**Basic Quantum Mechanics**

**Prof. Ajoy Ghatak**

**Department of Physics**

**Indian Institute of Technology, Delhi**

**Module No. # 07**

**Bra-Ket Algebra and Linear Harmonic Oscillator - II**

**Lecture No. # 04**

# Linear Harmonic Oscillator: Coherent State and Relationship with the Classical Oscillator

In our last lecture, we were discussing the eigen kets of the operator a in the linear harmonic oscillator problem.

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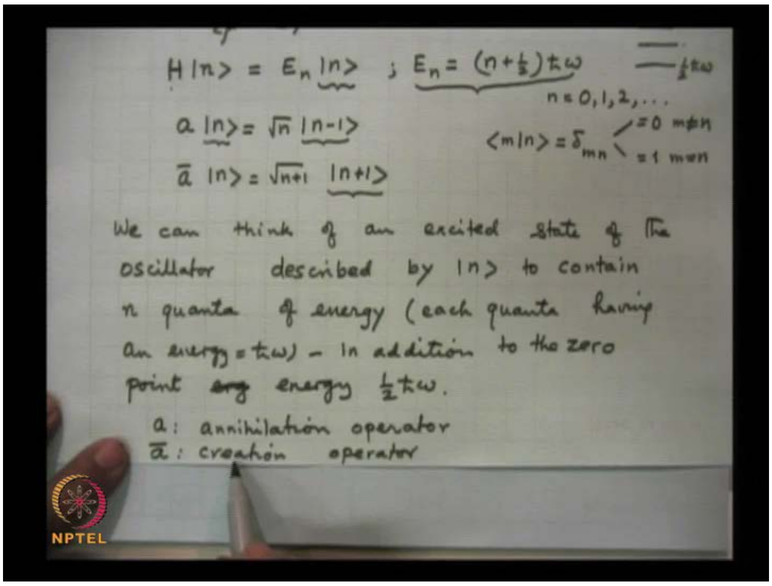


For the linear harmonic oscillator, the hamiltonian, as we all know, was equal to p square by 2 mu plus half mu omega square, x square, and we wanted to solve the eigen value equation H, and we found after using operator algebra, the solution of the eigen value, the Schrodinger equation is H n is equal to E n ket n, where E n is equal to n plus half h, cross omega.

These are the orthonormal eigen kets of the operator H, belonging to the eigen value E of n, where E n is equal to n plus half h cross omega and n is equal to 0, 1, 2, 3. So, the eigen value spectrum that we get is the same as we had obtained, by solving the Schrodinger equation. We had defined two operators a, and a bar, and we had shown that a ket n is equal to square root of n, n minus 1, and a bar ket n is equal to square root of n plus 1, operating on ket n minus 1.

Since, these are the ket n, are the eigen kets of the operator of the hermitian operator H, they form an orthonormal set of vectors that is n ket n is equal to delta m n, that is this is equal to 0, if m is not equal to n, and is equal to 1 if m is equal to n. Now, we can think of an excited state of the oscillator. Let me write it down. We can, because this is important, we can think of an excited state of the operator, excited state of the oscillator, described by ket n, to contain n quanta of energy, each quanta having energy equal to h cross omega.

Of course, this is in addition to the zero point energy; energy which is half h cross omega. You see, each state, you have this harmonic oscillator state, the ground state, which corresponds to n equal to 0 or no quanta. This is the zero point energy, half h cross omega, this is 3 by 2 h cross omega, 5 by 2 h cross omega and then 7 by 2 h cross omega. So, this has 1 quanta of energy h cross omega. This has 2 quanta of energy, 3 quanta of energy and so therefore, a operating on the state described by the ket n transforms it to a state n minus 1.

So, therefore, the operator a, is referred to as the annihilation operator (Refer Slide Time: 04:58). In the quantum theory of radiation, one has to use these operators extensively. This is referred to as the annihilation operator and similarly, a bar, when operating on a state n, sorry this has to be n plus 1(Refer Slide Time: 05:22). So, this goes to the state n plus 1. So, it creates a quantum and so this is known as a creation operator.

So, the operator a and a bar are known as the annihilation operator, and a bar is known as the creation operator, and we use this annihilation and creation operators in the quantum theory of radiation, quantum theory of phonons that is quantum theory of lattice vibrations. So, these are extensively used there.

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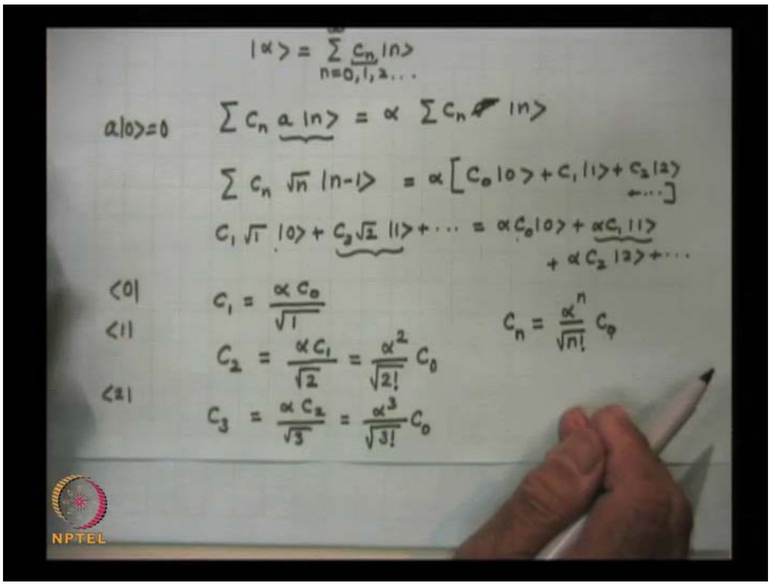


Now, what we did in our last lecture is we had started is that we wanted to solve this eigen value equation, a ket alpha is equal to alpha ket alpha, as we had mentioned in our last lecture. Here, a is a non hermitian operator; it is not a real operator. We started out and we wanted to solve this equation, the ket n, the eigen kets of the operator H form a complete set of functions, complete set of orthonormal functions.

So, any ket in the space can be expanded as eigen kets of the hamiltonian C n ket n, where n goes from 0 to infinity like from 0, 1, 2, 3 till infinity. So, let me substitute it there. Hence, we get, a operating on ket alpha is summation C n a ket n is equal to alpha. Here, alpha is a number, summation C n ket sorry ket n (Refer Slide Time: 07:30). We now know that a ket n is just C n under root of n, n minus 1. Of course, a ket 0 is a null ket, so that the first term does not exist, does become 0 and on the right hand side, this becomes alpha, C 0 ket 0 plus C 1 ket 1 plus C 2 ket 2, etcetera.

On the left hand side, we have the first term is 0, because a ket 0 is 0, so it is C 1 under root of 1 ket 0 plus C 2 under root of 2 ket 1 and so on. This must be equal to alpha times C 0 ket 0 plus alpha times C 1 ket 1 plus alpha time C 2 ket 2, etcetera. So, we must compare the coefficients. How do we understand them? Let me multiply by bra 0, so all of the term will vanish, except the first term here and first term here (Refer Slide Time: 08:59), so that C 1 must be equal to alpha C 0 by square root of 1.

Then we multiply by bra 1, all terms will cancel out, except for this term and this term. So, we will have C 2 under root of 2 is equal to alpha C 1 by under root of 2. I substitute C 1 here, so this becomes alpha square by square root of 1 into 2 that is 2 factorial into C 0. Now, if I multiply by bra 2, then I will get C 3; under root of 3, will become alpha C 2 by under root of 3 and so this becomes alpha cube 3 into 2 factorial is 3 factorial C 0. I leave it as an exercise for you to show that this in just trivial C n, the n-eth term will become alpha to the power of n square root of n factorial C 0. So, I have found out all C n, in terms of C 0. Therefore, we have we eigen ket.

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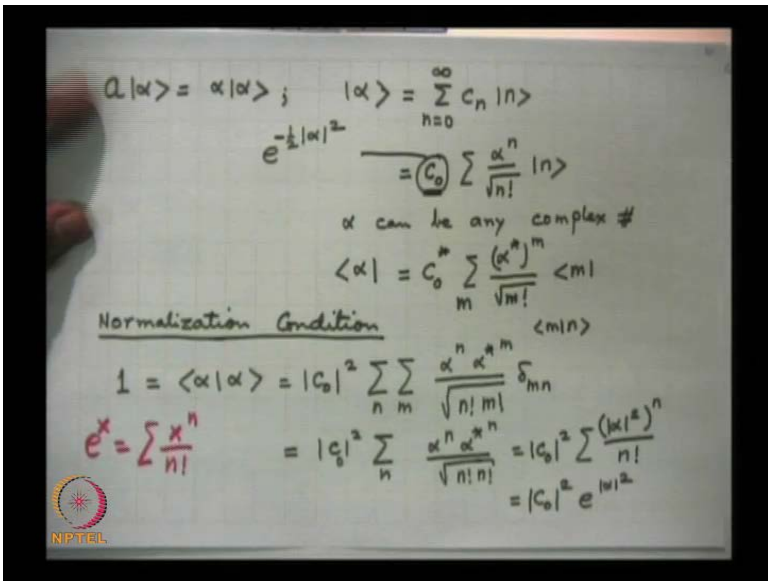


Our objective was to solve a ket alpha is equal to alpha ket alpha, and we assumed alpha to be expanded in terms of the orthonormal eigen kets of the operator a n is equal to 0 to infinity. We found that C n was equal to C 0, summation alpha to the power of n by square root of n factorial ket n. Now, this is valid for all the alpha; alpha can be real, alpha can be imaginary and alpha can be complex; so, all eigen values. You can see that alpha can take complex values and that is because the operator a, is not a real operator, a and a bar are not equal.

We have the complete set of complex numbers that could be its eigen value. So, alpha can be any complex number. The number sign, we usually denote by this. I still do not know what the value of C 0 is and so I try to normalize this ket. So, I had bra alpha, if I take the conjugate imaginary of that. So, this will be C 0 star, alpha star, and I will multiply this way. So, I write this as to the power of m, because n and m are bra m, because these are dummy variables. So, I can write p, I can write m or I can write one, anything that we feel like.

I have the normalization condition; the normalization condition will lead to 1 is equal to bra alpha ket alpha equal to C 0 times C 0 is mod C 0 square into double summation for n and m, alpha to the power of n, alpha star to the power of m, divided by square root of n factorial, m factorial. Here, bra m will become bra m ket n and that we know is the Kronecker delta function which is delta m n (Refer Slide Time: 13:53).

If I first sum over m, this is sum over. Then only the m equal to n term will survive. So, if I sum over n, only the m equal to n term will survive. So, for each value of n, C 0 square, I get n, alpha to the power of n, alpha star to the power of n, divided by, n and m are becoming equal, so n factorial times n factorial. Sorry, this is C 0 square; C 0 square summation, alpha mod square to the power of n, divided by n factorial. As you know that e to the power of x is equal to x to the power of n by n factorial summed over all values of n where n equal to 0 1 2 3.

This becomes equal to C 0 mod square, e to the power of mod alpha square. Therefore, we get the result. Sorry, if you write this, so we obtain C 0 within a phase factor, which is equal to e to the power of minus half, alpha mod square. Therefore, if I substitute the value of C 0 here, this is C 0 and I just prove that this is equal to minus half e alpha square.

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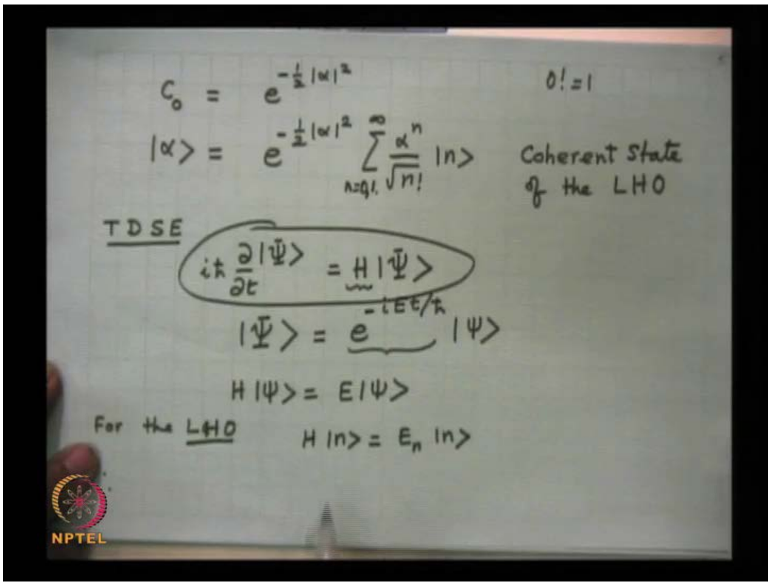


So, we obtain for the coherent state and this is a very important result, which I would like all of you to remember, which is e to the power of minus half alpha square and then alpha to the power of n square root of n factorial ket n, n going from 0, 1, 2, 3 and so on. Zero factorial is 1, as I am sure all of you know that zero factorial is equal to 1.

This state is known and is usually refer to as a coherent state and I will tell you the importance of this coherent state of the linear harmonic oscillator. Now, I consider the time dependent Schrodinger equation. So, I have i h cross delta by delta t psi is equal to H psi. If I assume a solution as we did, the solution that separates the time part, e to the power of minus i E t by h cross ket small psi, let us suppose, where this does not depend on time. Then one can easily show that H psi will be equal to e psi (Refer Slide Time: 17:56).

For any Hamiltonian, which does not depend on time, one can use the method of separation of variables, taking the time part out, substitute back in this equation and obtain this eigen value equation. So, for the linear harmonic oscillator problem, I have solved this equation, we know the solution of this equation and those are H ket n is equal to E n ket n. Therefore, the most general solution is and let me write it down here. Therefore, the most general solution of the time dependent Schrodinger equation will be the sum over all the superposed state.

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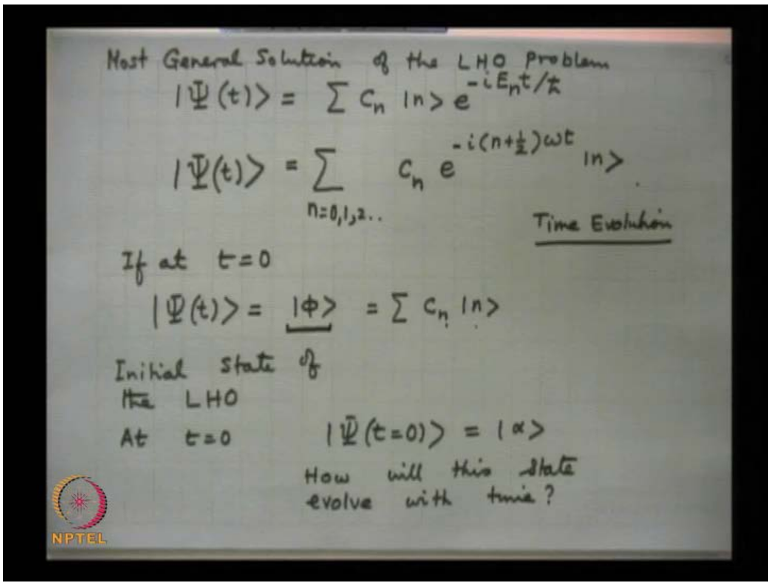


Therefore, the most general solution will be psi of t will be equal to summation C n ket n, e to the power of minus i E n t by h cross. Just as we did, while solving the Schrodinger equation, so this is the most general solution. Let me write it down. This is the most general solution of the linear harmonic oscillator problem. I know the value of E n. So, I substitute it here, so summation n equal to 0, 1, 2, 3, so on. C of n, this is n plus half h cross omega, h cross h cross cancels out, so I get minus i n plus half omega t ket n.

This is the time evolution. This above equation (Refer Slide Time: 20:25) represents the time evolution of a general state; time evolution. So, if at t equal to 0, psi of t is something like phi. Then what we should do is we should expand this, in terms of the orthonormal, in terms of the normal modes of the system, in terms of the orthonormal kets of the harmonic oscillator. So, let at time t equal to 0, psi of t is equal to phi, that is this describes the initial state of the of the linear harmonic oscillator. So, this is how we solved a variety of problems of the initial state of the linear harmonic oscillator.

Let us suppose it describe by the state by the ket phi. Then what we should do is we should first expand it and then if I am asked the question that how will this state evolve with time. The answer is simple. Each term, I multiply by e to the power of minus n plus half omega t ket n, and that will be the time evolution of the state. So, let us suppose that at t equal to 0, psi at t equal to 0 is the coherent state. Let us suppose, it is the coherent state, one of the eigen states of the operator a, and I ask that how will this state evolve with time? That is my question and there will be answer in a moment.

So, initially the linear harmonic oscillator is in the coherent state described by the ket alpha and I want to now ask the question as to how it will evolve with time.

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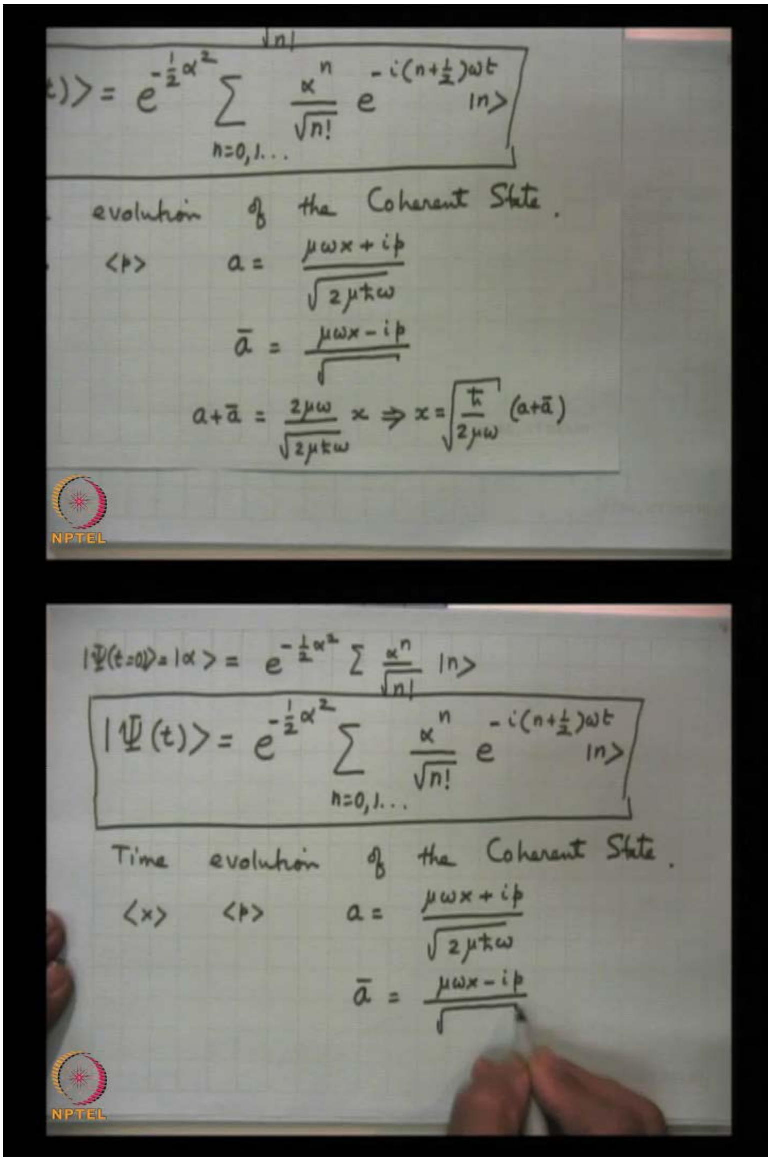


Now, the state alpha ket alpha, we have already shown that this is equal to e to the power of minus half, summation alpha to the power of n by n factorial into ket n; let me assume alpha to be real just for the sake of simplicity. So, initially this is at psi t equal to 0; so this is at ket psi t equal to 0. This is the initial state of the oscillator. How will it evolve with time?

So, psi of t, this will be equal to e to the power of minus half alpha square, summation and this goes from n equal to 0 1 2 3, alpha to the power of n, divided by square root of n factorial, e to the power of minus i n plus half omega t ket n. This state, this wave function defines (Refer Slide Time: 24:33) the time evolution of the coherent state. Now, why I am stressing this? So, what we will do is so this is my time evolution of the coherent state. Now, let me first find out the expectation value of x (Refer Slide Time: 25:11), the position coordinate and the expectation value of the x component of the momentum.

You recall that the operator a, and a bar, they have defined as a is equal to mu omega x plus i p divided by under root of 2 mu h cross omega. And a bar was mu omega x minus i p, divided by the same under. So, in order to obtain an expression we add these two. So, we get a plus a bar, this is equal to 2 mu omega divided by 2 mu h cross omega x. So, this tells us that x is equal to h cross by 2 mu omega under the root a plus a bar.

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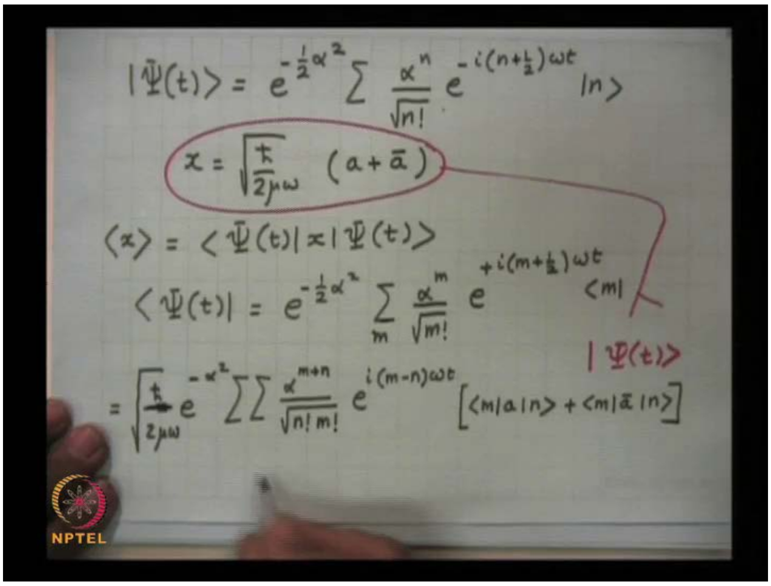


Let me rewrite this again. First, I rewrite the coherent state. The time evolution of the coherent state is psi of t is equal to e to the power of minus half alpha square, summation of x to the power of n by n factorial multiply e to the power of minus i into n plus half omega t ket n, summation is over this thing. Then x is equal to under root of h cross by 2 mu omega multiply by a plus a bar or a bar plus a. It does not really matter.

Now, I want to find the expectation value of x. Therefore, this is bra psi t, x psi t. So, first I have to calculate bra psi t, remember that alpha is real and I have assumed alpha to be real. So, I write bra psi t, the conjugate imaginary of ket psi t, so e to the power of minus half alpha square and let us do this little carefully. Now, since I will be using both n and m, so these two summations will get mixed up. So, let me write down the summation over m. So, alpha to the power of m square root of m factorial e to the power, the complex conjugate of that is e to the power of plus i m plus half omega t and then bra m.

Now, I put this thing here now here (Refer Slide Time: 28:47) and then operate this on psi t. It is really a slightly big expression, but let me try to calculate this. So, e to the power of minus half alpha square multiply by e to the power of minus alpha square is e to the power of minus alpha square and then there is a double sum and it is because of this double sum that I took the dummy index as n here and as m here. So, we will have alpha to the power of m plus n, square root of n factorial and m factorial. This will be m minus n, both the half will cancel out, e to the power of m minus n omega t. Actually, I can take this h cross by 2 omega outside.

Let me quickly put this under root of h cross by 2 mu omega. I can put it outside and this will be in two terms. Please, see this bra m a ket n; a plus a bar plus, bra m a bar ket n. These are known as the matrix element, the m n-eth matrix element of the operator a, so we know that a ket or let me let me work this out separately.

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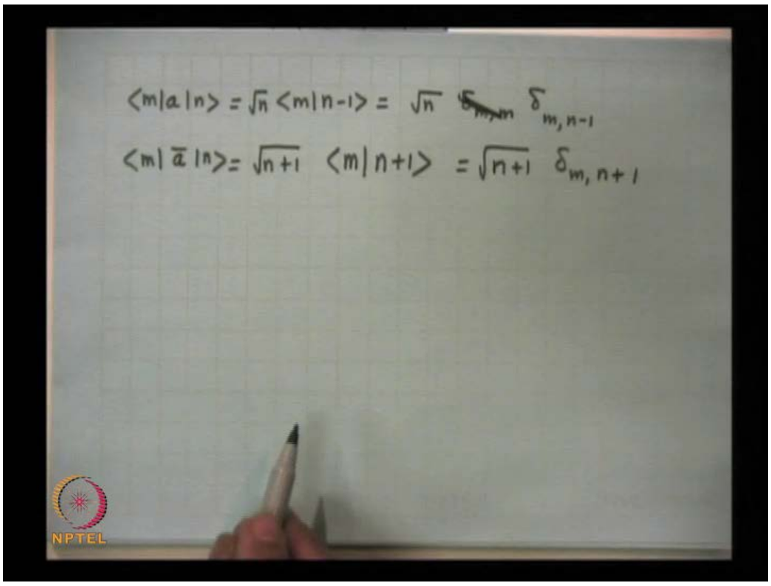


I want to first calculate m a n. This is very simple. So, this will be m square root of n which I take outside, n minus 1 which is equal to square root of n delta m comma m minus 1, sorry, this is delta of m comma n minus 1.

Similarly, you have m a bar n, so this will be square root of n plus 1, m n plus 1, so this will be square root of n plus 1 delta of m n plus 1. Now, I substitute these two expressions here.

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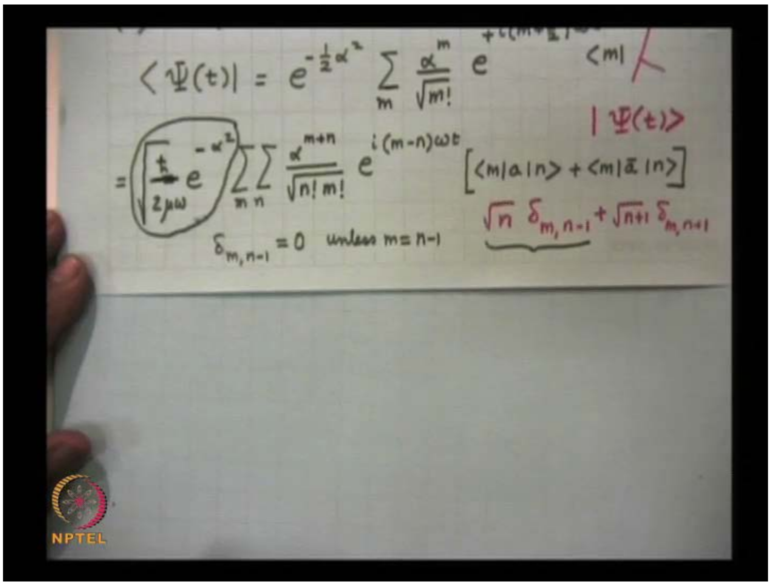

Please see this will be square root of n delta of n comma n minus 1, and this term will be plus square root of n plus 1, delta of m comma n plus 1. Now, please see, if I take just consider the first term, then in this sum over m and n, I sum over m and only the m equal to n minus 1 will survive, because delta m comma n minus 1 is 0, unless m is equal to n minus 1. In fact, this is equal to 1 when m is equal to n minus 1. So, the m equal to n minus 1 term will survive.

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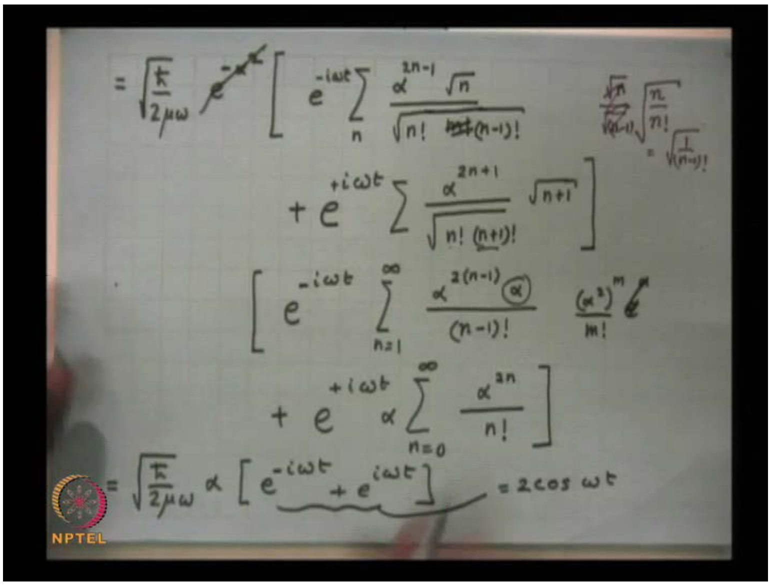

Therefore, if I take the first term here we will obtain we will obtain h cross, let me write it down like this. So, h cross by 2 mu omega, e to the power of minus alpha square, I sum over m first, so only the m equal to n minus 1 will survive. So, this term if I take outside, so this will be summation alpha to the power of n minus 1 plus n that is 2 n minus 1.

And this will be multiplied by square root of n, divided by square root of n factorial, m factorial, m minus n will be n minus 1 minus n. So, this will be minus i omega t and that is it. Let me first evaluate the second term, which is this is square root of n plus 1, delta of m n plus 1. So, only the m equal to n plus 1 term survives. So, that this will be plus sign here and this is a sum over n, this will be e to the power of, sorry this m factorial (Refer Slide Time: 35:07), this m is n minus 1, so this will be n minus 1 factorial.

And this term will be e to the power of m, which is n plus 1, so n plus 1 minus n is 1. So, this is plus i omega t, sum over, m is n plus 1. So, alpha 2 n plus 1, divided by square root of n factorial and m factorial is n plus 1 factorial, multiplied by square root of n plus

1. Now, in the first expression, you see square root of n, divided by square root of n minus 1 factorial, I am sorry this n over n factorial, square root of that, so this is equal to 1 over n minus 1 factorial, because this is 1 into 2 into 3 into n minus 1 into n. So, n and n cancels out, so this becomes n minus 1 factorial and similarly, this will be, this term will be, this will cancel out with this, and will give you only n factorial.

Therefore, I close the bracket here and I obtain e to the power of minus i omega t and let me write it down as alpha to 2 n minus 1. So, this is 2 n minus 2, but this is 2 n minus 1, so must have an alpha factor here, divided by n minus 1 factorial times n minus 1 factorial. So, that is just n minus 1 factorial and this goes from n equal to 1 to infinity, because n equal to 0 term is 0 plus the second term will be e to the power of plus i omega t, summation n equal to 0 to infinity. If I take the alpha outside, so I get alpha here, alpha to the power of 2 n alpha square raise to the power of n, and divided by n factorial and n factorial, so just n factorial.

If I take this alpha outside, so this is just alpha, sorry, alpha square raise to the power of say m divided by m factorial (Refer Slide Time: 38:24). So, this is all e to the power of alpha square, this is also e to the power of alpha square. So, e to the power of alpha square will cancel out with e to the power of minus alpha square. Both will have alpha outside. So, this will become h cross by 2 mu omega alpha into e to the power of minus i omega t plus e to the power of i omega t. So, this becomes 2 cos omega t and this is equal to 2 cos omega t.

You know that cos theta is equal to e to the power of i theta plus e to the power of minus i theta divided by 2, so this is the expression for cos omega t. This was the expectation value of x.

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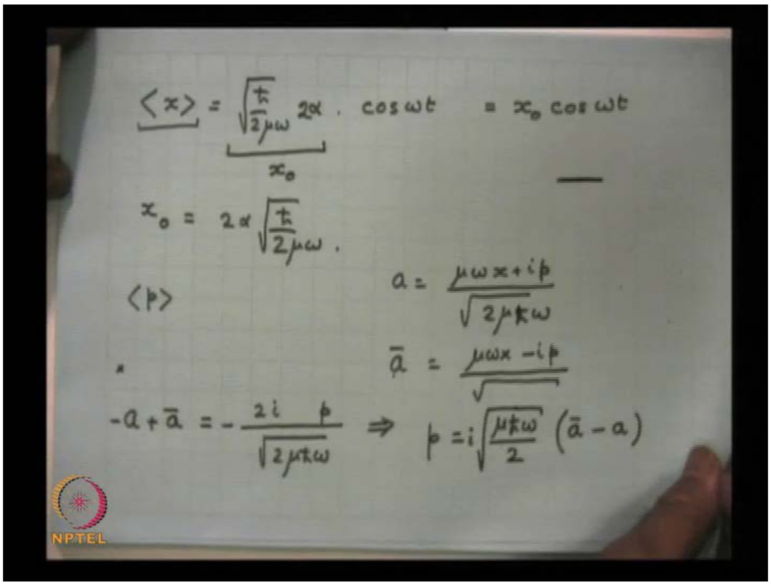

I rewrite it once again, so I get x is equal to h cross by 2 mu omega alpha, times 2 cos omega t. So, if I write this as x naught then this becomes equal to x naught cos omega t so that the expectation value of x, the initial wave packet, the expectation value of x oscillates with amplitude x naught. So, this is what a classical oscillator looks like, because you know that in a classical oscillator, you have like a simple bob of a simple pendulum, it oscillates back and forth. Therefore, the coherent state, the evolution or the time evolution of the coherent state is just like that of a classical oscillator.

You have x naught is equal to 2 alpha under root of h cross by 2 mu omega. Now, before I conclude this lecture, let me mention that similarly, I can find out the expectation value of p. We had a is equal to mu omega x plus i p divided by 2 mu h cross omega, and a bar was equal to mu omega x minus i p divided by the same. So, if I subtract this from this, I get a minus a bar is equal to 2 i p divided by 2 mu h cross omega.

If I multiply both sides by minus sign, I get minus here and plus here and minus here, and then I multiply by i, so that the right hand side becomes plus, so as we had done before that this becomes p is equal to under root of mu h cross omega by 2 times i and will be equal to and multiplied by a bar minus a. So, this is the representation of the operator t.

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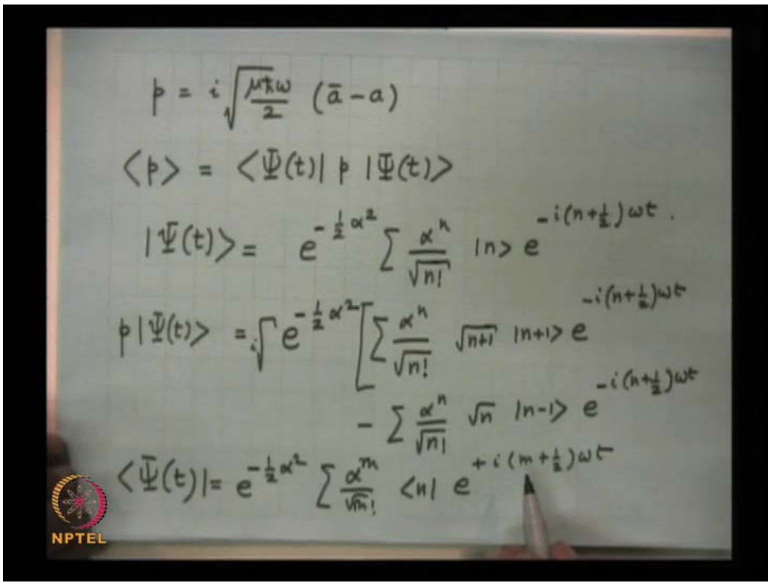


Having found that, let me rewrite this once again, so p is equal to i mu h cross omega by 2 a bar minus a. If I want to find out the expectation value of p, then this will be bra psi t p and then psi t. This is the expectation value of psi t and we know what is psi t. This is equal to e to the power of minus half alpha square, summation alpha to the power of n, divided by square root of n factorial, multiplied by ket n, e to the power of minus i n plus half omega t.

If I then use this, then we use again the relation that a bar minus a operating on psi t, will be a bar, and if I operate a bar minus a on psi t, then this will be e to the power of minus half alpha square, summation alpha to the power of n, square root of n factorial, a bar ket n will be square root of n plus 1, ket n plus 1, e to the power of minus i n plus half omega t. There will be another term, because this will be p times operating on psi t, so p will be multiplied by a factor i, under root of mu h cross omega by 2. Then you will have a term, so a operating on ket n will be alpha to the power of n, n factorial, square root of n, n minus 1, e to the power of minus i n plus half omega t.

Then one sum will be over, so then you have to evaluate what is psi of t, and psi of t will be equal to e to the power of minus half alpha square, alpha to the power of n, square root of n factorial, bra n, e to the power of plus i and then n has to be replaced by m, so that it does not get confused with that; m plus half omega t. So, I multiply it and I re do the same kind of analysis.

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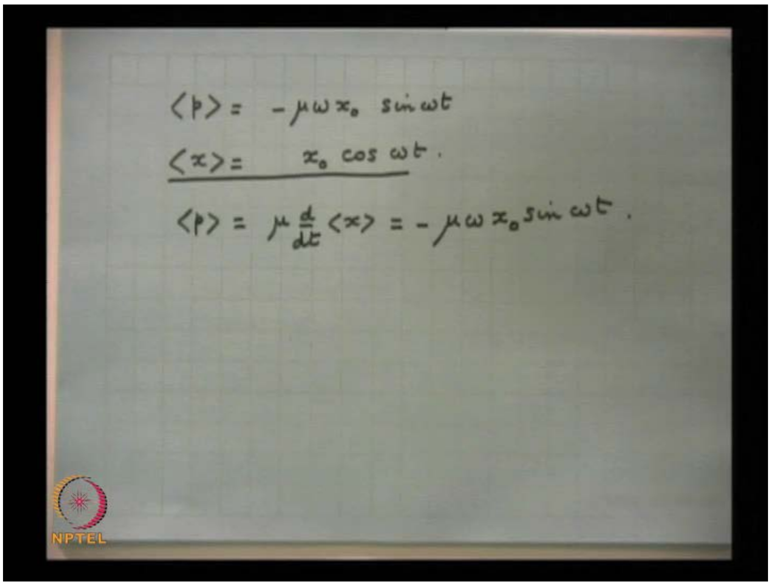


And it is just exactly similar to what we had done for the evaluation for the expectation value of x. I leave it as an exercise for you to show that the expectation value of p, will come out to be minus mu omega x naught sin omega t. It will come out like this.

You recall that expectation value of x came out to be x naught cos omega t and we know that in classical mechanics. Thus we find that the expectation value of the momentum becomes mu times the velocity, d by d t of x. So, d by d t of x and this is equal to if I differentiate this will come out to be minus mu omega x naught sin omega t, and this will be the expectation value of x, and therefore, the coherent state, the coherent state, which represents the classical oscillator. Now, maybe, we just stop here and start the next lecture form this point onwards.

Thank you