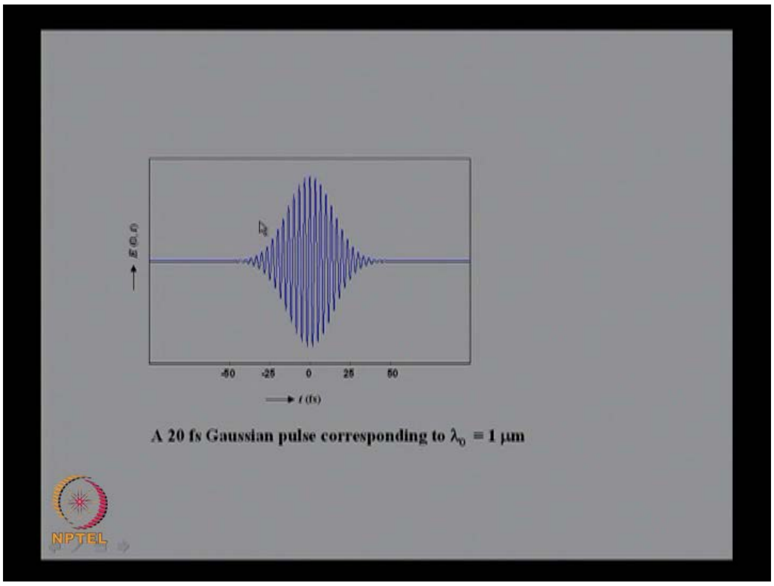
And let us suppose I have a 1 nanosecond of duration pulse so, the frequency spread delta omega is of the order of 1 over tau. So this is of the order of 10 to the power of 9 hertz, which is about a millionth of the frequency so, delta omega by omega is a very small number so delta omega by omega will be equal to 10 to the power of 9 divided by 3 into 10 to the power of 15 so, this is about 10 to the power of minus 6 in fact less than 10 to the power of minus 6. So it has a very, very small width an almost monochromatic so, let me do this again.

(Refer Slide Time: 52:46)



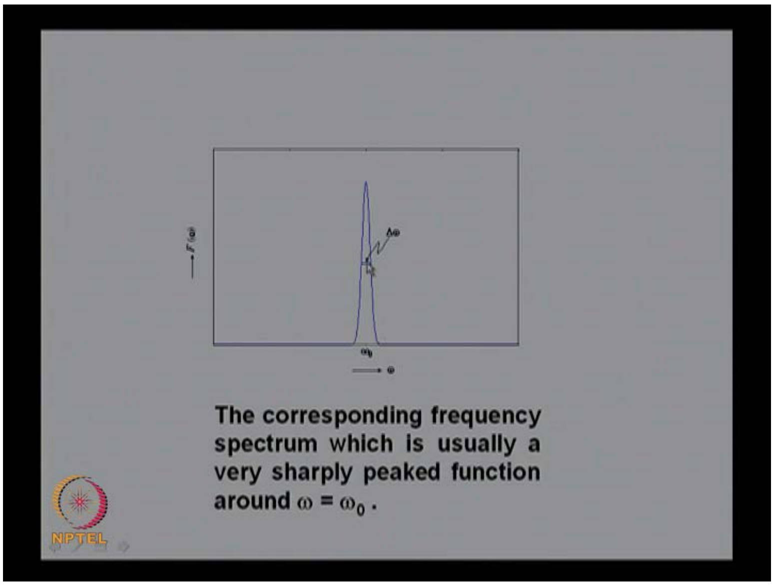
So let me have a we consider an optical pulse with Gaussian envelope. So the duration of the pulse is of the order of tau and the central frequency is around omega 0 is at omega 0. The spectrum the Fourier transform pulse is given by of omega equal to so much and if you substitute this here, we have shown that the frequency spectrum is so much. The spectral width is of the order of tau.

(Refer Slide Time: 53:24)



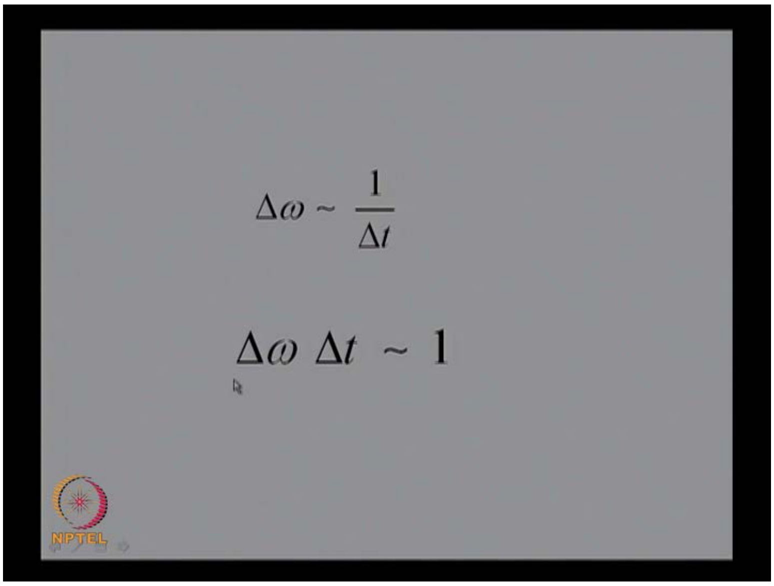
So, this is something like a let us suppose we consider a Gaussian pulse which corresponds to 1 micro meter wavelength and it is a 20 second 20 fem to second pulse. This is actually the how the electric field will look like.

(Refer Slide Time: 53:42)



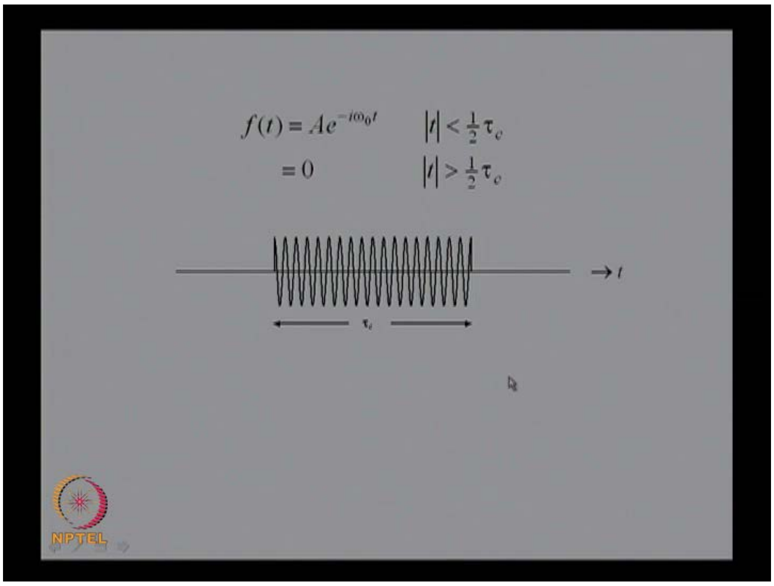
And its corresponding Fourier transform will be very sharply peaked the Fourier transform of a Gaussian is a Gaussian will be extremely small delta omega will be very small.

(Refer Slide Time: 53:57)



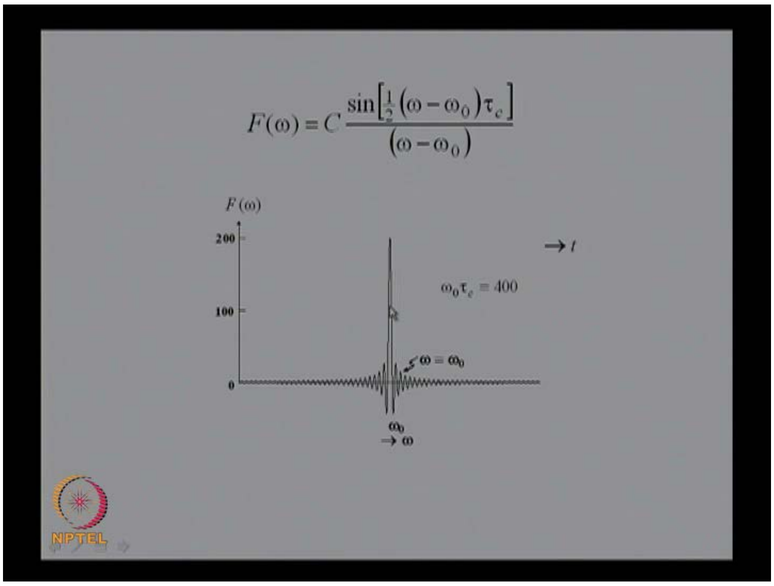
And we will have the uncertainty relation that delta omega delta t is of the order of 1.

(Refer Slide Time: 54:05)



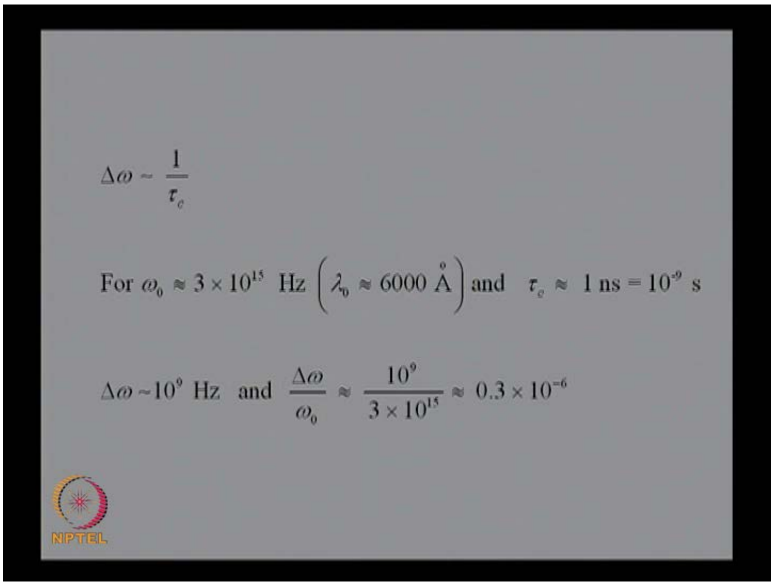
And we can consider similarly, another time dependent function where tau c, where where we have f of t is 0 for all times which is greater than plus half t c or minus half t c. So tau c is the duration of the pulse. So this is known as the coherence time and in between it is A into e to the power of minus i omega 0 t.

(Refer Slide Time: 54:37)



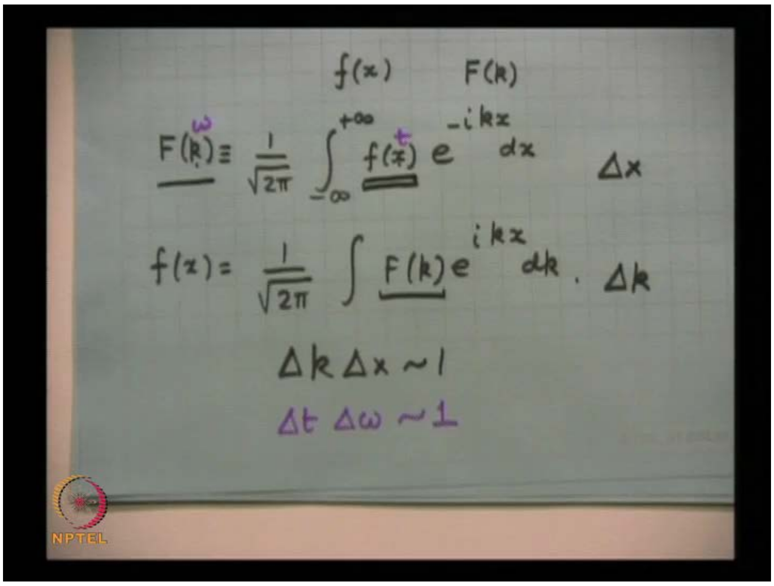
If you take the calculate the Fourier transform, it is a very simple exercise it will be very sharply peaked around omega equal to omega 0, in this particular case I have assumed omega 0 tau c is of the order of 100. So longer the value of tau c greater will be the sharpness of F of omega and greater will be the mono chromaticity of the pulse.

(Refer Slide Time: 55:09)



So for example, the example that had just now considered for omega 0 is equal to 3 into 10 to the power of 15 hertz, the corresponding wavelength is 6000 armstrong. If I consider a 1 nano second pulse then the corresponding frequency 6,000 will be 1 over tau c 1 over tau c is 10 to the power of 9 hertz and so the spectral purity of the pulse is point 3 into 10 to the power of minus 6. Now so I summarize this part of the lecture by saying that if I had any time dependent function or a space dependent function.

(Refer Slide Time: 55:59)



So I have f of x I write its Fourier transform as F of k and I write this F of k, I define this as equal to 1 over root 2 pi minus infinity to plus infinity f of x e to the power of minus i k x into dx. Then its inverse Fourier transform will be f of x will be 1 over under root of 2 pi F of k e to the power of i k x dk. I can replace x by time and then k with omega so, that is also possible and we can time dependent function and write down the corresponding frequency spectrum of the pulse.

If the function f of x has a localization of the order of delta x then F of k the localization is delta k. Then in general delta k delta x is of the order of 1 similarly, I can define for instead of x as I have just now mentioned this is t and then this will be the omega spectrum and if the time duration of the pulse is delta t. Then its corresponding frequency spectrum is delta omega and that is 1. So this complete so you can carry out this with other examples, that you take an arbitrary well behaved function f of x calculate its Fourier transform show that, if it is localized within a distance of the order of delta x, then its Fourier transform will be localized within a distance of the order of 1 over delta x thank you.