Basic Quantum Mechanics

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Module No. # 08

Angular Momentum - II

Lecture No. # 01

Angular Momentum Problem using Operator Algebra

In this and the following lectures, we will be discussing the Eigen values spectrum of an angular momentum operator using operator algebra. In the last lecture, we had discussed that if there are two observables denoted by the operators alpha and beta, and if they commute, then they will have simultaneous Eigen kets.

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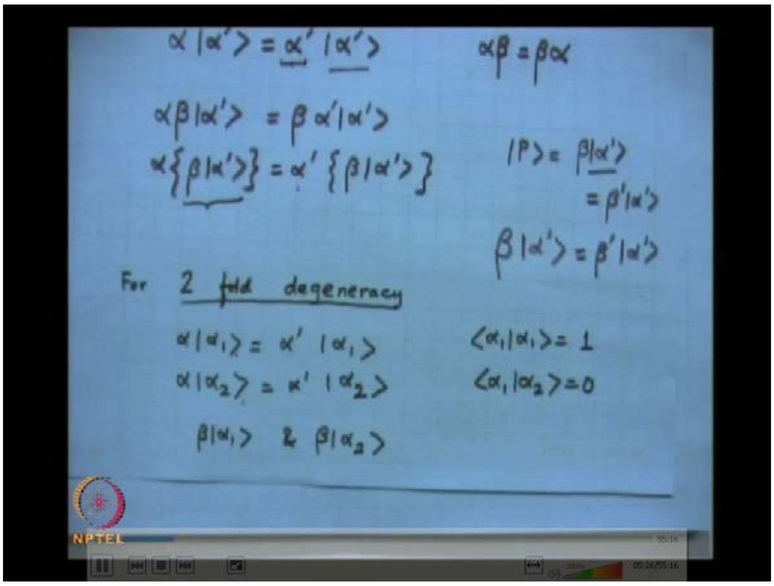

In fact, we considered first, that the operator alpha has a non degenerate Eigen value and with that Eigen value, we depend denoted by alpha prime. Then, if the two operators alpha, beta commute with each other; then we wrote down that, alpha beta ket alpha prime is equal to beta alpha ket alpha prime, but ket alpha prime is an Eigen ket of the operator alpha. So this is, just a number multiplied by a number, alpha prime is the Eigen value and therefore, it is a number and so this becomes alpha prime beta alpha prime.

So, then we said that, the ket; which is represented by ket beta alpha, that ket p. If this is a non degenerate Eigen value then, ket p given by beta alpha prime must be a multiple of ket alpha prime, because this is an Eigen ket of the operator alpha, belonging to the same Eigen value alpha prime.

This equation tells us, that ket beta ket alpha prime is an Eigen ket of the operator alpha, belonging to the same Eigen value and if it is a non degenerate Eigen value, then it must be a multiple of alpha prime and therefore, this equation tells us that, ket alpha prime is also an Eigen ket, beta ket alpha prime is equal to beta prime ket alpha prime and therefore, you will have ket alpha prime is a simultaneous Eigen ket of the operators alpha and beta. For alpha, it belongs to the Eigen value alpha prime and for beta; it belongs to the Eigen value beta prime.

We continue our discussion and we say that, let us suppose, this is a degenerate Eigen value; that is, let us consider this simple case by the degeneracy is twofold, so for 2 fold degeneracy, we can have two linearly independent kets, but that we denoted as alpha alpha 1 alpha prime alpha 1 and alpha alpha 2 alpha prime alpha 2. That is ket alpha 1 and ket alpha 2 are 2 Eigen kets of the operator alpha, belonging to the same Eigen value alpha prime and of course, they are orthonormal to each other. That is, we can always choose that, alpha 1 alpha 1 is equal to alpha 2 alpha 2, this is 1 and alpha 1 alpha 2, this is equal to 0.

There are set of orthonormal Eigen kets, so this corresponds to what we have a twofold degeneracy? Similarly, we can have threefold degeneracy. Now, beta alpha prime is an Eigen ket of the operator alpha. So therefore, beta alpha 1 and beta alpha 2 will also be Eigen kets of the operator alpha, and therefore, they must be linear combinations of alpha 1 and alpha 2.

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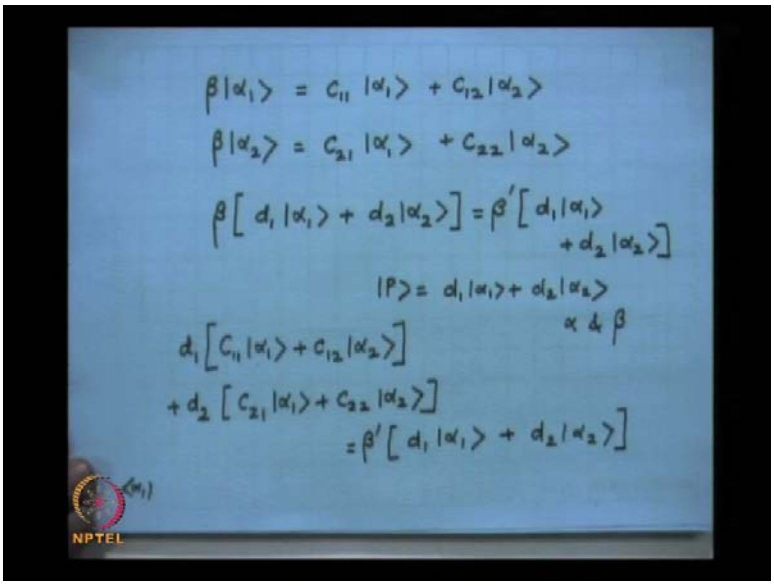

That is, beta alpha 1 is equal to C 1 1 ket alpha 1 plus C 1 2 ket alpha 2 and b beta alpha 2 is equal to C 2 1 ket alpha 1 plus C 2 2 ket alpha 2. I hope; I have made myself clear; I will repeat once again, it is a slightly critical that beta alpha 1 and beta alpha 2 are also Eigen kets of the operator alpha. So they must be expressible, as linear combinations of alpha 1 and alpha 2.

Now, let the Eigen ket; let the ket, which is an Eigen ket of the operator beta? Been denoted by beta operating on d 1 ket alpha 1 plus d 2 ket alpha 2 where, d 1 and d 2 are numbers. Let this be, a multiple of this d 1 ket alpha 1 plus d 2 ket alpha 2. Since, alpha 1 ket alpha 1 and ket alpha 2 are Eigen kets of alpha, belonging to the degenerate Eigen value; any linear combination is also an Eigen ket of the operator alpha.

We now, look for that linear combination, which are also Eigen kets of the operator beta. So, let that linear combination d 1 ket alpha 1 and d 2 ket alpha 2, with that linear combination; which is also an Eigen ket of the operator beta, belonging to the Eigen value beta prime. So here, d 1 d 2 are just numbers. So, we are assuming that ket p, which is denoted by d 1 ket alpha 1 plus d 2 ket alpha 2 is a simultaneous Eigen ket of alpha and beta.

So, let me work this out then, I will try to give you an example so d 1 beta ket alpha 1. So, beta ket alpha 1 is C 1 1 ket alpha 1 plus C 1 2 ket alpha 2 plus beta plus d 2 beta ket alpha 2 so that is, equal to C 2 1 ket alpha 1 plus C 2 2 ket alpha 2. Now, we must remember all these, here this is an operator alpha and beta are operators, but here they are all numbers. So, this is equal to beta prime d 1 alpha 1 d 1 ket alpha 1 plus d 2 ket alpha 2.

Now, ket alpha 1 and ket alpha 2 are orthonormal kets and therefore, the we can; I can multiply by bra alpha 1 then, this term will go to 0; this term will go to 0; and this term will go to 0 and we will obtain please see this, that if, I the coefficient of ket alpha 1; this will be d 1 C 1 1 and from this side, will be minus beta prime plus this ket alpha 1 is here, plus d 2 C 2 1 this is equal to 0 and then, we will have d 2 d 1 C 1 2 sorry let me let me not use this.

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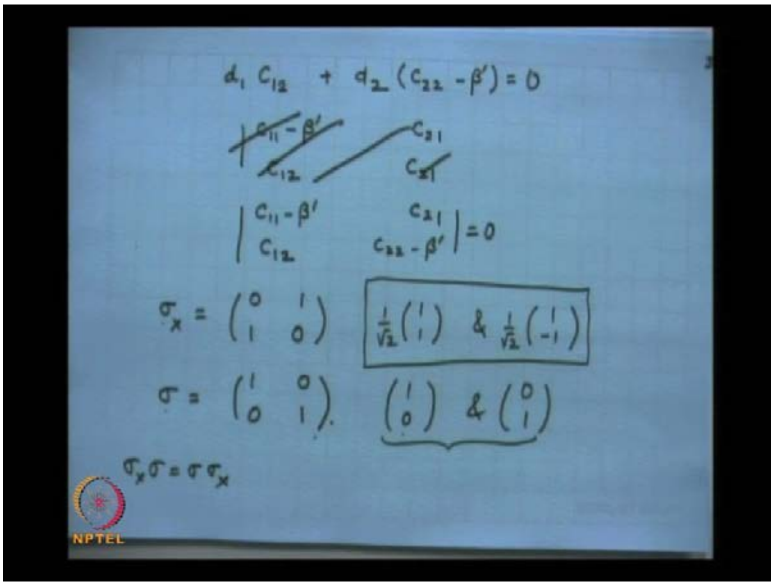

So now, if I equate the coefficient of ket alpha 2, then we will have d 1 C 1 2 plus d 2 C 2 2 minus beta prime, this is equal to 0. This is a set of linear homogeneous equations, so for non trivial solutions, the determinant must be 0. So, we will have C 1 1 minus beta prime C 2 1 C 1 2 C 2 1, sorry sorry I am sorry, this will be yeah that is alright, so this was alright C 1 1 minus beta prime C 2 1 C 1 2, and then C 2 2 minus beta prime; this is equal to 0.

So, this will give me two values of beta prime 2 values of beta prime and for each value of beta prime, we will have two ratios so that; we can always construct as set of orthonormal Eigen kets, which are simultaneous Eigen kets of the operators alpha and beta and let me take the same example, as we had taken in our last class. Let me take the example, like sigma x is equal to 0 1 1 0 and sigma is equal to 1 0 0 1. Now here, sigma x and sigma commute with each other, they are both hermitian matrices therefore, we must have a complete set of complete Eigen kets, but not all Eigen kets of sigma will be Eigen kets of sigma x. You can see that, the Eigen kets of sigma x as we had discussed yesterday in my last lecture, are 1 1 and 1 minus 1.

You can put a root 2 so that, they are not normalized. On the other hand here, these are simultaneous Eigen kets, but these are also Eigen kets 1 0 and 0 1, so, these are Eigen kets of sigma, but not Eigen kets of sigma x, but I can always choose a linear combination. So that, which are simultaneous Eigen kets of sigma and sigma x and that linear combination of this particular case, is this. These are the 2 orthonormal vectors, which are simultaneous Eigen ket vectors of sigma x and sigma.

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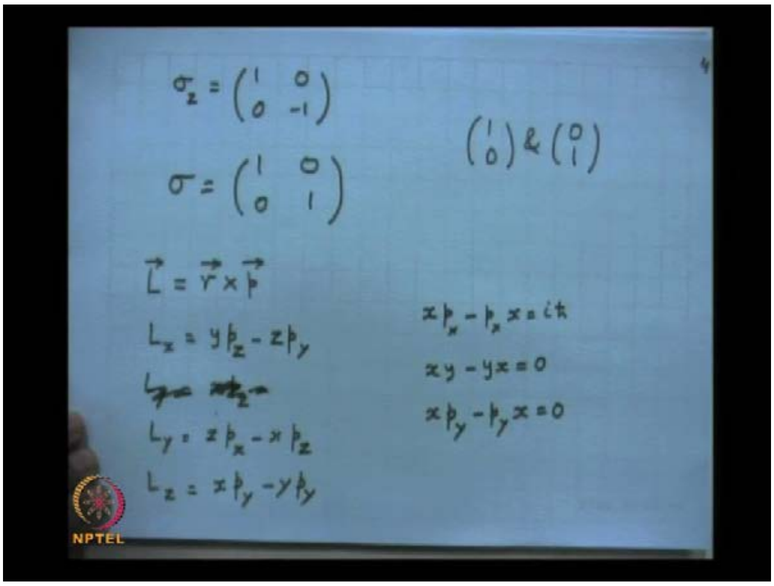

Similarly, we said; we had said that, if we take this matrix, these are the Pauli spin matrices; this 1 0 0 minus 1 and then, sigma again the same matrix 1 0 0 1 and then, sigma z and sigma commute with each other, but the simultaneous Eigen kets are 1 0 and 0 1. So, 1 0 and 0 1 are simultaneous Eigen kets of sigma z and sigma. This and this are simultaneous Eigen kets of sigma and sigma x and sigma.So we have shown that, if two observables; if two linear operators commute then, we can always construct a complete set of simultaneous Eigen kets.

Now, we will discuss the operator algebra, associated with components of the angular momentum operator. Now, we will first; as a we had mentioned in my last lecture; we will use this classical definition of the angular momentum operator; we will start with the classical L is equal to r cross p, so that L x is equal to y p z minus z p y and similarly, L y is equal to x p z minus no I am sorry it should be y, so this will be; L y will be z p x minus x p z they have to be at the cyclic order and L z will be x p y minus y p x.

However, we must be very careful that x and p x do not commute x p x minus p x x is equal to i h cross. On the other hand, x and y commute x y minus y x is 0 or x p y also commute minus p y x is equal to 0. With this, we will now commute; we will now calculate, the commutation relation between L x and L y.

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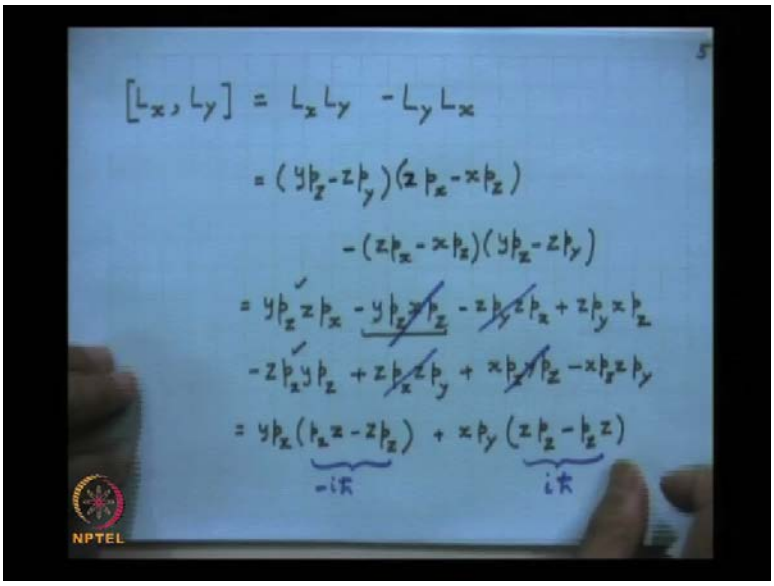

So, let me calculate, the commutation between the commutator L x L y. So, this is equal to, we have do this little carefully, putting things in proper order. So, L x L y minus L y L x, and L x is equal to y p z minus z p y minus sorry multiplied by L y; L y is z p x minus x p z and the same thing, in the reverse order in the reverse order that is, minus z p x minus x p z, and then y p z minus z p y. So, let me write out the terms fully, there will be four terms here; and four terms here and we must write them in proper order, and let me first write the first term y p z z p x minus y p z x p z minus z p y z p x plus z p y x p z and this will be minus z p x y p z and then, this times; this will be plus z p x z p y and then, this, this, that is plus x p z y p z and the last term will be minus x p z z p y.

So you will see that, for example, this term; all of them commute with each other all of them; all of the terms commute with each other. So, x I can put them in any order; I can take the x here, so x y p z p z x y p z p z. So, this term will cancel out with this term. Similarly, here, I can take z here; z square p x p y; this is z square p x p y. So, this term will cancel out with this term, but here, we have to be careful; we see that, here you have p z z and z p z so we have to be very careful, so we can take the these two terms outside.

So, we will have y p x p z z y p x minus z p z so this is, this term and this term. Similarly, here, if i take out x p y so then, this will be z p z minus p z z. So, this we; as we all know, this is equal to i h cross and this is the opposite minus sign of that, so this is minus i h cross, so we will have.

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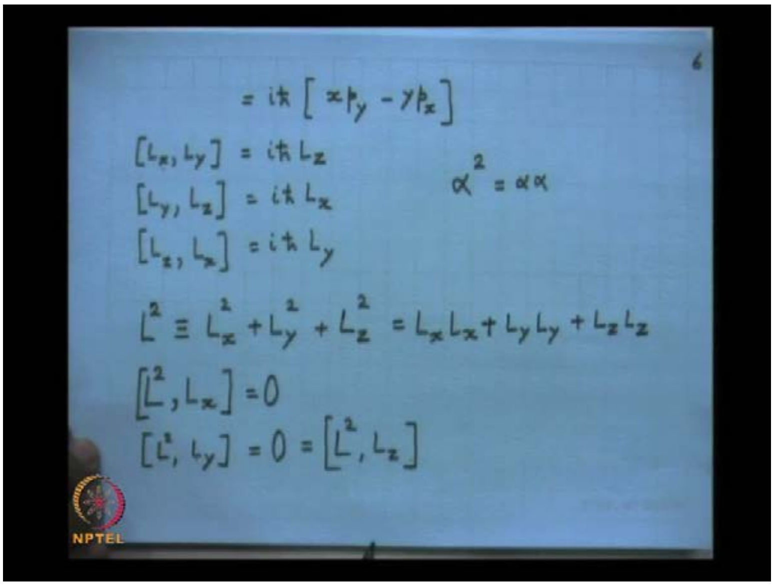

If I take i h cross outside, so we will have i h cross x p y minus y p x. So, this is nothing, but L z, so we get i h cross L z. So the left hand side, as we had written before was L x comma L y so that, L x and L y the x component of the angular momentum and the y component of the angular momentum do not commute and the commutation relation is and the commutation relation is L x L y is equal to i h cross L z.

Similarly, you can write in cyclic order, that L y L z y z x so i h cross L x and then, this will be z x is equal to i h cross L y. These are the important commutation relations and which, we will take as the starting point of our analysis. We will not worry about, how we have got this commutation relation? We will start with this and build up the operator algebra, to find out the Eigen values and Eigen functions of the L square and the L z operator. So, L x easy way to remember is, this is x y z and then y z x they are all in cyclic order z x y x y z y z x z x y so these are the three very important commutation relation.

As I had told you earlier, if I have an operator alpha; the square of the operator is defined as alpha times alpha. So, the operator L square is defined as L x square plus L y square plus L z squared. So, what do I mean by L x squared? That means L x L x plus L y L y plus L z L z. I will show, that L square commutes with L x and because there is nothing secret about x, so L square also commutes with L y and L square also commutes with L z. So although, L x L y and L z do not commute with each other, but each one of them commutes with L squared.

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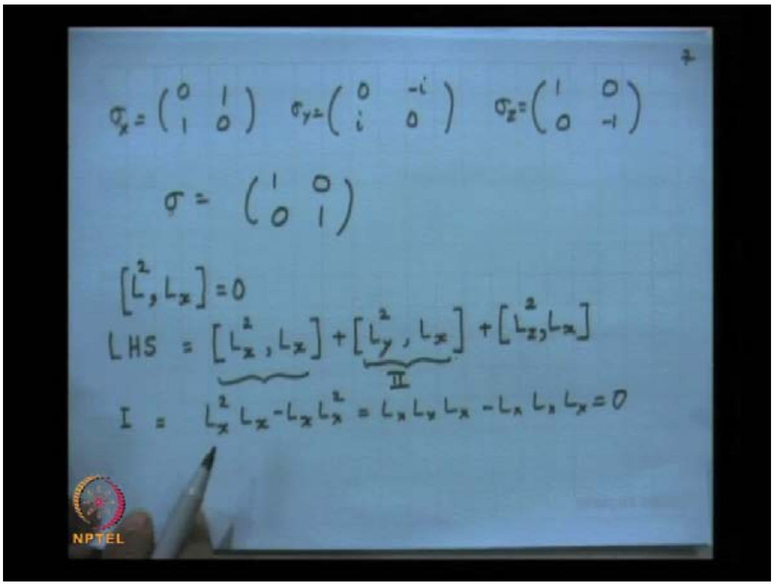

So, it was something similar to that we consider the matrices that 1 0 1 1 0 0 minus i i 0; these are actually the pauli spin matrices 1 0 0 minus, these three matrices do not commute with each other. So, these are denoted by; you must remember them sigma x, sigma y, and sigma z. However, they each one of them commute with the sigma matrix, which is 1 0 0 1. Sigma x commutes with sigma, sigma y commutes with sigma, sigma z commutes with sigma.

So, let me try to show that L square commutes with L x. I will show that this is equal to 0. Now, L square is equal to say let me consider the left hand side; so left hand side is equal to L square is equal to L x square L x plus L y square plus L z square.

So, L z y square L x plus L z square L x. Now, this is obviously 0, because the first term is L x square L x minus L x L x squared. So, if you write it out in detail, so this is L x L x L x minus L x L x L x, so this is 0. Let me calculate the second term, the this is the second term, and similarly, I will give you, one can calculate the third term, and one can show that, the addition of these two terms are independently not 0, but the addition of these two terms will be 0.

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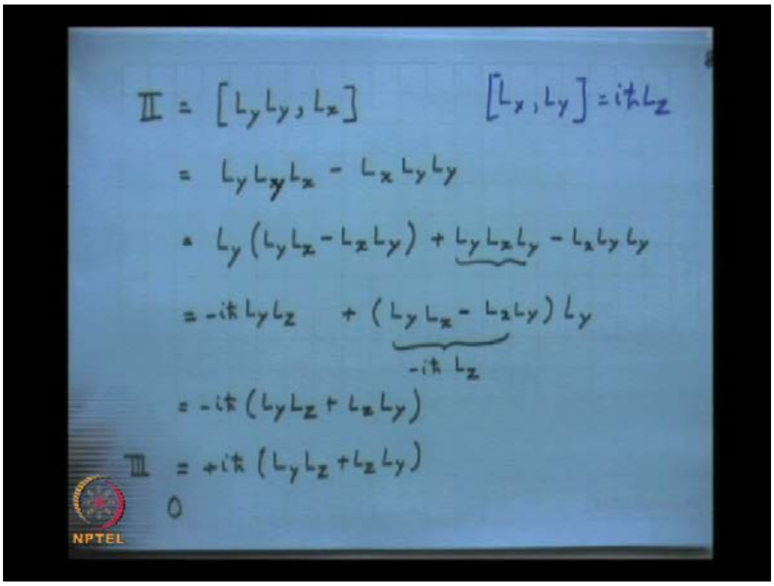

So, let me consider calculate the second term, which is equal to L y square L y square so that is L y L y comma L x. So, this is equal to L y L y L x minus L x L y this is y this is y, L x L y L y. So, what we do is? That I add and subtract that is L y, I write this down as L y L x minus L x L y. So, I added a term minus L y L x L y, so I add this term L y L x L y minus this term; I hope you have understand L x L y L y; I have added this term and subtracted this term.

So, this will be from the previous analysis, we had shown that, L x. In fact, we had shown little earlier, that L x comma L y is equal to i h cross L z. So, this is L x L y minus L y L x so this is, the negative sign of that therefore, this will be minus i h cross L y L z. We must write them in proper order, that is very necessary and then, I take L y on the right side so I get, L y L x minus L x L y multiplied by L y. So this is also, equal to minus i h cross L z so I get, minus i h cross L y L z plus L z L y.

I leave it as an exercise for you, to calculate the third term; which is L z square L x and carry out the same analysis and the third term will come out to be the opposite of this, that is plus i h cross L y L z plus L z L y and the total will be 0. So that, we have established that, L square commutes with L x.

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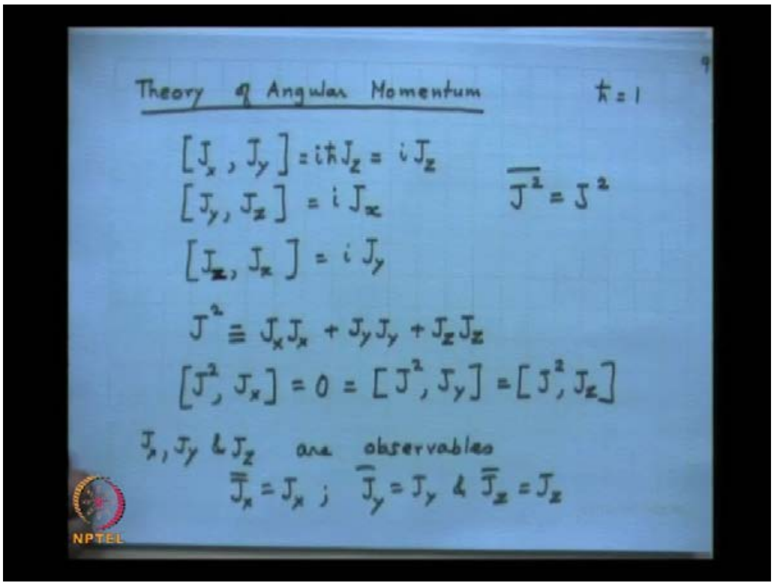

Similarly, so we had three commutation relations, that we have derived and so the starting point of the theory of angular momentum start with the commutation relation. We had earlier used differential operator algebra to write down the operator representation of the angular momentum operators. We had written that, L z operator was equal to minus h cross delta by delta phi; we do not worry anything; we just use the commutation relation and with that sense, it is slightly different from what we did earlier? And indeed, we will get slightly different results and therefore, the convention is that since, we are depending only on the commutation relations; we use this symbol J representing the angular momentum operator.

So, we replace L L x L y and L z by J x J y and J z and we start, with the theory of angular momentum by saying that, J x J y and we use the system of units, in which h cross is 1 so we say, J x comma is equal to i h cross J z but we assume h cross equal to 1 so i J z. Similarly, J y comma J z is equal to i J x and J z J x is equal to i J y. Further, we define J square J x J y J z; further we define, the operator J square as J x square plus J y square plus J z squared. That is J x J x plus J y J y plus J z J z; that is J x square plus J y square plus J z square and we assume and as we have shown that, J square commutes with J x; commutes with J y; and commutes with J z.

Further; since, x J x represents the x component of the angular momentum. It is an observable therefore, it must be represented by a real by a hermatian operator; by a real operator therefore, because J x J y and J z J x J y and J z, because these are observables, J x bar is equal to J x that is, they are all real operators. J y bar is equal to J y and J z bar is equal to J z and of course, J square bar is J squared.

So, J x J y and J z and J square are real operators. So, we assume just what is written on this paper and nothing else. Three things, we therefore, assume first that J x and J y do not commute and these are the commutation relations; we have got it in a certain way, but now, this is our assumption; this is what we did even in the harmonic oscillator problem?

We wrote down the hamitonian and then, we assume only the commutation relation between x and p. Similarly, here, we assume, the commutation relation between J x J y J y J z and J z J x and that J square J x J square commutes this and the third assumption is, that since, they are all observables; they are all represented by real linear operators and therefore, since J square commutes with J x. I can have a complete set of simultaneous Eigen kets of J square and J x or of J square and J y or of J square and J z.

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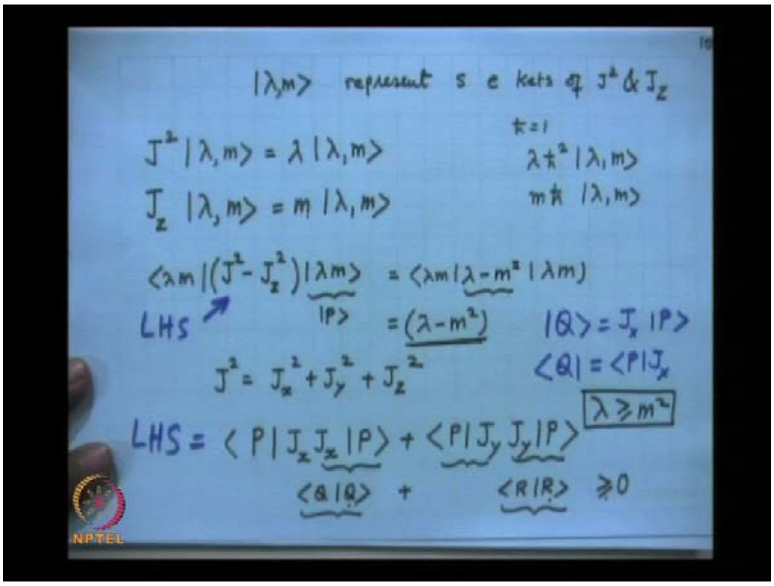

Let me assume, that ket lambda m is represent simultaneous Eigen kets of J square and J z. Belonging to the Eigen values, lambda and m respectively. So, what I am trying to say is that, J square actually, it is lambda comma m is not lambda times m; these are two numbers.

So, ket lambda m is a simultaneous Eigen ket, of the operators J square and of the operator J z. Let the and the Eigen values and the corresponding Eigen values are denoted by lambda m and J z operating on lambda m is equal to m lambda m. Lambda m are numbers actually, if I, we have; we must remember that; we have taken; we have assumed h cross is equal to 1. So actually, what we are saying is? This is lambda on the right hand side is actually, lambda h crosses square lambda m the h crosses square is hidden and this is m h cross lambda comma m.

So, I want to solve this Eigen value equation; I want to solve this Eigen value equation that is, we want to obtain simultaneous Eigen kets. A complete set of simultaneous Eigen kets, corresponding to the operators J square and J z. corresponding to the operators J square and J z. First, I consider this operator J square minus J z squared; J z square is J z J z operating on lambda m and this is lambda m. So, these are orthonormal kets, because the belong to real hermitian operators; they are Eigen kets of the hermitian operator. So please see, J square operating on ket lambda m; this is equal to the left hand side remains the same, lambda and J z operating on lambda m is m.

If, I operate again by J z. It is m operating m J z this thing, so m squared; so this becomes m lambda minus m square ket lambda m. This is just a number, because these are numbers; these are just Eigen value; this is the operator, so this becomes and if, these are orthonormal kets then, this becomes lambda minus m squared. Now, let just for the sake of convenience, I represent this by ket p.

So, J squared; as we remember is equal to J x square plus J y square plus J z squared. So, J square minus J z square is just, J x square plus J y squared, so this left had side so the left hand side this quantity; left hand side is equal to bra P J x square J x J x plus J y J y. So, I write this as ket P; I hope this is clear plus ket p bra p J y J y.

Let this be ket Q J x ket Q; J ket Q is defined as so ket Q is defined as J x ket p. So, the conjugate imaginary of that is bra Q; this is bra p J x bar, but J x bar is J x, because it is an observable. So, bra p J x is just bra Q, so this is bra Q plus J y ket Q. Let us suppose, this is ket R then, this is bra R so my left hand side is bra Q ket Q plus bra R ket R.

So, the left hand side is greater than or equal to 0; equal to 0 if bra ket Q is a null ket and if, ket R is a null ket. These are two positive definite quantities or can be zero also, unless each 1 of them 0, the sum is always positive. So, we get the remarkable result, that lambda must be greater than equal to m squared. So we find that, from just by operator algebra that, this Eigen value of J square lambda must be greater than or equal to m squared.

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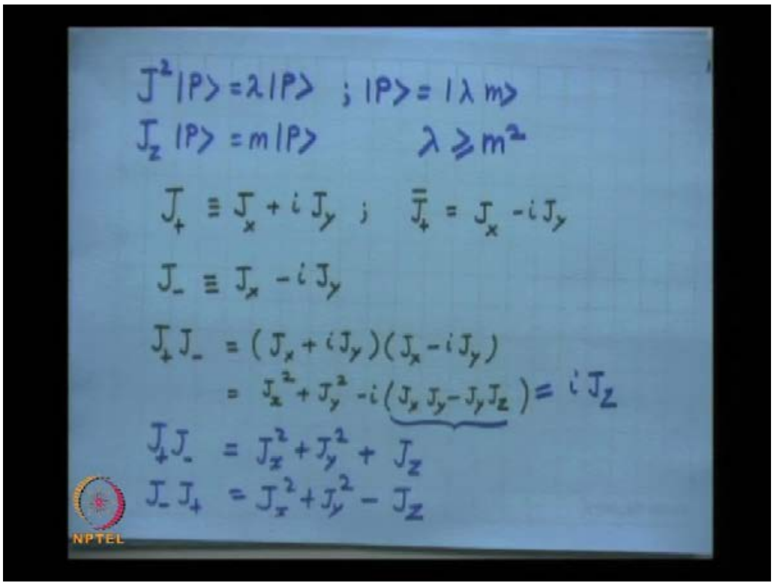

Now, as we had done; so we had the very important result that first of all, we write down that let ket p; which is ket lambda m is a simultaneous Eigen ket of the operator J square and J z. So, we will ket p is equal to ket lambda m then, we had derived the very important relation, that lambda must be greater than m squared. Now, we define in the harmonic oscillator problem, we had defined two operators a and a bar, which we had denoted by; I mean represented by; what are known as annihilation and creation operator? Here, we defined two operators, which will which are which will be called ladder operators.

The two operators J plus, which is equal to defined to be equal to; when I have 3 equal to sign, that means defined to be equal to J x plus i J y and the adjoint of this is J x bar, which is J x minus i J y. So, we defined another operator J minus which is the adjoint of this operator; which is J x minus i J y. So for example, J plus J minus is equal to, you have to be careful J x plus i J y into J x minus i J y. Now on the first side, you will say this is a plus i b and this is a minus i b so if, i multiply these 2 out; I will get a square plus b squared, but that is not correct; you have to be very careful in multiplying in proper order.

So, this will be J x J x; which is J x square then, J i times i is minus 1 minus minus plus J y square and then, minus i if, I take outside; it will be J x J y minus J y J z, because the operators do not commute. So, this is the commutator of J x comma J y; so this, this, this, we had shown that, this is equal to this part; was equal to i h cross actually, so i J z. So, i times i is minus 1; so this becomes plus J z; so this becomes J plus J minus becomes J x square plus J y square plus J z.

I leave it, as an exercise for you to show that, you take it in the reverse order; that is J minus J plus will be equal to J x square plus J y squared, but here, it will be a minus sign. I will leave it as a very simple exercise for them.

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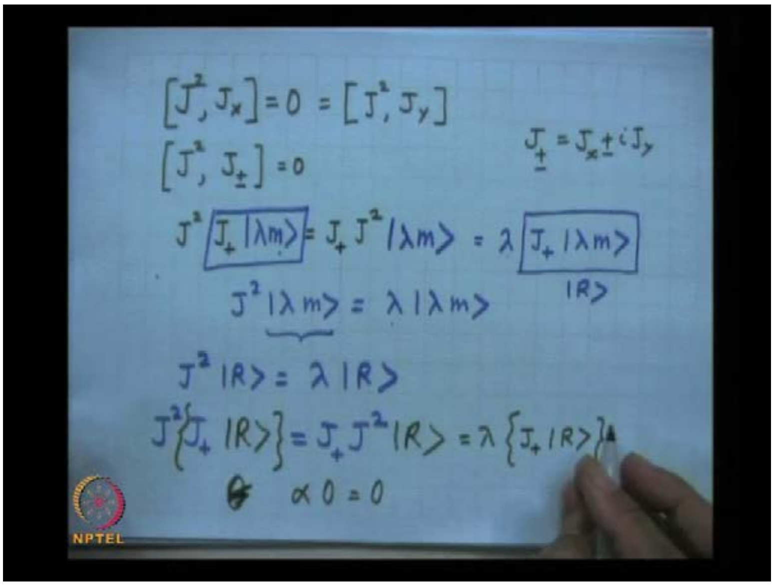

Now, let me consider the commutation relation. We know that, J square commutes with J x and J square also commutes with J y; so J square will commute with J plus. J plus is if, you remember that, we had defined the operator J plus as J x plus i y and J minus as J x minus i J y. So, J square will commute with J plus and J minus.

So please see this, J square J plus is equal to 0 or J square J minus is equal to 0. Therefore, J square J plus leaves some space here, is equal to J plus J squared. I multiply; I operate this on the Eigen ket lambda m; I operate this on lambda m and you see, J square operating on this thing is; this is the Eigen ket. J square operating on ket lambda m is equal to lambda operating on this thing. So, this is an Eigen ket, so this becomes lambda and lambda I can take anywhere, so lambda J plus ket lambda m. Thus if, what we had started out with? J square ket lambda m is equal to lambda ket lambda m.

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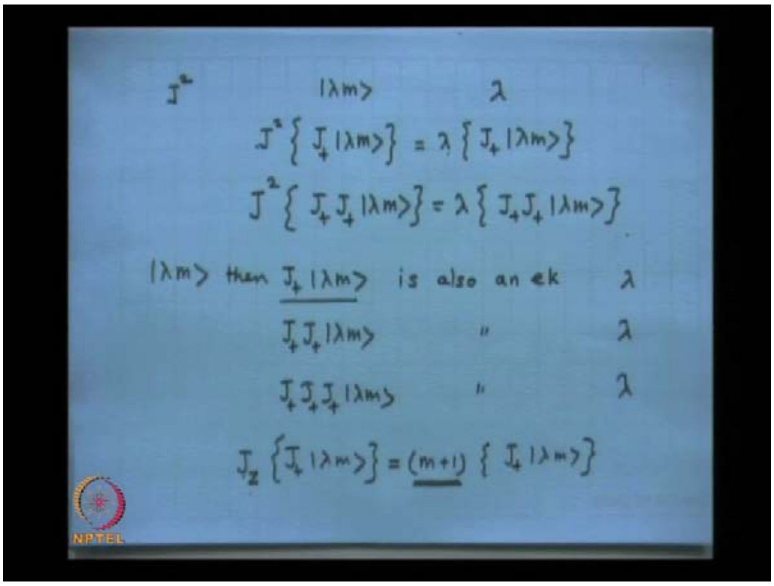

Thus, if ket lambda m is an Eigen ket of the operator J squared, belonging to the Eigen value lambda then, J plus ket lambda m; because J plus ket lambda m is also an Eigen ket of the operator J squared, belonging to the same Eigen value lambda. So, we must expect the degeneracies; we must expect degeneracies. Similarly, let this be, I denote by ket R. Let us suppose, so I have J square ket R is equal to lambda ket R.

Now, I again do like this. So, J square J plus is equal to J plus J squared; I operate this on ket R. Now, J square ket R is lambda; this is lambda J plus ket R, so I put this inside the bracket I put this inside the bracket so then, J plus ket R is also an Eigen ket. Provided, it does not become a null ket; if it becomes a null ket then, of course, it is a trivial solution, because null ket is always an Eigen ket and anything can be an Eigen value.

I have; as I have mentioned earlier; if, I have any operator alpha, operating on null ket is always 0. So, this is a trivial solution, so if, you have therefore, we conclude that, if ket lambda m is an Eigen ket of J square belonging to the Eigen value lambda then, J plus ket lambda m is also an Eigen ket of the operated J square belonging to the same Eigen value lambda and similarly, provided this is not a null ket. Similarly, J square operating on J plus J plus ket lambda m will be again, lambda J plus J plus ket lambda m.

So, we must expect degeneracies. So, if ket lambda m is an Eigen ket then, J plus ket lambda m is also an Eigen ket belonging to the same Eigen value lambda. Provided it is not a null ket. Then, J plus J plus ket lambda m is also an Eigen ket belonging to the same Eigen value lambda provided this is not a null ket and then, J plus J plus J plus ket lambda m is also an Eigen ket belonging to the same Eigen value lambda.

In the next lecture, we will show that, J plus ket lambda m is also an Eigen ket of the operator J z, but now belonging to the Eigen value m plus 1 J plus 1 ket lambda m. So therefore, we will show that J plus ket lambda m is a simultaneous Eigen ket of the operator J square and J z. For the operator J squared, it belongs to the same Eigen value lambda, but for the operator J z. It belongs to the Eigen value m plus 1. Thank you.