Basic Quantum Mechanics

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Module No. # 07

Bra-Ket Algebra and Linear Harmonic Oscillator– II

Lecture No. # 02

Dirac's Bra and Ket Algebra The Linear Harmonic Oscillator

In the previous lecture, we had developed the algebra, relating to the Bra and Ket notation and we had also defined consider linear operators and define the adjoint of a linear operator.

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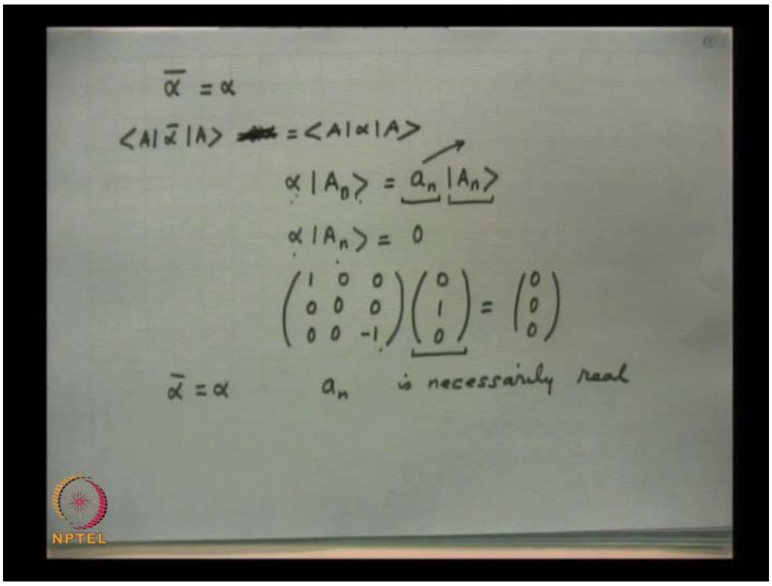

For example, if we consider the linear operator alpha, then it is adjoint is denoted by alpha bar and if alpha bar is equal to alpha. That means that alpha bar ket A bra A is equal to bra A alpha A and if this is valid, then alpha bar is equal to alpha, and we say that the operator is real operator or a self adjoint operator.

Then we said that if I consider the eigen value equation that is alpha ket A n is equal to A n ket A n. This is a number, alpha operating on a ket, which is by definition in non null ket, otherwise it is a trivial solution. So, if alpha operating on a ket, gives a multiple of the same ket, then we say that a n is an eigen value of the operator alpha and ket A n is the corresponding eigen ket.

We can have a situation that alpha ket A n is a null ket, is equal to 0. Then if alpha is not a null operator and if ket A n is a not a null ket, then this is a perfectly valid equation, and 0 is an eigen value. I gave an example that if I have an, say 3 by 3 matrix, say (1, 0, minus 1), the eigen values are (1 0 and minus 1). I just consider a simple matrix, then this operating on (0 1 0), will be (0 0 0). So, this is a well behaved; this is an eigen ket of the this operator, belonging to the eigen value 0.

We said that when if alpha bar is alpha, then a n is necessarily real. Now, Dirac argues in his famous book, the reference to which I have given in my previous lecture, and he argues like this, all observables, anything that we can measure, like the position momentum, energy, angular momentum or anything, they are to be represented by real linear operators" and when we make a measurement a precise measurement of that observable, then we will get one of the eigen values, and then since we will get only a real number, all eigen values must necessarily be real, and Dirac further argues that it is an axiom that all observables must be represented by a real linear operator.

So, I repeat Dirac writes in his book, that any observable like the x coordinate, the position or the y component of the momentum or the z component of the angular momentum or the energy or something like that, any quantity that can be measured is represented by a real operator or a self adjoint operator. When you make a measurement of that observable, you get one of the eigen values; we will illustrate this by solving the harmonic oscillator problem.

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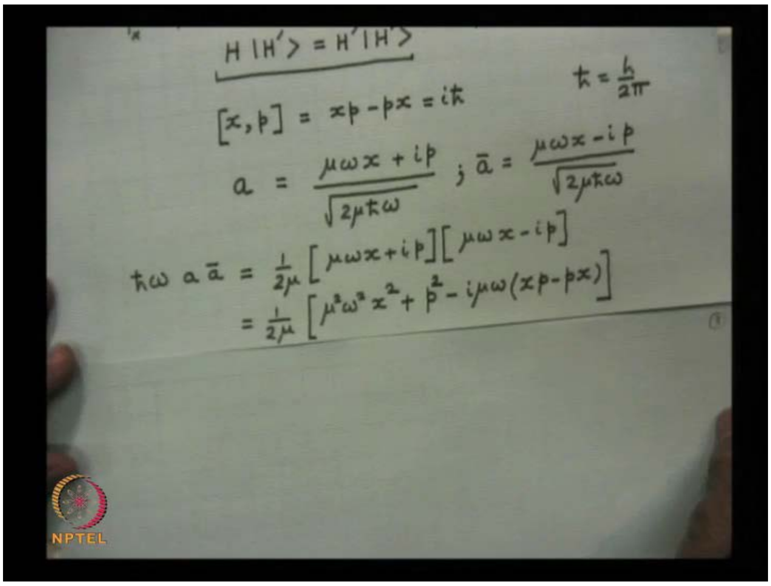

Let me consider the linear harmonic oscillator problem. We had solved the Schrodinger equation for that linear harmonic oscillator problem, in which the hamiltonian, the operator representing the total energy, is given by p square by 2 mu, plus half mu omega square x square, where p is actually the x component of the momentums, and x is the position coordinate. Of course, mu is the mass, omega is the frequency. So, these are constants, now half is of course, a constant. This is the kinetic energy and this is the potential energy.

Dirac says, since x and p, actually it is p x, and since it is a one-dimensional problem, we replaced by p, and he says that since x and p can be measured, these are real operators. That is x bar by definition is x, y bar is y and p bar that is p x bar is p and of course, H bar is equal to H.

What we would like to do is that we would like to solve this eigen value equation. H, he writes it in this form, H ket H prime is equal to H prime ket H prime (Refer Slide Time: 08:06). Our objective for the next 1 hour or so will be to solve this eigen value equation. Now, the only thing that he assumes that we had derived sometime back is the commutation relation. You recall that we had said that that x, p, this is known as the commutator of x and p. This was equal to x p minus p x, the 2 operators do not commute, and this is equal to i h cross, where h cross is the Planck’s constant divided by 2 pi.

So, we will just assume this, and use the Bra-Ket algebra to solve this equation, and this procedure is due to Dirac himself, and this really a very elegant procedure. Dirac defines two operators. These are not real operators, he defines it as, a equal to as mu omega x. This is a constant, omega is a constant, x is of course, the x coordinate, plus i p, where i is square root of minus 1, divided by under root of 2 mu h cross omega. Then the adjoint of these operator is a bar, and is equal to mu omega x, i will be replaced by minus i, p bar is p, x bar is x, and these are of course, numbers, real numbers divided by 2 mu h cross omega.

He defines this two operators and the recalculates h cross omega a a bar. So, you have, if you multiply this by this, you will get 2 mu h cross omega is the denominator. The h cross omega will cancel out, so you will get 1 over 2 mu of mu omega x plus i p multiplied by mu omega x minus i p. Now, it is something like a plus i b and a minus i b. So, do not be in a hurry to write this as a square minus b square, because x p is not equal to p x. So, one has to be little careful, so let us do this carefully.

So, we will have 1 over 2 mu, the first term will be mu square, omega square x square, and this i times i is minus 1, and into minus 1 is plus 1, so p square. And if I take minus i mu omega outside, so it will be x times p minus p times x. This is the crucial part of the calculation. Sorry, this is p square, so i p times minus i p is p square.

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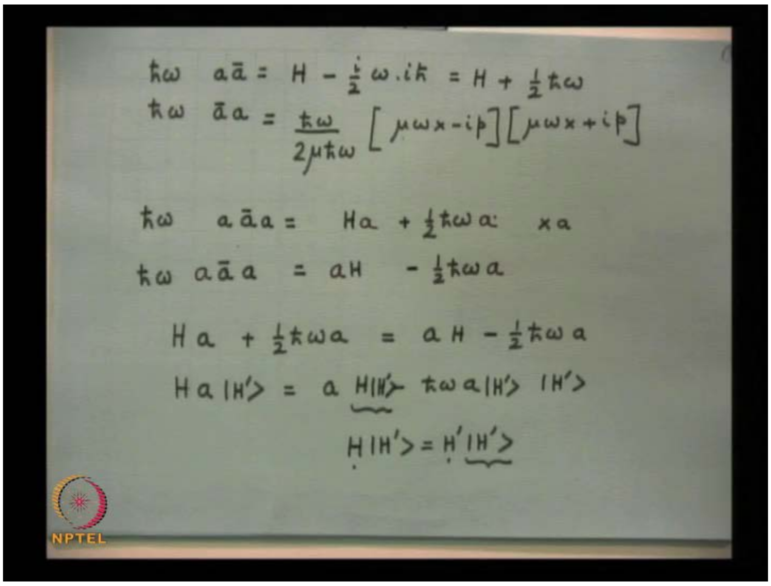

So, we will have if you write this, so you will have h cross omega. Please, leave some space here. Here, a, a bar, this is p square by 2 mu plus half mu omega square x square. So, that is my H and this is the H, the Hamilton (Refer Slide Time: 13:00), which is p square by 2 mu plus half mu omega square x square. So, this is H minus i by 2, the mu and mu cancel out, omega and x p minus p x is i h cross. So, this is i h cross. So, i times i is minus 1, minus and minus is plus 1. So, we will obtain this is equal to H plus half h cross omega.

Similarly, I can write down for h cross omega, a bar a and this is equal to again h cross omega divided by 2 mu h cross omega. So, a bar a will be mu omega, x minus i p multiplied by mu omega x plus i p. Everything will remain the same, except that instead of x p minus p x, it would be p x minus i p (Refer Slide Time: 14:29). So, here there will be a change in sign. So, mu square, omega square x square plus p square, will remain the same, except since this is in reverse order, so this will change sign. So, I leave that as an exercise. I hope you can work this out yourself.

We will get two equations. One is the top equation, let me rewrite it. So, h cross omega, a, a bar is equal to H plus half h cross omega and that is it. The second one will be h cross omega, a bar, a, leave some space, is equal to H minus half h cross omega.

What I do is that I multiply this equation, on the right by a, so I get a here. I get a here and a here. Here, you pre multiply by a, so you get, a H. So, the left hand sides are equal and you therefore, get H a plus half h cross omega a, is equal to a H minus half h cross omega a. So, I take this to that side, so you will get H a, is equal to a H minus h cross omega a. Operate this on the eigen ket of H prime. So, a operating on H prime, H operating on H prime, and this operating on H prime.

We know that H prime ket H prime is an eigen ket. H ket H prime is an eigen ket, so this is just a number, ket H prime is an eigen ket of the operator H, belonging to the eigen value H. This is what we had started out that we want to solve this eigen value equation. So, we say that let ket H prime be an eigen ket of the operator H with H prime as the corresponding eigen value.

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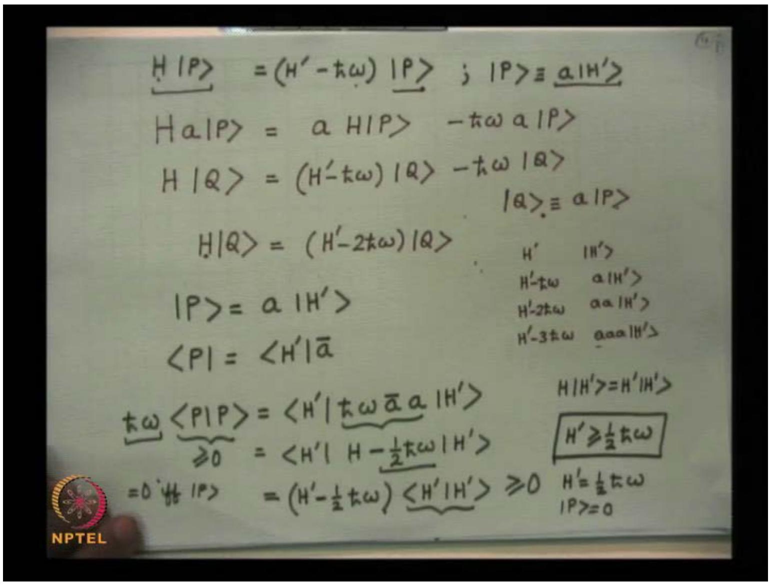

So, we obtain from here, if I rewrite this slightly, so we get H operating on a H prime. That is let me write down as P. This will be a H operating on H prime, will be H prime. H prime is a number, which I can put outside, so H prime and then minus h cross omega times ket P, where ket P is defined to be equal to a ket H prime. This is an extremely important equation (Refer Slide Time: 19:06) and it says that if ket H prime is an eigen ket of the operator H, then ket P, which is equal to a ket H prime, is also an eigen ket of the operator H, belonging to the eigen value H prime minus h cross omega, provided this is not a null ket, because if it is a null ket it is a trivial solution.

So, I repeat and you must understand this that if ket H prime, of course, this is not a null ket; ket H prime is an eigen ket of the operator H belonging to the eigen value H prime, then ket P, which is equal to a ket H prime, is also an eigen ket of this same operator H, corresponding to the value eigen value H prime minus h cross omega ket P.

Now, I can proceed further. You remember that we had the relation that H a was equal to a H minus h cross omega a. We had derived this relation. Now, instead of ket H prime, I now operate this on ket P, and this on ket P, and this on ket P, and we will obtain H ket P is equal to H prime minus h cross omega. So, this will be equal to H prime minus h cross omega a ket P. Let me write this as ket Q. So, this is H operating on ket Q. This will be equal to minus h cross omega ket Q, where ket Q, let me write it carefully, ket Q is defined to be equal to a ket P, and so this becomes H Q is equal to H prime minus 2 h cross omega ket Q.

Thus, if ket P is a non null ket and is an eigen ket of the operator H, then ket Q, which is defined as a ket P is also an eigen ket of the operator H, belonging to the eigen value H prime minus 2 h cross omega.

Therefore, we see that H prime is an eigen ket, if the corresponding eigen value is eigen ket is H prime. Then H prime minus h cross omega, the eigen ket is a ket H prime and H prime minus 2 h cross omega. Then this will be a, a ket H prime and similarly, you can proceed indefinitely, provided any one of them a, a, a, H; provided any one of them does not become a null ket.

Now, let me do a little more algebra. See the beauty of the algebra. Let me write down ket P is equal to a ket H prime, therefore, bra P is equal to H prime, a bar. So, I write this down as bra P ket P, a scalar product. So, let me multiply this by h cross omega. So, we get H prime h cross omega, a bar, a H prime.

Now, if you look up your notes, then we had derived H sorry a bar a, h cross omega a bar a (Refer Slide Time: 24:16) was equal to H minus half h cross omega, so we had derived this and this is equal to H prime H minus half h cross omega. So, you see this is just a number, so I can take it out. H operating on H prime is a multiple of H prime, so this becomes H prime minus half h cross omega bra H prime ket H prime.

Now, you see this is of course a positive number, and see the beauty of the logic. This is of course a positive definite. This is greater than or equal to 0. This is greater than or equal to 0 if and only if P is a null ket. So, this tells us that H prime must be greater than or equal to half h cross omega and H prime is equal to half h cross omega, then this will become 0, then ket P must be a null ket.

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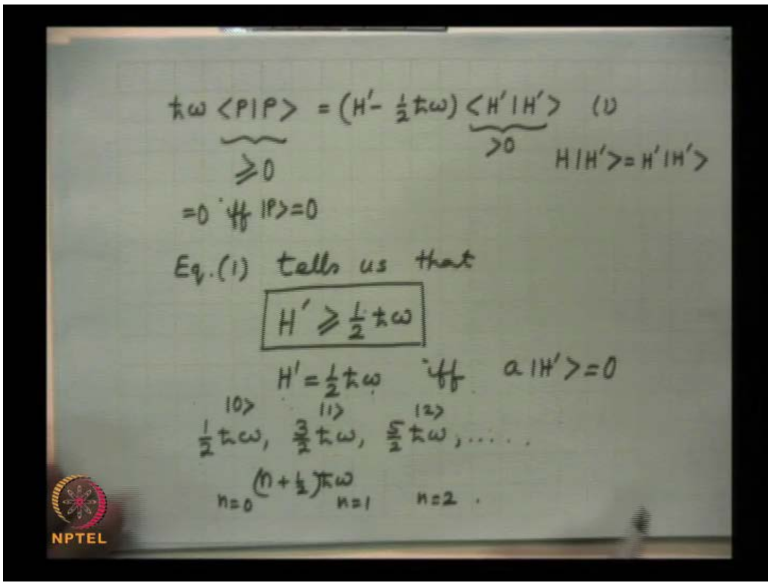

Let me repeat this and it is a very important argument. I have here h cross omega P, P is equal to H prime minus half h cross omega bra H prime ket H prime. Now, H operating on H prime is H prime and this is of course, a non null ket. Otherwise, it is a trivial solution. So, we assume that ket H prime is an eigen ket of the operator H, and the corresponding eigen value is H prime. So, this says that this is positive, definitely positive definite.

This is greater than or equal to 0 and equal to 0, if and only if ket P is a null ket. Therefore, this equation tells us, this equation one tells us (Refer Slide Time: 27:16) that H prime must be equal to or greater than, must always be greater than or equal to H, it can be H prime is equal to half h cross omega if and only if a ket H prime is a null ket.

We see (Refer Slide Time: 27:53) that we just now obtained that if H prime is an eigen ket, the eigen value, then H prime minus h cross omega, then H prime minus 2 h cross omega, H prime minus 3 h cross omega; this cannot go on indefinitely. This cannot go on indefinitely, because then it will violate this inequality (Refer Slide Time: 28:18) and therefore, at some stage, it must terminate and it can terminate only, if a ket H prime is a null ket. That can happen only at half h cross omega and this is absolutely beautiful.

So you see that you will have eigen values, as half h cross omega, 3 half h cross omega and 5 halves h cross omega and so on. So, if I write this as n plus half h cross omega, then n is 0 here, n is 1 here, n is 2 here, and if I write the corresponding kets as ket 0, the eigen kets, ket 1, ket 2.

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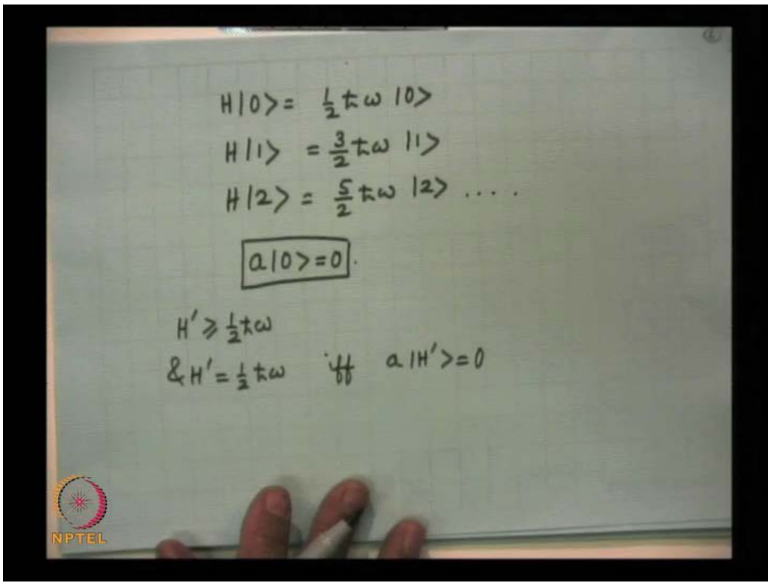

That means what I am trying to say is that H ket 0 is equal to half h cross omega ket 0. H ket 1 is equal to 3 by 2 h cross omega ket 1; H ket 2 is equal to 5 by 2 h cross omega ket 2 and so on. We will show that it has to go indefinitely and that I have not yet proved.

Then there cannot be an eigen value minus half h cross, for that to happen, we must have a ket 0 must be a null ket. So, we said, I go back that if I showed that H prime is always or greater than or equal to half h cross omega. We first prove that and H prime is equal to half h cross omega, if and only if, a ket H prime is a null ket. So, from half we can go to minus half, but that is not possible and that is not possible if a ket 0 is a null ket.

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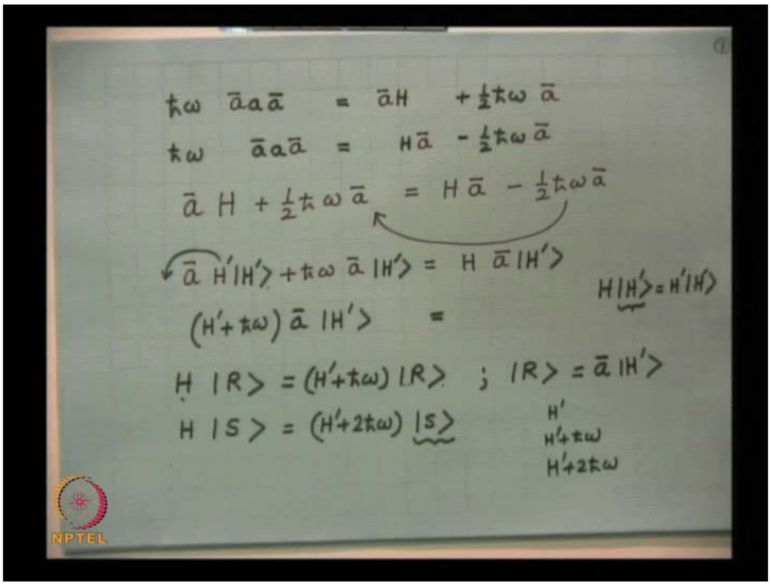

Let me go through this algebra once again little differently. So, we had obtained h cross omega a, a bar, leave some space if you are writing, is equal to H plus half h cross omega. Then you have h cross omega a bar a is equal to H minus half h cross omega. Now, what we did was that in order to make the left hand side equal, I multiplied a here and a on the left here, a on the right here and a on the left here, so then it became left hand side became equal.

What we will do now is we could have also multiplied on the left by a bar and on the right by a bar. So, if I multiply this on the left by a bar, then this is a bar. This is of course a number, so it does not matter where I write, and on the right by a bar; it becomes H a bar and this. If I add this, so therefore, the left hand sides are equal. So, I get a bar H plus half h cross omega a bar is equal to H a bar minus half h cross omega a bar.

So, what I do is I take this to this side, so you get a bar H. Leave some space here, so plus h cross omega, a bar, half plus half is 1, leave some space here, this is equal to H a bar. Now, what you do is that you operate this on ket H prime, which is an eigen ket of the operator H. So, if you recall H ket H prime is equal to H prime ket H prime.

You can see H operating on H prime is H prime and this becomes a multiple. Then I can shift it here, because it is just a number. So, my left hand side becomes H prime plus h cross omega, a bar ket H prime. This is equal to the same. So, let me turn the sides. So, you get H, say ket R, is equal to H prime plus h cross omega ket R, where ket R is equal to a bar H prime. So, we have that.

Please listen to the logic of the argument; if ket H prime is an eigen ket of the operator H then ket R, which is equal to a bar ket H prime is also an eigen ket of the operator H. Now, belonging to an eigen value H prime plus h cross omega. Similarly, I can then again do the same kind of algebra, instead of operating this by H ket H prime, I will this by ket R, and I will obtain H ket S is equal to H prime plus h cross omega plus h cross omega. So, plus 2 h cross omega ket S. So, this will be again an eigen ket eigen value equation provided ket S is not an eigen, not a null ket.

If H prime is an eigen value then H prime plus h cross omega is also an eigen value, H prime plus 2 h cross omega, provided at any stage we do not get a null ket, then it becomes a trivial solution. But I will just now show it can never become a null ket and that the proof is as follows.

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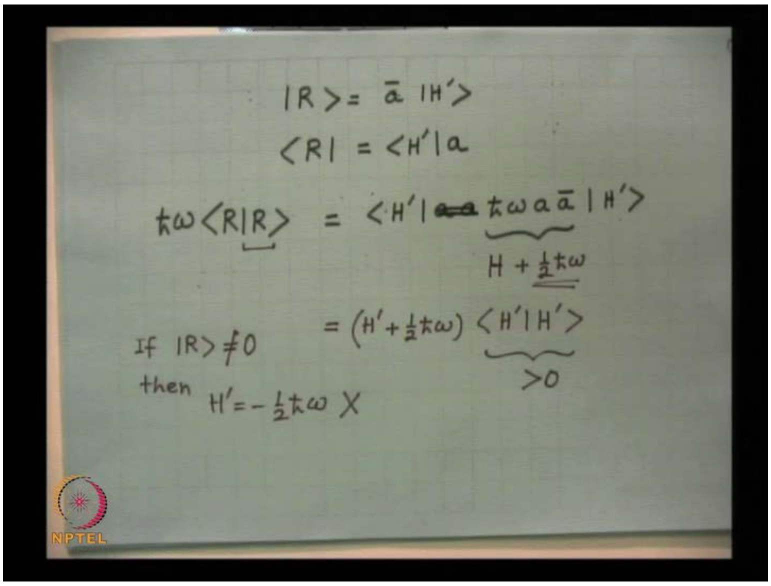

That if ket are is equal to a bar ket H prime, then bra R will be equal to bra H prime, a bar bar, which is a. So, if I write bra R ket R and if I multiply this by h cross omega, then I will get H prime a a bar sorry sorry I'm sorry h cross omega, a, a bar, and this was equal to, if you recollect, this was equal to H plus half h cross omega.

So H operating, if I let me just check this with my… yes this is correct, this is correct you will have that H operating on H prime is H prime, and this is just a number, so this becomes H prime plus half h cross omega. Now, this is always greater than 0 and if this becomes a null ket; if this is 0, then H prime becomes equal to minus half h cross omega. But, this violates the inequality that we had derived sometime back, which says that H prime must be greater than or equal to half h cross omega (Refer Slide Time: 38:34).

We had derived that H prime can at least the half h cross omega and that cannot be greater than that. But, here it says that if ket R is a null ket, then H prime must be equal to this. But, this is impossible, so ket R can never be a null ket. So, if ket are is never a null ket and if H prime is an eigen value.

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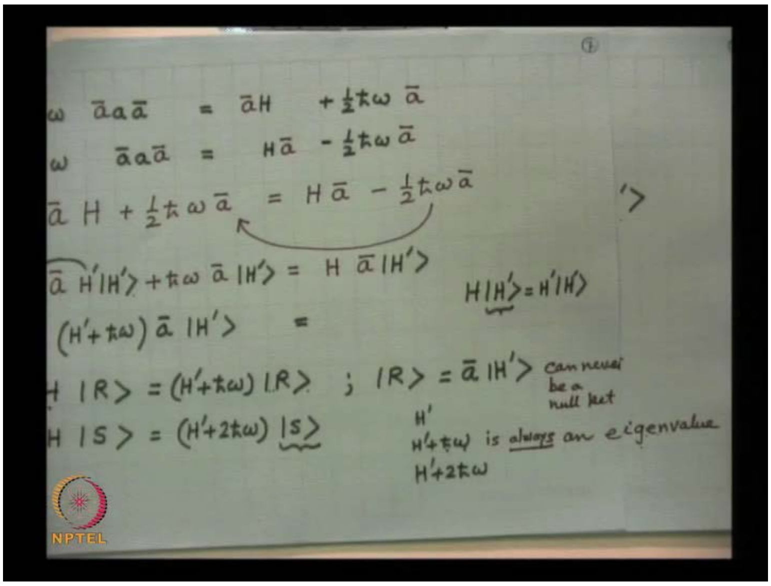

Then H prime plus h cross omega is always an eigen value. I write down is always an eigen value. So if H prime is an eigen value, H prime plus h cross omega is always an eigen value because ket R can never be a null ket. You will have to do this yourself once and then it will become easy and therefore, we finally, obtain the remarkable result. Just by using operator algebra that the eigen value spectrum,

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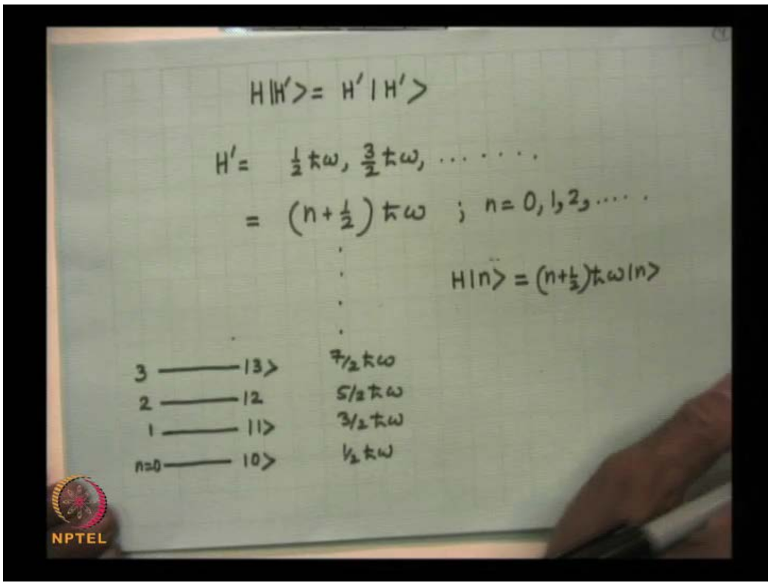

H operating on H prime, the solution of this equation is that H prime can take the values of half h cross omega, 3 halves h cross omega to infinity. It cannot be terminated anywhere, so there are infinite number of eigen values. Therefore, we write this as n plus half h cross omega where n is equal to 0, 1, 2, 3 etcetera, and we label the eigen values. The states are therefore, depend, so we have this n is equal to 0, we represent this by ket 0. This is n is equal to 1 and we represent this by the state ket 1, and then n is 2, which is ket 2. Then 3, which is ket 3, so if n is 3, and then the eigen value is 7 by 2 of h cross omega. This is 5 by 2 h cross omega and 3 by 2 h cross omega and half of h cross omega.

So, we get the same eigen value spectrum, as we had obtained by solving the Schrodinger equation, and we write this down as H n, the eigen value equation for the linear harmonic oscillator problem is written as n plus half h cross omega ket n.

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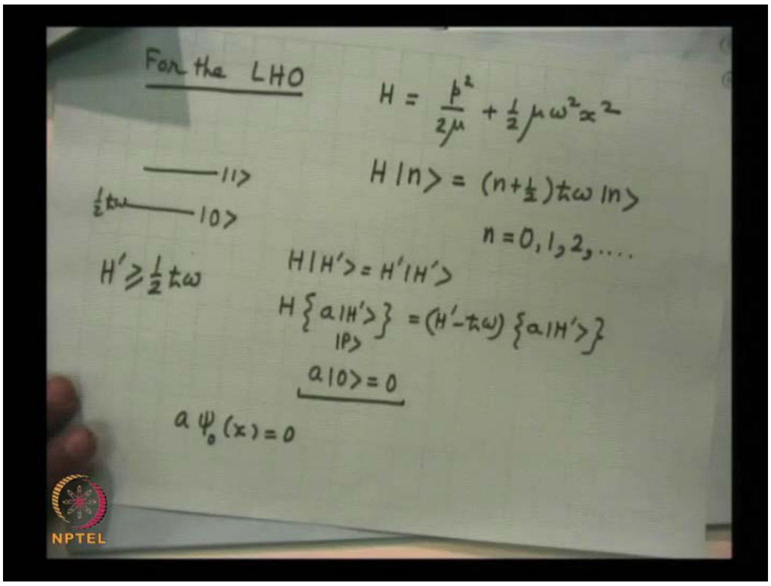

So, we finally write that for the linear harmonic oscillator the Hamiltonian is equal to p square by 2 mu plus half mu omega square, x square. Then H, the eigen value equations are written as H ket n is equal to n plus half h cross omega ket n, where n takes the values 0, 1, 2, 3, etcetera. So, this represents the complete eigen value spectrum of the of the harmonic oscillator problem.

Let me tell you a simple example that since ket 0 is the ground state, ket 1 is the first excited state. So, a ket 0, you see we had said that if H ket H prime is equal to H prime ket H prime, then H operating on a ket H prime was equal to H prime minus h cross omega; a ket H prime, this was p. Now, if H prime is equal to half h cross omega, then there cannot be any state below that. Therefore, a ket 0 must be a null ket; A ket 0 must be a null ket, otherwise there would be an eigen value of minus half h cross omega, which will violate this inequality that H prime must be greater than or equal to half h cross omega. So, what was a? So, if the Schrodinger representation, therefore, a psi 0 of x is equal to 0.

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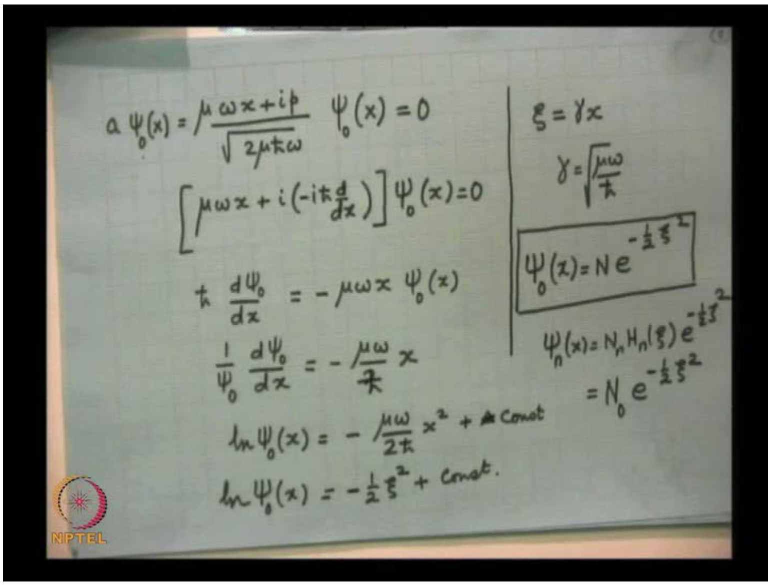

Let me see what it leads to. So, a was equal to mu omega a psi 0 of x, which is equal to mu omega x plus i p by under root of 2 mu h cross omega, for psi 0 of x must be 0. This is just a constant, so you get mu omega x plus, if I write down i p as minus i h cross d by dx psi 0 of x is equal to 0. So, this i square is minus 1, minus of minus is plus 1. So, you get h cross d psi 0 by dx is equal to, if you take the other side, minus mu omega x psi 0 of x. So, if I divide by psi 0, so you get 1 over psi 0 d psi 0 by dx, minus mu omega by 2, sorry it is h cross into x and that is it.

If I integrate this (Refer Slide Time: 45:57). So, you get log of psi 0 of x is equal to minus mu omega by 2 h cross x square plus a constant of integration. So, psi 0 of x and if you recall that we had by solving the linear harmonic oscillator problem, we introduced a dimensionless variable xi is equal to gamma x, where gamma was equal to under root of mu omega by h cross. So, you have log of psi 0 of x was equal to minus half xi square plus a constant. So, you will get if you write this (Refer Slide Time: 47:01) is the ground state wave function psi 0 of x within a multiplicative constant e to the power of minus half xi square.

Using operator algebra, we are been able to find to determine that we had obtained psi n of x as N of n, H n of xi, e to the power of minus half xi square. Now, for n equal to 0, so this will be N 0, since H 0 of xi will be 1, so e to the power of minus half xi square. So, we obtain the same wave function by using the fact that a psi 0, that since there is the half h cross omega is the lowest eigen value, we must have a psi 0 of x which will be 0 and therefore, we get the long state wave function.

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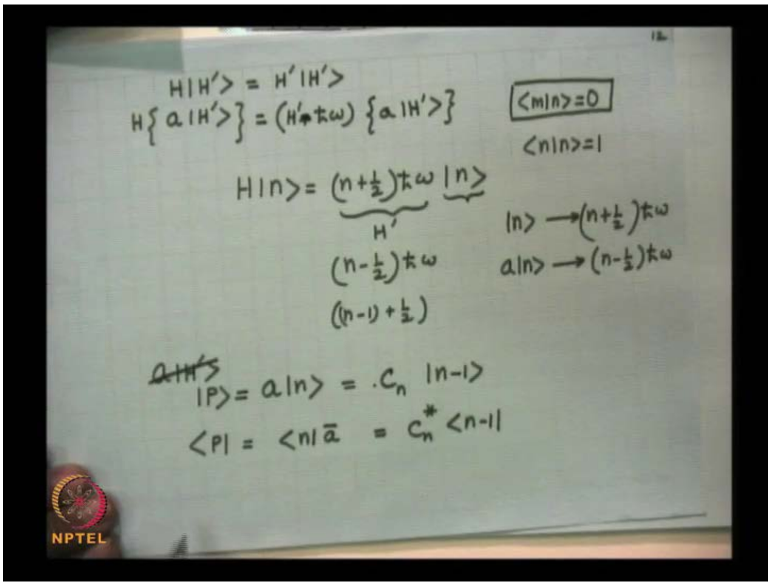

Now, we will do a little bit more algebra and we had said that a ket H prime is an eigen ket, if ket H prime is an eigen ket belonging to the eigen value H prime, then a ket H prime is an eigen ket belonging to H prime plus h cross omega. Sorry, it is minus h cross omega; a ket H prime. Therefore, we had written down H ket n is equal to n plus half h cross omega. So, this is my H prime. Therefore, if I reduce minus h cross omega, so this becomes n minus half h cross omega. So, this is equal to n minus 1 plus half.

Therefore, a ket H prime or rather a ket n, must be a multiple of ket n minus 1. I hope you all understand this; ket n corresponds to the eigen value n plus half h cross omega. And a ket n, if it is not a null ket, then this corresponds to an eigen value of n minus half h cross omega. So, n has been reduced to n minus 1, therefore, a ket n must be a multiple of n minus 1.

Let me write this down as ket P, therefore bra P is equal to n a bar, is equal to C n star, bra n minus 1. I forgot to tell you one thing that since these eigen kets belong to different eigen values and since there is only one eigen ket belonging to one eigen value; different eigen kets belonging to different eigen values, must be orthogonal; m n must be 0. We had proved this that since H is a self adjoint operator, eigen kets belonging to different eigen values must necessarily be orthogonal. I can always multiply each one of them such that n n is 1. So, we had the orthonormality condition.

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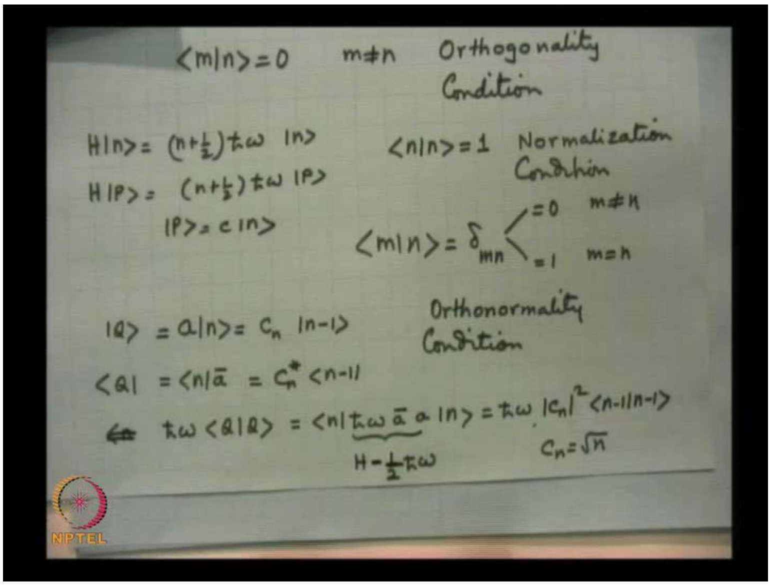

So this is the m n equal to 0, for m not equal to n. This is the orthogonality condition and this follows from the fact that we have different eigen values and eigen vectors corresponding to different eigen values must necessarily be orthogonal. We have proved that eigen vectors belonging to different eigen values must necessarily be orthogonal and since you have H ket n is equal to n plus half h cross omega ket n, and you can multiply the equation by c. Let us suppose, some constant H times ket P is equal to n plus half h cross omega ket P, where ket P is a multiple of n. So, any multiple is also an eigen ket. So, you can always choose the multiplicative constant such that bra n ket n is equal to 1.

This is the normalization condition and I can put this two together to write down such as m, n is equal to delta m n and that delta m n is equal to 0, if m is not equal to n; is equal to 1 if m is equal to n. This is known as the orthonormality condition.

Therefore, we had here that a ket n must be for the lower eigen value, a multiple of n minus 1. So, if I denote this by ket Q let us suppose, then bra Q is equals to n a bar is equal to c n star bra n minus 1. If I multiply this by h cross omega from both sides, so I get bra n bra Q sorry sorry h cross omega bra Q ket Q, which is equal to bra n h cross omega a bar a ket n. This is equal to h cross omega mod c n square bra n minus 1 ket n minus 1. This is equal to 1 and this is (Refer Slide Time: 54:59) equal to H minus half h cross omega.

From this point onwards, we will continue in our next lecture, so we will operate this H on ket n and we will find that there is n plus half h cross omega, so half and half cancels out. You will be left with n square and so we will get C n is equal to square root of n. You may just try to work this out yourself and so that when we come for our next lecture we will go through those. Those are very extremely important relations and they are of extremely importance not only in quantum mechanics but, in laser theory, in quantum optics and in many other diverse areas.

Thank you.