Basics Quantum Mechanics

Prof. Ajoy Ghatak
Department of Physics
Indian Institute of Technology, Delhi

Module no # 07

Bra-Ket Algebra and Linear Harmonic Oscillator - II

Lecture no # 03

The Linear Harmonic Oscillator using Bra and Ket Algebra (Contd..)

Previous lecture, we had introduced the Bra and Ket algebra, the concept of linear vector spaces and had solved the harmonic oscillator problem.

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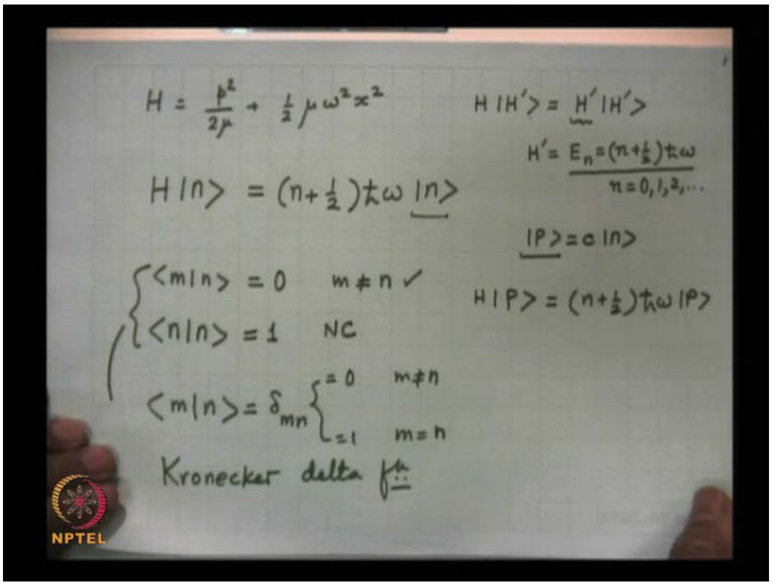

For the harmonic oscillator, the Hamiltonian was given by p square by 2 mu plus half mu omega square x square. Our objective was to solve the eigen value equation. This represents the eigen value equation and H prime is the eigen value and ket H prime represents the corresponding eigen ket.

We had shown that the eigen values H prime is equal to the E n. This is equal to n plus half h cross omega, where n takes the values 0, 1, 2, 3 etcetera. So, we obtained the same eigen values and we have obtained while solving the Schrodinger equation.

Corresponding, to the eigen ket H, eigen value H prime, the eigen ket is denoted by ket H prime. So, we denote this by H ket n is equal to n plus half h cross omega ket n. This is the eigen value and corresponding to the eigen value n plus half h cross omega, ket n are the eigen kets.

Now, since H is a Hermitian operator. All the eigen values are of course real. But, we had shown that eigen kets belonging to different eigen values, must be orthogonal. That is m ket n must be equal to 0, if m is not equal to n. Furthermore, since this is a linear of equation, any multiple of ket n, let us suppose, I write ket p is equal to some multiple of ket n. Then, if I multiply both sides by the complex number C, then H ket p, p is also an eigen ket, belonging to same eigen value n plus half h cross omega. So, the eigen kets are determined within the multiplicative constant and we choose the multiplicative constant such that they are normalized.

This is the orthogonality condition and this is the normalization condition. We can combine these two (Refer Slide Time: 03:46) and we obtain we can write m n is equal to delta m n, which is the Kronecker delta symbol, which is 0, if m is not equal to n, and is equal to 1, if m equal to n. This is known as the Kronecker delta function. So, these are the eigen values and eigen ket of the operator H.

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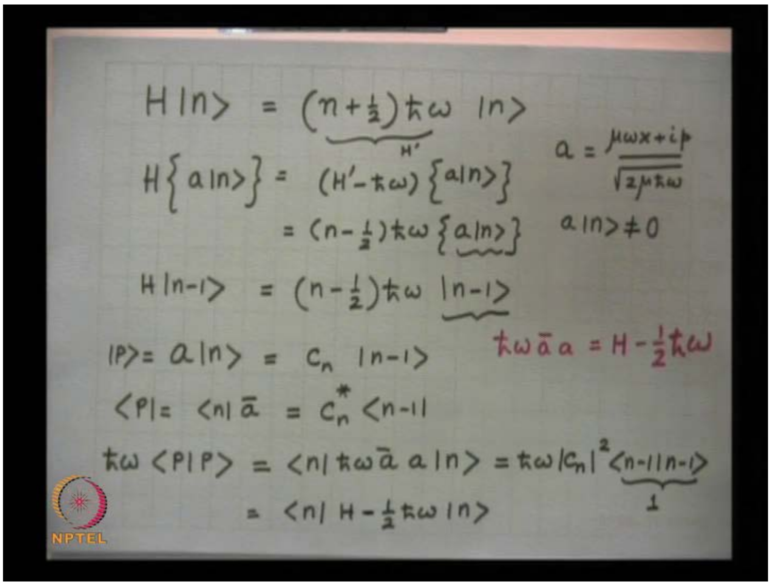

Now, I rewrite this equation, so I have H ket n is equal to n plus half h cross omega ket n. We can also introduce two operators, which is a and a bar. So, a was equal to mu omega x plus i p, under root of 2 mu h cross omega and a bar was the adjoint of this. We had shown that H a ket n, so if this is H prime, and then this will be the eigen value and is H prime minus h cross omega, a ket n.

If ket n is the eigen ket of the operator H, and the corresponding eigen value is the H prime, then a ket n is also an eigen ket of operator H, belonging to the eigen value H prime minus h cross omega, provided of course, as I have mentioned that a ket n is not a null ket, because this is a null ket then this is a trivial equation.

So therefore, if I subtract this, so this becomes n minus H prime minus h cross, minus n minus half h cross omega, a ket n. Now, we also know that if I replace n by n minus 1, so if I replace n by n minus 1, this becomes n minus half h cross omega. So, this must be multiple of this.

I should have a ket n must be a multiple of a minus of ket n minus 1. I write this as ket p and if I take bra p, then this will be equal to n a bar is equal to C n star bra n minus 1, and if I write h cross omega bra p ket p, then this will be equal to n h cross; this is just the number, so I can put it anywhere I feel like, a bar a ket n. On my right hand side, will be h cross omega mod C n square, bra n minus 1 ket n minus 1, but since they are normalized, so this is equal to 1.

Now, quite sometimes back, in my last lecture, I had shown that this was equal to n H minus half h cross omega. I had shown that h cross omega that h cross omega a bar a, was equal to H minus half h cross omega. Therefore, let me write down the right hands side. Sorry, is it coming here. (No volume between: 08:34-08:54)

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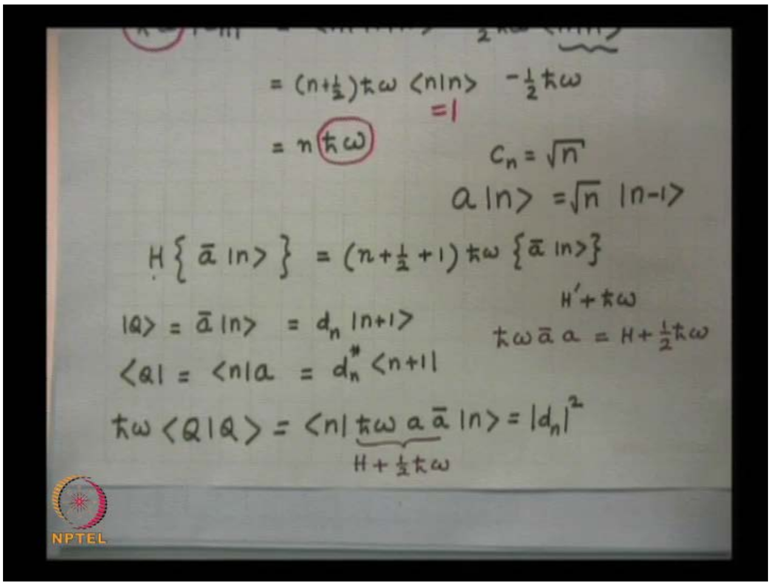

The right hand side is h cross omega mod C n square and that is equal to n H ket n minus half; this is just a number, so I can take it out bra n ket n.

H ket n is equal to n plus half h cross omega, bra n ket n, so this is 1, and this is also 1, so this is minus half h cross omega, so this quantity is equal to 1. I can, because the kets are normalized, so this will be equal to n h cross omega. So, this factor and this factor cancel out (Refer Slide Time: 10:10) and we obtain within phase factor C n is equal to under root of n.

Therefore, we get the result that a ket n is equal to square root of n, n minus 1. This is a very important result that we must remember (Refer Slide Time: 10:37). Now, once if we start with the same equation and let me we start with the same equation, H ket n is equal to n plus half h cross omega, and instead of a, we write a bar.

So, a bar ket n is an eigen ket, corresponding to the eigen value n H prime plus h cross omega. Then the eigen value n plus half h cross omega, therefore, we obtain that H operating on a bar ket n, will be equal to H prime plus h cross omega. So, this is n plus half plus 1, h cross omega. So, it is H prime plus h cross half h cross omega.

This will be a bar ket n, so a bar ket n is an eigen ket of the operator H, belonging to the eigen value H prime plus h cross omega. Therefore, a bar ket n, must be a multiple, let me write down say d n of n plus 1. So, let us suppose this is ket Q, then bra Q will be equal to n a bar bar, which is a, which is d n star bra n plus 1. Just as we did last time, if we now multiply h cross omega bra Q ket Q, this is equal to bra n h cross omega a, a bar ket n. This is equal to mod d n square bra n plus 1 ket n plus 1 and that is 1.

Now, in lecture before this, we had said that h cross omega a bar a, was equal to H plus half h cross omega. So, this quantity is H plus half h cross omega.

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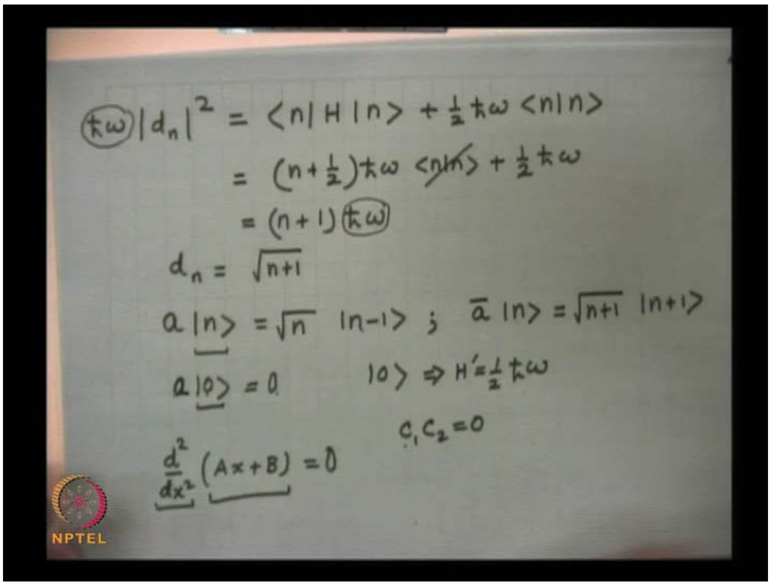

We rewrite this and we obtain mod d n square and is equal to n H ket n plus half h cross omega n ket n. So, this is equal to n plus H, operating in, will be n plus half h cross omega, and then will be bra and ket n, which will be one. This is 1 plus half h cross omega.

I am sorry, (Refer Slide Time: 14:53) there is one thing that when I multiply this by h cross omega, I should have multiplied this also by h cross omega, because I wrote bra Q ket Q multiplied by h cross omega. So, I missed out writing the h cross omega last time. This is your h cross omega d n square. So, this is 1 and this is half h cross omega. So, this becomes n plus 1 h cross omega. So, this h cross omega here and this h cross omega cancel out, and if I take the square root, so you will get d n is equal to square root of n plus 1.

So, we get two very important relations. The first was a ket n is equal to square root of n n minus 1, and the second one is a bar ket n is equal to square root of n plus 1, n plus 1.

Since, the operator a raises to by 1, number the value of n, and here it lowers the value of n. These are known as ladder operators or creation and destruction operator.

Now, in this case if we write a ket 0, then n is 0. So, this is a null ket, so this is the ground state and one cannot go below that. Ket 0 is the eigen ket; belonging to the eigen value H prime is equal to half h cross omega. So, here once again, a is a non null operator, is even when I have two number say C 1, C 2. If this is 0 then we can say either C 1 must be 0 or C 2 must be 0.

But in operator algebra, this argument is not valid. Here, a is non null operator, ket 0 is a non null ket, but a operating on ket 0, is a null ket, it is something like this. If you differentiate d 2 by dx square of A x plus B, this is a non null operator. This is a non null function, but if you differentiate this two, this is always 0. So, here you have this, a ket 0 is a null ket. It says that you cannot have an eigen value less than half h cross omega. Now, with this algebra, we can very easily calculate many of the quantities.

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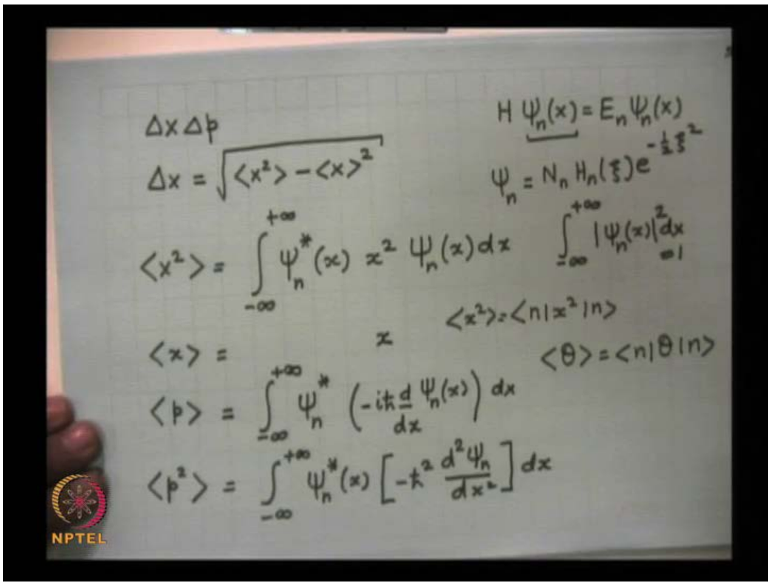

For example, let us suppose we want to calculate delta x delta p for the harmonic oscillator states. Now, what do I mean by delta x delta p? Let me first just consider delta x, so delta x is the x square minus x average square under the root.

If you use Schrodinger type of wave functions, let us suppose we have H psi n of x is equal to E n psi n of x, where psi n of x as you recall for the harmonic oscillation problem was N of n, H of xi e to the power of minus half xi square. These are again the normalized eigen functions. So, what will be my x square? I have to calculate like this. x square will be this, so let us suppose this is psi n of x, so this will be from minus infinity to plus infinity psi n star of x, x square, psi n of x dx. These are normalized, so actually this is divided by xi n star xi n.

But we have assumed these are normalized wave functions, minus infinity to plus infinity xi n x mod square dx is equal to 1. So, this is my x square. Similarly, x will be just the same quantity but, x square replaced by x, and as we had discussed, the p will b minus infinity to plus infinity psi n star, I am removing, and then I must use the differential representation, so minus i h cross d dx of psi n of x dx. Similarly, p square will be minus infinity to plus infinity, psi n star of x, and p square will be, here minus and minus plus then minus, so this will be minus h cross square d 2 psi n by dx square. Now, using the Hermit gauss function, it is of course, possible to calculate these integrals but, it is in general they are very cumbersome.

What we will do now is to use operator algebra and calculate n x square n. This is my x square, so any operator the expectation value of any observable, will be at the n th state will be n, o, n. Let me illustrate this

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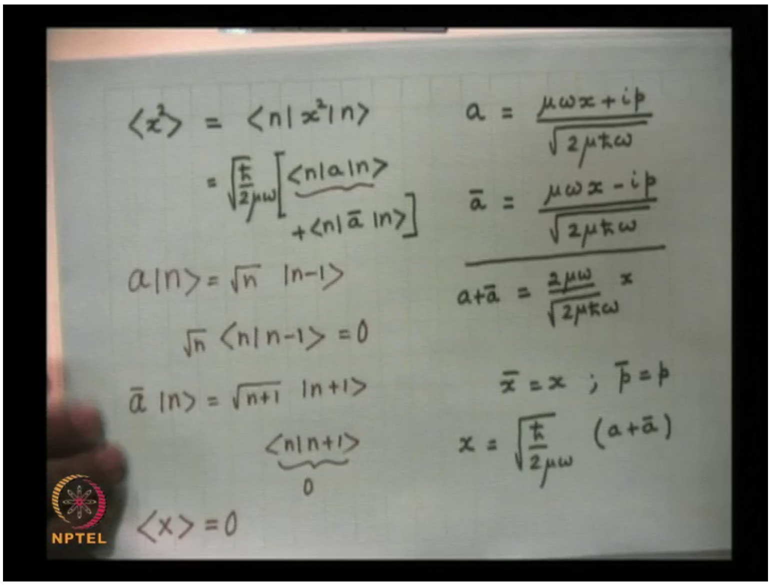

Let me calculate what is x average? x average will be the average value of x, will be n x n. Now, you may recall that the operator a was equal to mu omega x plus i p divided by under root of 2 mu h cross omega, and then a bar was equal to mu omega x, x bar is equal to x and this is minus i p, because x and p are observable, so x bar is equal to x and p bar is equal to p.

So, divided by under root of 2 mu h cross omega, if I add them up, then I will get a plus a bar is equal to the p and p will cancel out, so 2 mu omega, so this will be 2 mu omega divided by under root of 2 mu h cross omega times x. Therefore, x will be equal to under root of h cross by 2 mu omega a plus a bar. This is the operator, so we will have this. This is equal to under root of h cross by 2 mu omega, n of a n, a plus a bar, so plus n a bar ket n.

Now, this is very easy to calculate, as we just calculated that a ket n was equal to square root of n, n minus 1. So, this quantity will be bra n, n minus 1, and this is square root of n. So, this is 0. So, similarly, a bar ket n, is equal to square root of n plus 1. So, this will be bra n ket n plus 1, and this is because they are orthonormal kets, so this is also 0. So, we get the result expectation value of x is 0, corresponding to a particular state, eigen state of the harmonic oscillator. Let me calculate x square. So, what is x square? It is x times x.

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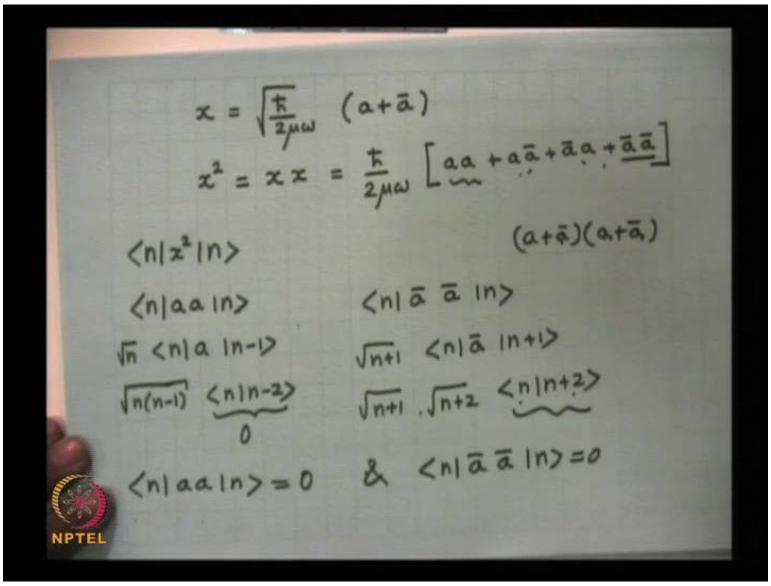

So, let me write down the expression for x. This is equal to h cross by 2 mu omega, a plus a bar, so x square, the square of an operator is just x times x. This will be equal to h cross by 2 mu omega, a plus a bar whole square. So, we must keep the order. So, this will be a, a plus, a a bar plus, a bar a plus, a bar a bar. One must be very careful. One must not write 2 a a bar or a square plus a bar square plus 2 a a bar. No, because this a plus a bar has to be operated on a plus a bar. So, this will be a a plus, a a bar plus, a bar a plus a bar a bar.

Now, I want to evaluate bra n x square ket n. Now, the first term will be bra n, you can do it. So, this will be a operating on that will be square root of n, n a n minus 1, and this will be square root of n minus 1, so n into n minus 1, n n minus 2. This will be 0, because these are eigen kets belonging to different eigen value, so the scalar product is 0. Similarly, a bar a bar, so you have n a bar a bar ket n. This will be square root of n plus 1, n a bar n plus 1, and this will be square root of n plus 1 times square root of n plus 2 n n plus 2.

Since, this two numbers are not equal, so this will also be 0. Therefore, we get the relation that n, a, a, n is 0, and n, a bar, a bar is also 0. But, these matrix elements corresponding to this, the scalar product corresponding to this, will not be 0 as I will show you in a minute.

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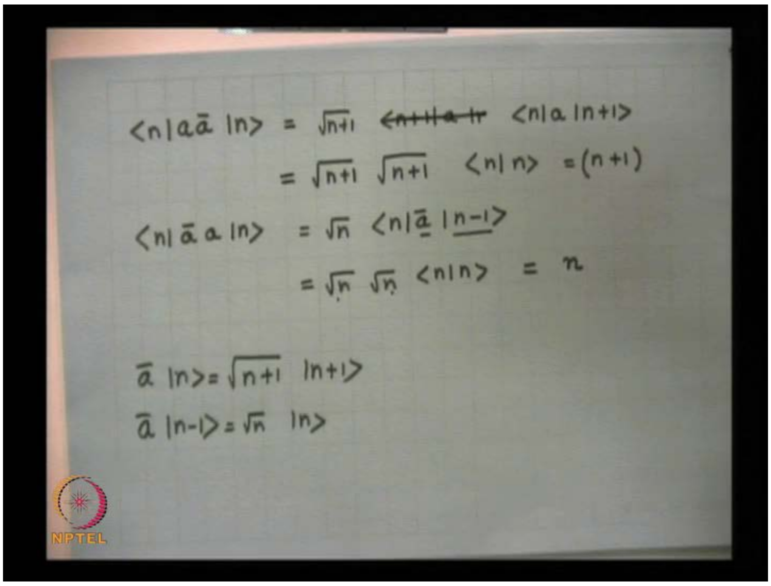

We will have a, a bar ket n bra n, so this will be a bar ket n, will be equal to square root of n plus 1, is really very simple. I am sorry, so this is bra n, a, n plus 1. Now, this will be n plus 1, a, operating on n plus 1, is again n plus 1. So, this will be n, which is 1. So, this is equal to n plus 1.

Let me do the other one; n, a bar a ket n, so a ket n, will be square root of n, n minus 1 and a bar ket n minus 1 will be square root of n, n, n, because remember the square root of n equal to square root of n plus 1, n plus 1. So, if I replace n by n minus 1, so n minus 1 will be square root of n, n. This you must remember. So, this will be equal to square root of n, times square root of n, is just n. So, if I add this two, so this becomes equal to n plus n plus 1.

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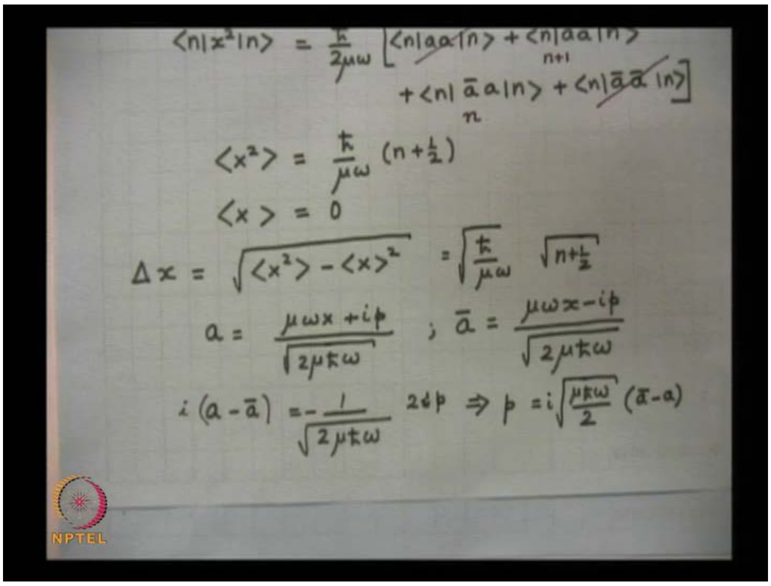

So, in this expression, let me therefore, we have this therefore, n x square n, will be equal to h cross by 2 mu omega. So, this is 0 and so let me write it down; n, a, a, n, plus n, a bar, n plus n a bar, a, n plus n, a bar, a bar, n.

We have just now shown that this is 0 and this is 0. This is equal to a, a bar that equal to n plus 1, and this is equal to n. Therefore, this will be 2 n plus 1, and if I take the half inside, so you will get h cross by mu omega, n plus half. So, my expectation value of x square is this. x is 0, therefore, delta x is equal to x square minus x average square, but this is 0. So, you will have under root of h cross mu omega under root of n plus half.

One result we have obtained. Similarly, we have, we wrote down the two operators; a, which was equal to mu omega x plus i p divided by under root of mu 2 h cross omega, and a bar is equal to mu omega x minus i p by under root of 2 mu h cross omega. Now, one can immediately see, in order to obtain an expression from x, we add them up. In order to obtain a expression from p, you must subtract them.

You get a minus a bar, will be, if I take out 2 mu h cross omega outside, then this will be i p minus of minus i p. Therefore, this will be 2 i p. So, I multiply both sides by i, so you get p is equal to and there will be minus sign here, so it will be under root of mu h cross omega by 2. Then i and then a bar minus a, and you see if I multiply this by i, so if I multiply this by i, this becomes a minus sign here, and this will be i and if I take the minus sign here then a bar minus a.

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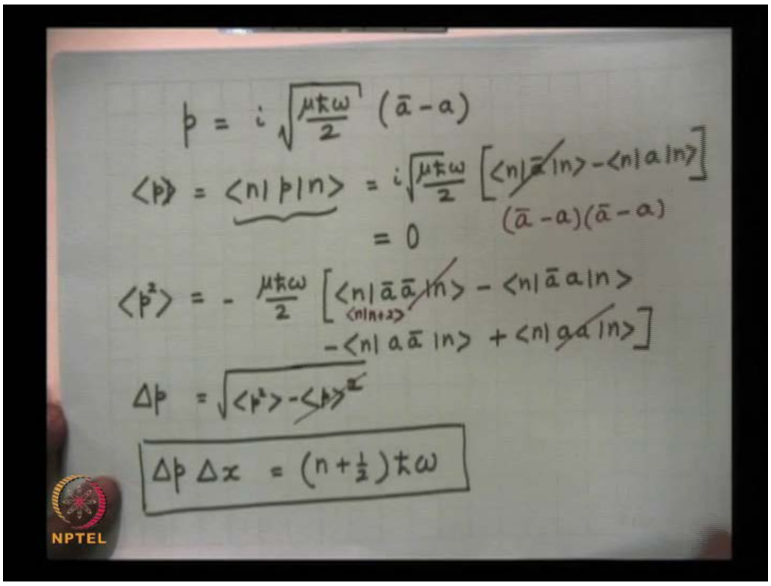

Therefore, we finally, obtain that the operator, the representation of the operator p is equal to i mu h cross omega by 2 a bar minus a. Now, it is trivial, it is really, once we have done to calculate the expectation value of p and expectation value of p square. So, you have expectation value of p is equal to bra n, ket p, ket n. So, this will be i under root of h cross omega by 2 n a bar n, minus n, a, n. As we have done, before this is proportional to ket n plus 1. So, this is 0, this is proportional to ket n minus 1, so this scalar product of that is 0, so this is 0.

And the second one is equally straight forward. So, p square will be equal to n p square. So, if I square this, I will get minus mu h cross omega by 2, and then there will be 4 terms, and this if I multiply, a bar minus a, with a bar minus a, so it will be a bar minus a multiplied by a bar minus a. So, once again, we have to be very careful and we have to write them in proper order. So, a bar a, a bar a bar ket n, then operating on this, will be minus n a bar a ket n, and this term minus n, a, a bar ket n. this operating on this will be… (Refer Slide Time: 36:53).

I hope you know realize that this term will be n plus 1, some multiple, and then a bar n plus 1 will be proportional to n plus 2. So, we will have a term which is n, n plus 2, which is 0, so this term is 0, similarly, this term is 0, and I can take the minus sign outside. I leave it as an exercise for you to calculate and you obtain an expression, which will be something like this is.

So, we can calculate then delta p, which will be equal to p square minus p average square. But, this quantity is 0 and this quantity we have just evaluated. If you multiply these two out, delta p and delta x, you will get n plus half h cross. The uncertainty product is the minimum for the ground state and as we go to higher and higher states, the uncertainty product increases. This is the result, which we have obtained in a fairly straight forward manner by using the operator algebra.

The operator algebra allows us to calculate the matrix element. This particular quantity is known as the matrix element of the operator p (Refer Slide Time: 38:28) and one can easily calculate the uncertainty product. Now, so this completes one and we had said in my in our last lecture.

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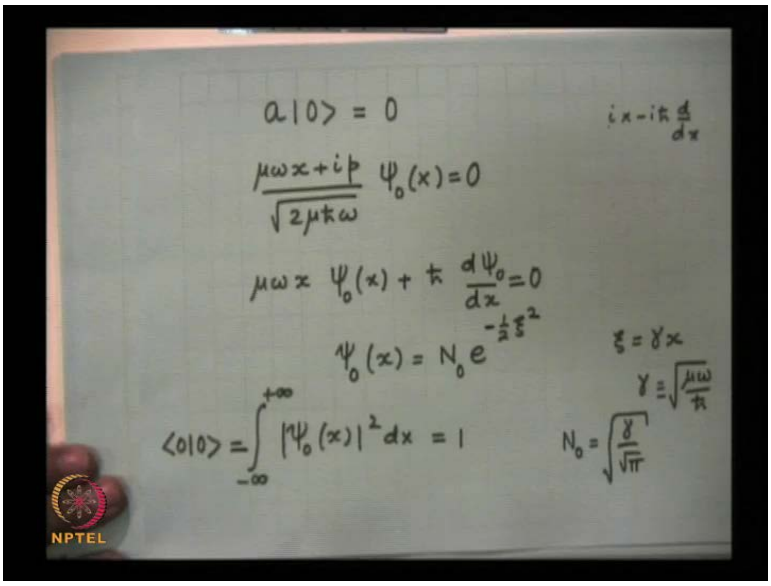

We have said that a ket 0 is a null ket, and therefore, corresponding Schrödinger wave function will be such that what is a operator omega x plus i p by under root of 2 mu h cross omega psi 0 of x is equal to 0. So, this term goes off. So, we have obtained mu omega x, psi 0 of x, and p is, i times minus i h cross d by dx. So, i square is minus 1, minus, minus plus 1, so plus h square d psi 0 by dx. We integrated this equation and this is a very simple integration and we have obtained for psi 0 of x, some normalization constant e to the power of minus half xi square.

So, we write this as N 0, where xi is equal to gamma x and gamma is equal to mu omega by h cross, defined to be equal to. N 0 will be gamma by root pie, square root of that. If you integrate this, so it is normalization is with respect to x that is minus infinity to plus infinity psi 0 mod x whole square dx is equal to 1. We write this in terms of the Bra-Ket algebra that ket 0 bra 0 ket 0, this is equal to this.

The relationship between the Schrodinger wave function and bra and ket algebra is thorough the scalar product. This is a scalar product. Now, I want to find higher order wave functions. So, we now use that we have psi 0 of x, we know, I want to know the higher order wave functions.

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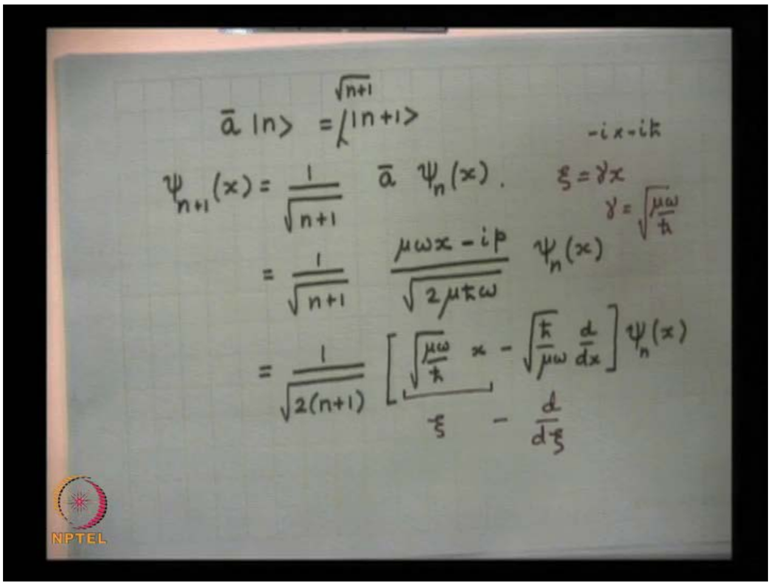

We have a bar ket n is equal to n plus 1. Therefore, we have psi sorry multiplied by square root of n plus 1. So, psi n plus 1 of x, will be equal to 1 over square root of n plus 1, a bar psi n of x. So, you will have a bar is 1 over under root of n plus 1, a bar is mu omega x minus i p divided by under root of 2 mu h cross omega, operating on psi n of x. So, if I take 2 outside, so I will get under root of 2 n plus 1, and then it will be mu omega by h cross x, and this is minus i of x into minus i times of h cross. So, this will be minus minus plus and i square is minus 1, so h cross by mu omega d of dx psi n of x.

We have xi is equal to, we had written down the xi is equal to gamma x, where gamma is equal to under root of mu by omega mu omega by h cross. So, this quantity is xi and this quantity is d by d xi.

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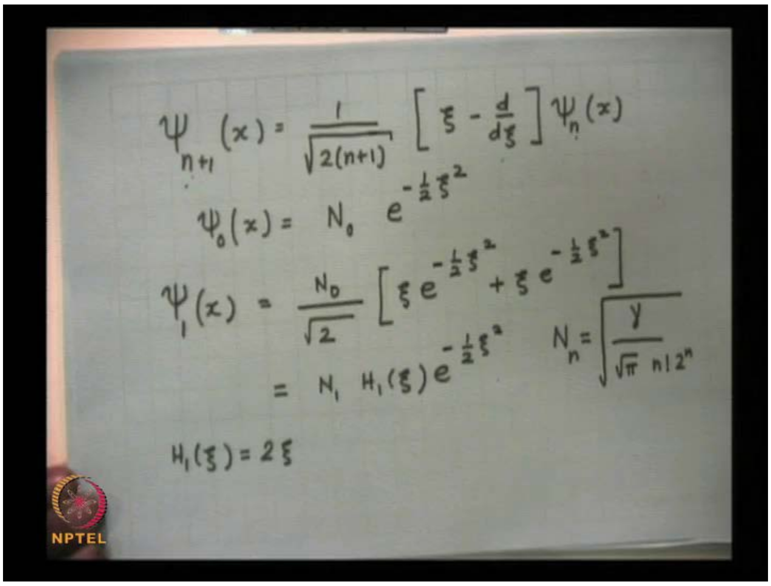

Therefore, we get the very important result that from the operator algebra that xi n plus 1 of x is equal to under root of 2 n plus 1, xi minus, d of d xi, psi n of x. So, if I know xi n of x, by simple differentiation I can find out xi n plus 1. But, I do know psi 0 of x, psi 0 of x, we know which is equal to N 0 e to the power of minus half xi square. So, if I substitute it there, so then psi 1 of x, will be N 0 and divided by N 0. So, this is 2 into 1, and the first term is xi into e to the power of half xi square, and if I differentiate this, so this will be xi square, 2 xi by 2, so plus xi, e to the power of minus half xi square.

So, H 1 of xi, e to the power of minus half xi square, but the general formula for N of n is equal to gamma under root of pie, n factorial divided by 2, so two to the power of n, under the root. So, H 1 of xi, as you know is equal to 2 times xi. Now, I know psi 1 of xi so I can find out xi 2 of xi, and they will be automatically normalized. They will be automatically normalized and once I have psi 2 of xi, then I can differentiate it again and I obtain psi 3 of xi and so on.

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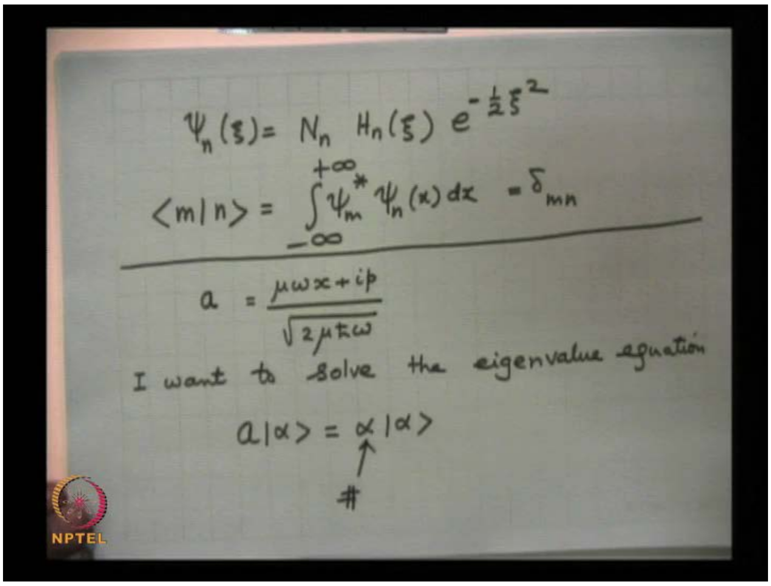

And the final result is as you all know that that psi n of xi is equal to N of n, H n of xi, e to the power of minus half xi square, and here the relationship with the ket's is through the scalar product m ket n, is equal to psi m star psi n dx, from minus infinity to plus infinity, and that as you all know is a Kronecker delta function.

So, we have now derived how to use the operator algebra to find out the eigen sates, the eigen function, the Schrodinger eigen functions of the corresponding to the linear harmonic oscillator problem. We, finally, do an extremely important calculation and that is find out the eigen values of the operator a. This is the annihilation operator that is a is equal to mu omega x plus i p, divided by under root of 2 mu h cross omega.

I want to solve the eigen value equation, a ket alpha is equal to alpha ket alpha, where alpha here is a number. Now, I want to solve this equation.

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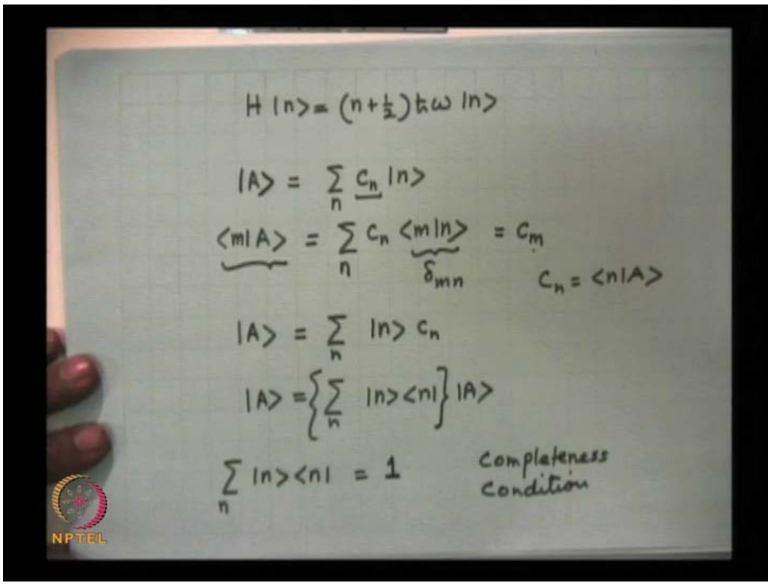

Now, as we had mentioned earlier that the eigen ket of the operator H, form a complete set of wave function that is n plus half h cross omega ket n, and Dirac argues this in a very beautiful manner. He says that in any state of a dynamical system, if you make a measurement of an observable, then it collapses to the eigen state of the observable. Since, and therefore, any state of a dynamical system should be expressible, as a linear combination of the eigen states of the observable. Therefore, the eigen kets of the observable must form a complete set of functions.

So therefore, any arbitrarily ket, say ket A can be described as a linear condition like this. How do I find this C n? This C n, I find as, if I multiply this equation by say bra n, so I get bra n ket A. This is equal to summation, this is n, m n. The summation is over n. This is equal to delta m n, so when I sum over n, for a given value of m, this comes out to be C m. So, I have any arbitrary ket A, if I sum over n, then C of n is equal to so C m is equal to m.

So I write this down as, this is just the number. So, I can write this as n times C n. I can always write that, because this is a number, I can put it anywhere I like. But, C m, from this equation is C n is equal n A. So, I write this as summation n, n a. Now, this is a linear operator. Since, this is valid for any arbitrary ket A, so this must be a unit operator.

So, we say that this is known as the completeness condition of orthonormal kets n summed over all values of n. This is a unit operator; this is known as the completeness condition. So, any state can be expanded in terms of the eigen kets of the Hamiltonian.

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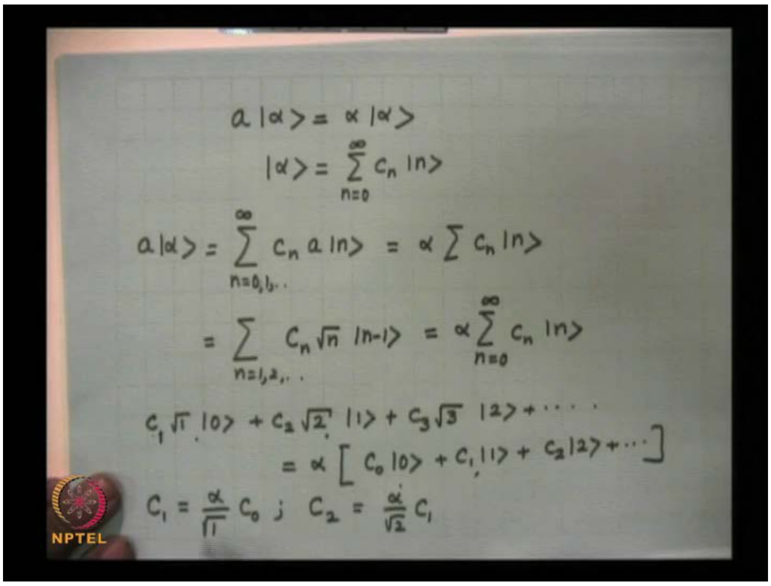

So, we rewrite this, a ket alpha. I want to solve this eigen value equation. I write alpha as summation C n ket n, so for n equal to 0 to infinity, like 0, 1, 2, 3 and so on.

Let me do this carefully, so a operating on this C n, are just numbers. So, a operating on ket alpha, is just summation C n a ket a n, so n equal to 0 to infinity, and this will be equal to alpha times C n ket n. a ket 0 is a null ket, and a ket n is as we have obtained n equal to now is 1, 2, 3, because n equal to 0 that is 0, so C n square root of n, n minus 1. This is equal to alpha, n equal to 0 to infinity C n ket n.

Now, let us expand this. Let us write this out, so we get C 1 square root of 1 ket 0, plus C 2 square root of 2 ket 1, plus C 3 square root of 3 ket 2, and this will be alpha times C 0 ket 0, plus C 1 ket 1, plus C 2 ket 2. Therefore, C 1 will be alpha by root 1, because this is a set of orthonornmal kets, so if I multiply by bra 0, only this term will remain and this term will remain.

So, C 1 will be equal to alpha by root 1 C 0. C 2, now if I multiply by bra one, so all the terms will disappear, except this term and this term. So, you will get C 2, will be equal to alpha by root 2, C 1. So, C 1 is equal to alpha by root 1.