Basic Quantum Mechanics
Prof. Ajoy Ghatak
Department of Physics
Indian Institute of Technology, Delhi

Module No # 03
Linear Harmonic Oscillator-I
Lecture No # 02
Linear Harmonic Oscillator (Contd.)

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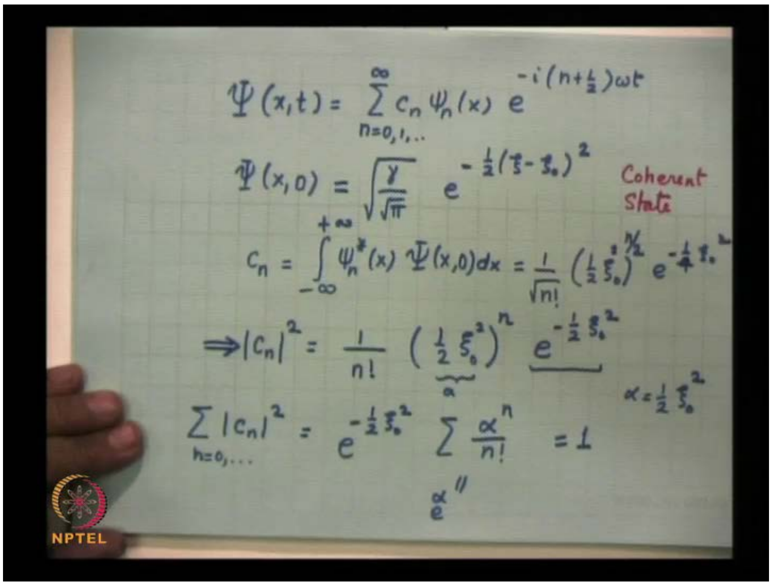

Continue our discussion on the solutions of the linear harmonic oscillator problem. In my last lecture, I had shown that the most general solution of the one-dimensional Schro dinger equation is given by c n psi n of x e to the power of minus i n plus half omega t where n goes from 0 to infinity. Then, we said, now let us assume that at time t equal to 0 the wave function of the oscillator associated with the oscillator is given by e to the power of minus half xi minus xi 0 whole square. Such a state is known as a coherent state of the oscillator, now, then we can find out the corresponding to psi of this form of the wave function at t equal to 0, I can find out c of n. And that will be psi n star of x and multiplied by psi of x comma 0 d x.

All limits are from minus infinity to plus infinity. If I carry out this integration as I mentioned last time one can get an analytical expression one over n factorial half xi not square half xi not square sorry half xi naught square raise to the power of 1 by 2 e to the power of minus half xi square sorry one quarter. So, therefore, mod c n square will be equal to 1 over n factorial half xi naught square raise to the power of n e to the power of minus half xi naught square. You can now see, that if i sum mod c n square over n and let us suppose this, I denote by alpha. So, this factor I take outside. So, e to the power of minus half xi naught square. So, this will be alpha to the power of n divided by n factorial, where alpha is equal to half xi 0 square.

So, this is just e to the power of alpha or e to the power of half xi naught square. So, that will cancel out with this, and you will get one. So, once i n c n square represents the probability of finding this system, the oscillator in the nth state. Now, I have an expression. So, this is normalization condition that n is equal to 0 1 2 3 to infinity, this is equal to 1, and that what it should be expected because the wave function initially at t equal to 0 is normalized . So, therefore, I asked myself that what will be the value of energy that I will get the answer is, that you can get any of the energies and there is a probability c n square for obtaining the n th Eigen energy.

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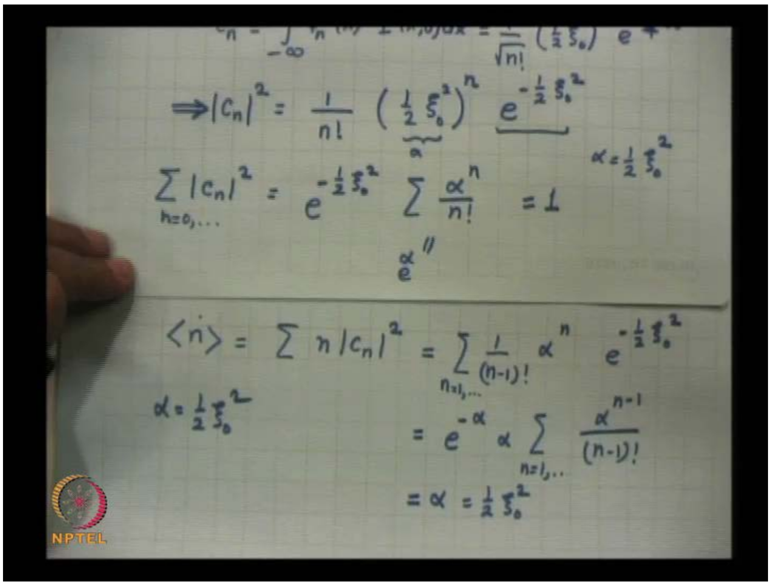

So, as we know the possible energy levels are e is equal to e n is equal to n plus half h cross omega. So, the expectation value of e that you will get will be the expectation value of n plus half within brackets multiplied by h cross omega. And, what is the expectation value of n? This will be summation n mod c n square and I have just now written the value of mod c n square. So, if you see this, that mod c n square is given by this. So, if you some this up, then it will be n by n factorial. So, that is n minus 1 factorial. So, n will go from 1 to infinity and this is alpha to the power of n, alpha to the power of n and e to the power of minus half xi naught square, I can take outside. So, this is e to the power of alpha. So, I take alpha outside.

So, e to the power of minus alpha outside and alpha outside. And so this will be alpha to the power of n minus 1 by n minus 1 factorial. So, this is n equal to 1 2 3 where alpha is equal to half xi naught square. So, this is e to the power of alpha. So, therefore, this will come out to be alpha which is equal to half xi naught square. So, this is the value, average value of n that you will get, similarly you can write down what is the expectation value of n square? I leave that as an exercise.

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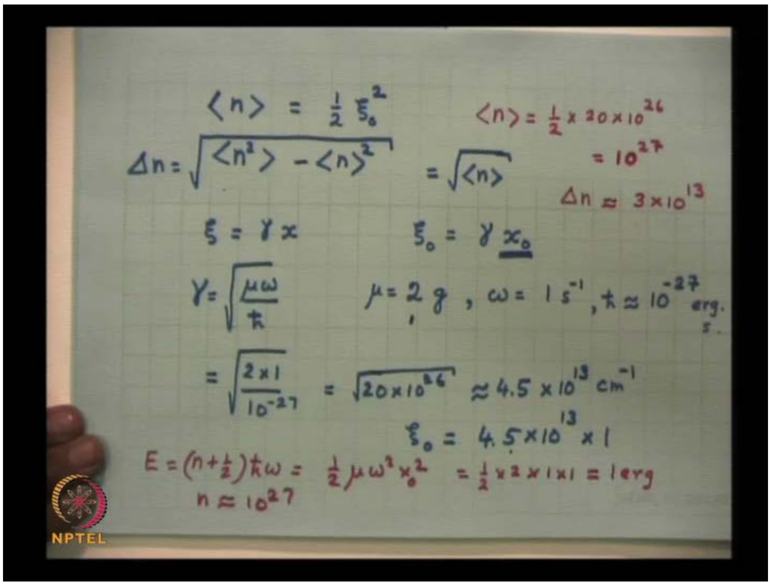

And, final result will be that the spread in the, so, the expectation value of n will be equal to half xi naught square. Similarly, you can calculate what is n square and then let us suppose the spread in the value of n, that is delta n is equal to this, I leave this as an exercise for you to calculate the expectation value of n square. And you will find that this will be square root of n, that is xi 0 divided by under root of 2. Now, let me give you a let me do some numerical examples. So, you will have let us suppose, that xi naught, you remember that xi is equal to gamma x. And so, therefore, xi naught is equal to gamma of x naught, now gamma, if you remember that was equal to underfoot of mu omega by h cross.

Now, let me take a very simple example that mu is equal to, let us suppose two grams, I am using c g s system of units, omega is equal to 1 second inverse and h cross is approximately 10 to the power of minus 27, it is 1 point something erg second. So, if you substitute it here. So, I consider a classical oscillator, the mass is two grams, omega is the time the 2 pi by t is about one. So, the time period is of the order of 1 second and h cross is of course, the Planck's constant divided by 2 pi is 10 to the power of minus 27 erg second. So, if calculate this. So, this will be 2 into 1 divided by 10 to the power of minus 27. So, this will be 2 into 10 to the power of 27. So, under root of 20 into 10 to the power of 26. So, this is about 4.5 under root of 20 into 10 to the power of thirteen meter inverse, centimeter inverse, sorry.

So, you see the value of xi naught is very large. So, if the amplitude of my displacement of the displacement of the oscillatory, say one centimeter. So, xi naught is equal to gamma times x naught. So, x xi naught will be gamma is 4.5 into 10 to power of 13 into x naught. Let us suppose it is one centimeter. So, I have a particle of mass 2 grams having a time period of the order of 1 second and the amplitude is 1 centimeter. So, the value of xi naught is very large. So, this is equal to, so therefore, the value of expectation value of n, please see this, the average value of n will be half xi naught square, 4.5 square was actually 20. So, 20 into the 10 to the power of 26. So, this is above 10 to the power 27.

So, the states which have the quantum number of the order of 10 to the power 27, it is excited and this is expected, because you see e is equal to n plus half h cross omega and the total energy of the classical oscillator, as you know is half mu omega square x, square sorry, x naught square which is amplitude. So, mu is 2 grams. So, half into 2 omega is 1 and x naught is 1. So, this is 1 erg h cross is 10 to the power minus 27. So, n must be 10 to the power of 27, this is what we get. So, therefore, we are exciting states with very large quantum number, this is actually, what I am discussing is Bohr's correspondence principle. That in the limit of classical mechanics, we have extremely large quantum numbers.

And, then how many states the delta n, the number of states which are approximately getting excited around 10 to the power of 27, is square root of that? So, this is of the order of square root of that will be 3 into 10 to power of 13. So, this is 30,000 US billion states or 30 trillion states.

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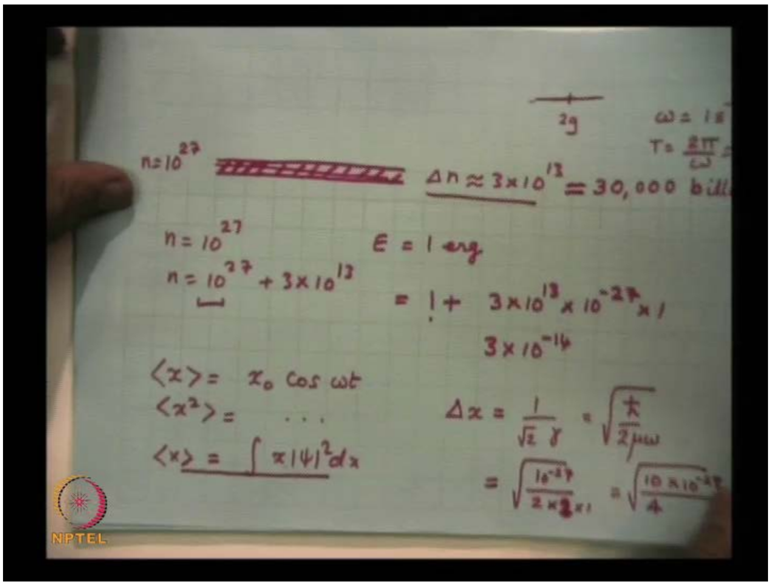

So, I hope I am able to make you understand that you have this is the ground state of the oscillator and this is n equal to 1, n equal to 2, n equal to 3. And you come here and it is n is equal to 10 to the power of 27. So, there are so many billions and billions and billions of states are there. Here a very large number of states get excited, this is the delta n, delta n is very large, delta n is of the order of 3 into 10 to the power of 13. So, this is about, this is equal to actually thirty into 10 to the power of 12, that is 30000 d billion states, 30 trillion states. But the spread of energy is very small because this is only 10 to the power of minus 13 of 10 to power of twenty seven. So, as n goes from 10 to the power of 27 to n to the power of 27 plus 3 into 10 to the power of 13.

The change in energy is extremely small, because here E I know was 1 erg and this will be 1 erg plus 1 plus how much? With if I take an n value of 3 into 10 to the power of 13 into h cross omega h cross is 10 to the power of minus 27 omega is 1. So, this is 3 into 10 to the power of minus 14. So, we are adding 1 to this number, which is extremely small in comparison to 1. So, therefore, the energy of the classical oscillator is very precisely defined, delta e is very small, delta x is very small, delta p is very small. And, that is the domain of classical mechanics in the domain of large quantum numbers. We have classical mechanics, we had in the just before we finished our last lecture, we had said that x was equal to x naught cos omega t and similarly we had found out s naught square x square is equal to something.

So, that delta x, we had found to be equal to 1 over square root of 2 gamma. Now, what was gamma equal to so this was mu omega by h cross. So, this will be h cross by 2 mu omega. Similarly, one can calculate delta p also. So, what was x equal to x was equal to x was equal to integral x mod psi square d x from minus infinity to plus infinity and that is a very straightforward integration to calculate. So, let me calculate delta x, I am assuming the amplitude to be one centimeter a 2 gram mass having a time period omega is equal to 1 second inverse. So, the time period is 2 pi by omega. So, about 6 seconds is the time period and the amplitude is 1 centimeter. So, my uncertainty in x is 10 to the power of minus 27 divided by 2 into half into 1 sorry 2 into 2 sorry, mu is 2 grams. So, this will be 4. Let us suppose I take 10 outside 10 into 10 to the power of minus 28 and this is 4.

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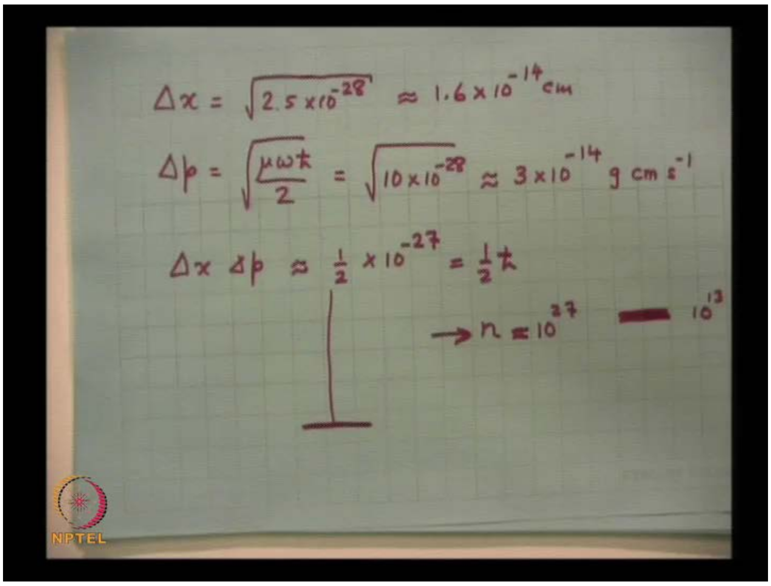

So, my delta x delta x comes out to be, please see this carefully. So, this comes out to be 2.5 into 10 to the power of minus 28 centimeters. So, 2.5 is about say 1.6 into 10 to the power of minus 14 centimeter. So, this is the uncertainty in x, similarly 1 can calculate the uncertainty in p. So, that comes out to be under root of mu omega h cross by 2, mu is

2 grams. So, 2 cancel out omega is 1 h cross is 10 to the power of minus27. So, this is 10 into 10 to the power of minus 28. So, this is about 3 into 10 to the power of minus 14, the unit of momentum will be gram centimeter per second, and if you multiply this out. So, this will come out to be delta x, delta p is about 5, that is half into 10 to the power of minus 27.

So, this is equal to half h cross, it is a very important result that I have derived, that for my classical oscillator, delta x delta p is of the order of h half h cross. So, is my uncertainty principle applicable to the classical oscillator that I see in my first year lab the answer is of course, yes. But both position and momentum are determined with a tremendous degree of accuracy, and that is a consequence of the fact that the value of h cross is extremely small. So, once again when I have a pendulum and I make it vibrate make it oscillate like this, then I asked myself to what state of the oscillator does it belong to? The answer is, it does not belong to a particular state. It belongs to a superposition of states, and then you tell me, what states have been superposed?

And the answer is, we are superposing states for which the quantum number is 10 to the power of 27 about. And in that quantum number the spread in n is 10 to the power of 13, 3 into 10 to the power of 13. So, for n equal to 10 to the power of 27 plus minus 10 to the power of 13. So, that is the number of states we are exciting hundred trillion states, around n equal to 10 to the power of27. So, although we are exciting a tremendously large number of states, but because h cross omega is an extremely small quantity. So, therefore, the energy is also very precisely defined. And, delta e by the spread in the energy divided by the expectation value of energy is extremely small, 10 to the power of minus 13 or so. So, that is my classical oscillator.

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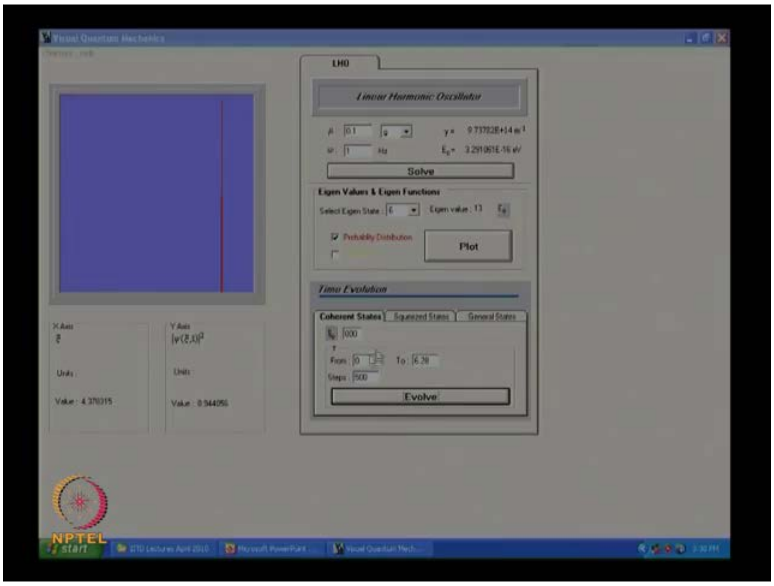

So, therefore, I will again go back to the slide that I had shown, that is very important. So, this is my coherent state and let me evolve. So, it is a packet it is a wave packet obtained by superposition of a thousand trillion states. So, here I have taken only xi 0 value which is about 1 thousand or something like that, but we have just now found that the value of xi 0 is extremely large. So, this packet will become extremely small in width and this is how the packet will evolve with time? So, I hope by now, you know the relationship of the classical oscillator, to the quantum oscillator. In quantum mechanics the energy levels are quantized, but the classical oscillator that we see in our first year lab, it corresponds to n equal to 10 to the power of 27. And the spread in the n value is also large, but it is extremely small in comparison to the value of n and so therefore, you will have an almost the energy is very precisely defined.

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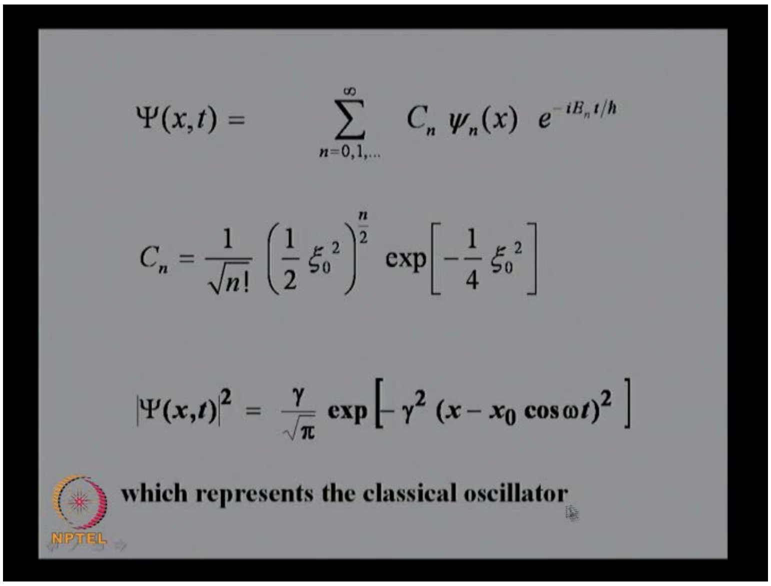

So, let me mention one more thing that we had discussed this, but this is the value of C n that we obtained.

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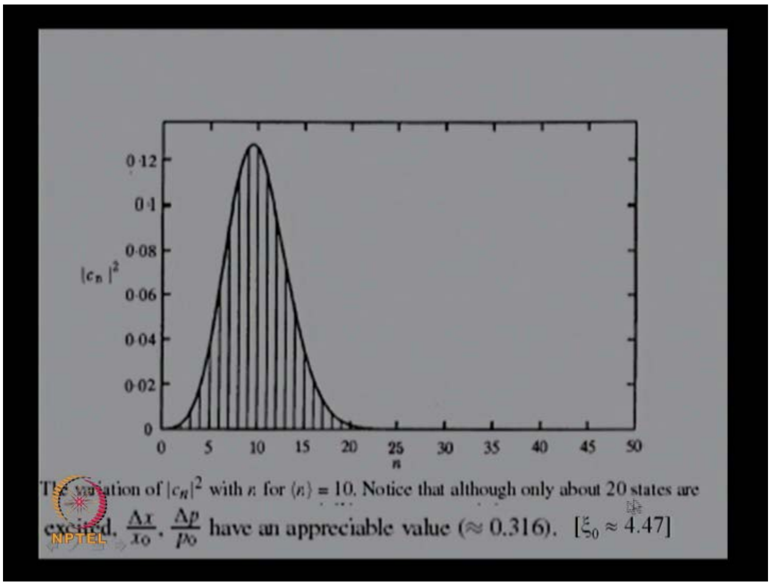

If for example the expectation value of n is 10, then these are, this is the on the vertical axis you have C n square and the horizontal axis it is n. So, the probability is maximum for around n equal to 10. So, these are the number of states that get excited about 10 12 of them, when average value of n is 10. For a classical oscillator, average value of n is 10 to the power of 27.

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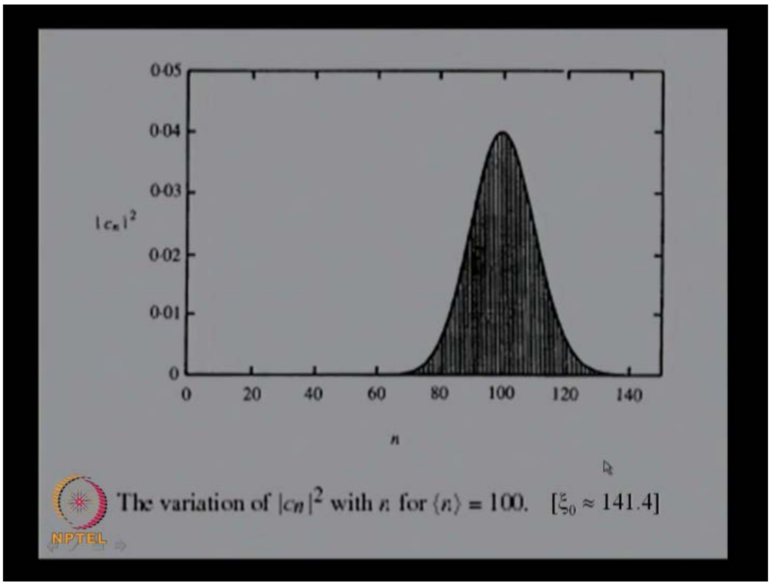

So, when n average value of n becomes 100, then it occupies about 100 states around n equal to 100. So, average the spread in n is about 10.

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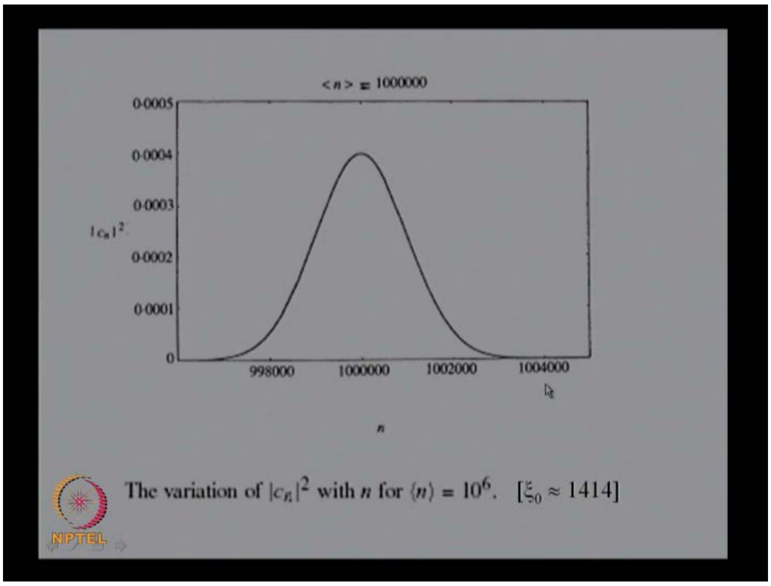

When n becomes 1 million, then around 1 million so many states are getting occupied the number of states are large, but the energy spread is extremely small. So, we finish this here. Now, we will go back to the solution of the Schrodinger equation. However, before we start solving the Schrodinger equation, I would like to solve the differential equation which is known as the confluent hyper geometric equation. And, the solution of which is not only important for the harmonic oscillator problem, but also for the hydrogen atom problem for the three-dimensional oscillator problem and many other problems. So, once if you understand, how to obtain the solution of the confluent hyper geometric equation. Then it is, it will be very easy for you to understand how to obtain the Eigen values corresponding to many problems of interest in quantum mechanics.

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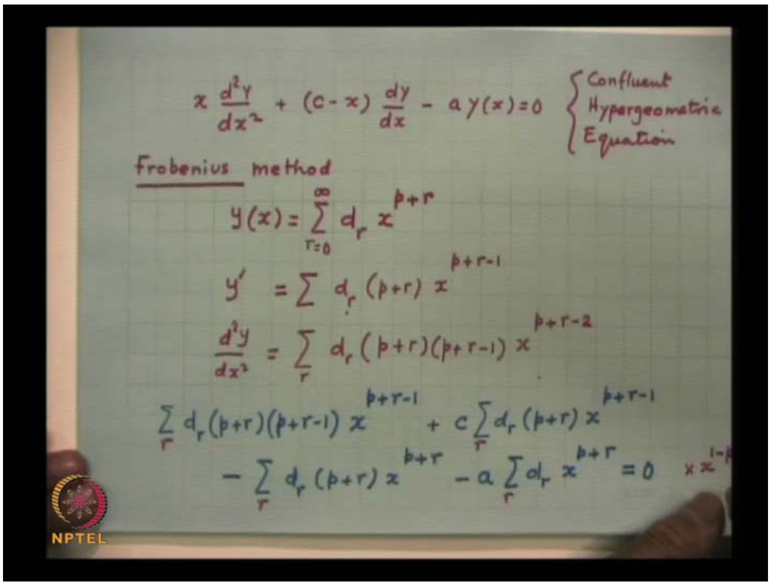

So, we try to solve the equation x d 2 y by d x square plus c minus x d y by d x minus a y of x is equal to 0. This equation where a and c are this is known as the confluent hypergeometric. Hypergeometric equation, actually the name is quite big, but the solutions are very easy to understand. So, I would just like you to patiently workout the solutions using the series solution method and once you understand that solution. The solution corresponding to the hydrogen atom problem and the harmonic oscillator problem and the three-dimensional oscillator problem will become very easy.

So, we will use the power series method which most of you may have already studied, this is also known as the Frobenius method. Frobenius method and the method involves, that I make a power series expansion of y of x in powers of x. So, I constants are usually it is written as c of r, where the since the constant c is appearing here. So, we will replace this by d r x to the power of p plus r. Where r goes from 0 to infinity, this is known, this is the fundamental equation which is used in the power series method expansion. So, I differentiate this once, and I differentiate this again. So, y prime of x which is equal to d y by d x the summations are all from 0 to infinity. So, this will be d s of r p plus r. I will do it carefully p plus r minus 1 and d 2 y d x square will be equal to summation r d r as I all of you know p plus r then I have to differentiate this.

So, I will get p plus r minus 1 x to the power of p plus r minus 2. Next step is I substitute this here; I substitute this here, and write down the equation. So, if I multiply by x. So, I will get summation d s of r, p plus r p plus r minus 1 x to the power of p plus r minus 1 because I multiplied by x. So, it becomes like this, plus c times this equation plus c times d y by d x. So, c times d y by d x that is d r 1 just has to do it patiently. I will do it once p plus r minus 1 minus x times this. So, x times this that is summation d r p plus r and x to the power of p plus r minus a y of x. That is a summation d r d r x to the power of p plus r equal to 0.What I do next is? I multiply the whole equation by x to the power of 1 minus p, the summation is over r summation is over r. So, I multiply the whole equation by x to the power of 1 minus p.

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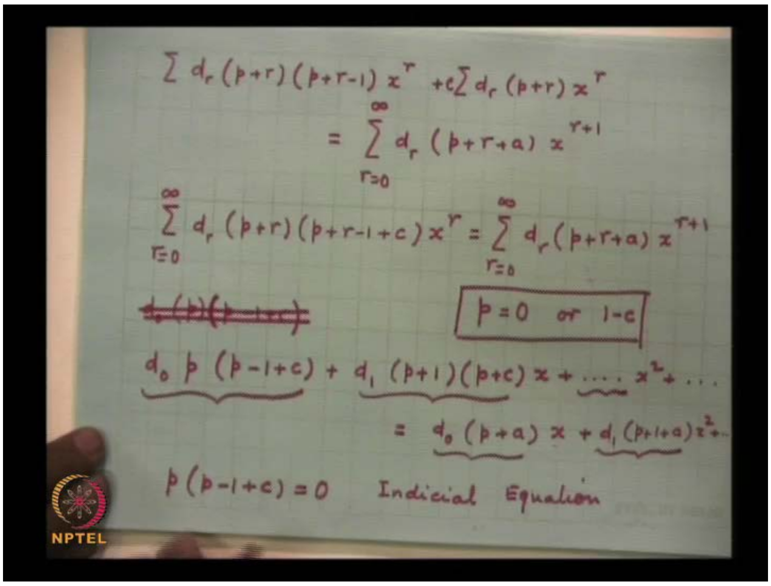

So, if I do that, what I will obtain is? The following summation d s of r I will have to write down this again p plus r p plus r minus 1 and the multiplication is 1 minus p. So, this is x to the power of r plus c d r p plus r x to the power of 1 minus p, that is x to the power of r minus, if I take these two expression on the right hand side. So, I will get is equal to d r p plus r plus a p plus r plus a times x to the power of r plus 1 r going from 0 to infinity, all of r's are going from 0 to infinity. So, just write these down. So, on the left hand side. So, let me combine these two. So, you will have summation please see this d's of r p plus r I can take common p plus r minus 1 plus c p plus r minus 1 plus c x to the power of r, r going from 0 to infinity.

And here it will be r equal to 0 to infinity d r p plus r plus a x to the power of r plus 1 on the left hand side. There is an infinite series on the right hand side there is an infinite series. Now, if this has to be valid for all values of x then the coefficient, all the coefficients must be equal.

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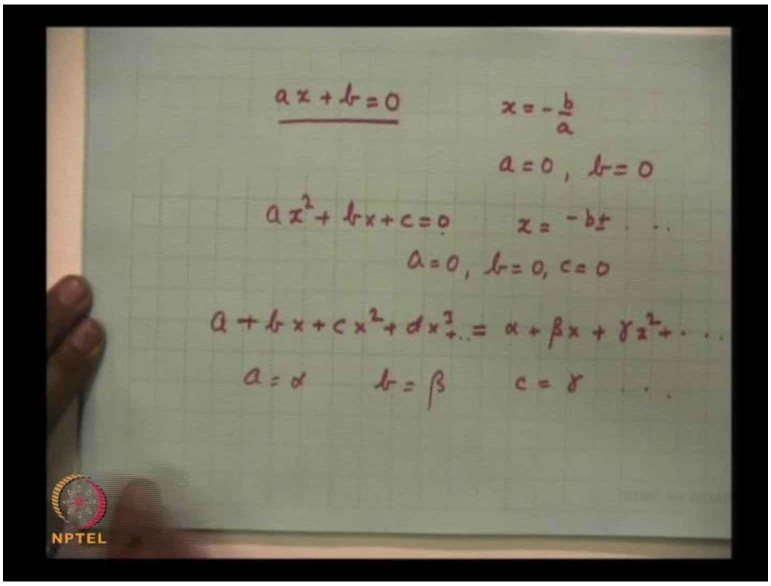

Now, let me illustrate this I am sure most of you know this, but let me illustrate this, that if I have an equation like this, say a x plus b is equal to 0. Then this equation is satisfied only at x is equal to minus b by a, for this equation to be satisfied for all values of x, you must have a is equal to 0 and b is equal to 0. Then and then only this equation will be valid for all values of x, let me give you another example I have to I have a quadratic equation a x square plus b x plus c equal to 0. Now, for given values of a b and c as you know the roots of the quadratic equation, you know minus b plus minus. So, on so forth this is satisfied for 2 values of x; however, if this equation is to be valid for all values of x then a must be 0, b must be 0, and c must be 0.

This is an identity, this has to be valid for all values of x. And so therefore, on the left hand side, let us suppose I have a plus b x plus c x square plus d x cube and so on. And, on the right hand side, you have alpha plus beta x plus gamma x square and so on. Then, for this to be valid, this equation to be valid for all values of x, a must be equal to alpha b must be equal to beta and c must be equal to gamma and so on. So, that is the property of an identity. So, you have here an equation. So, the first coefficient is d zero. So, d 0 r is 0. So, p multiplied p minus 1 plus c sorry x to the power of 0. So, let me scratch it out, sorry. So, I have d 0 p into p minus 1 plus c x to the power of 0, which is 1 plus d 1 p plus 1 r is 1 p plus c x plus something like x square plus x cube.

And on the right hand side, this is d 0 p plus a x plus d 1 p plus 1 plus a into x square and so on. So, the coefficient of this should be equal to this should be equal to this and since there is no x to the power of 0, here this should be 0, but d 0 cannot be zero. So, you must have p into p minus 1 plus c must be 0. This is known as the indicial equation, and therefore, we must have the roots of indicial equation are p is equal to 0 or 1 minus c, do these are the roots of the indicial equation. Then, you will have from this equation, if you write this down. I can rewrite this in the following manner.

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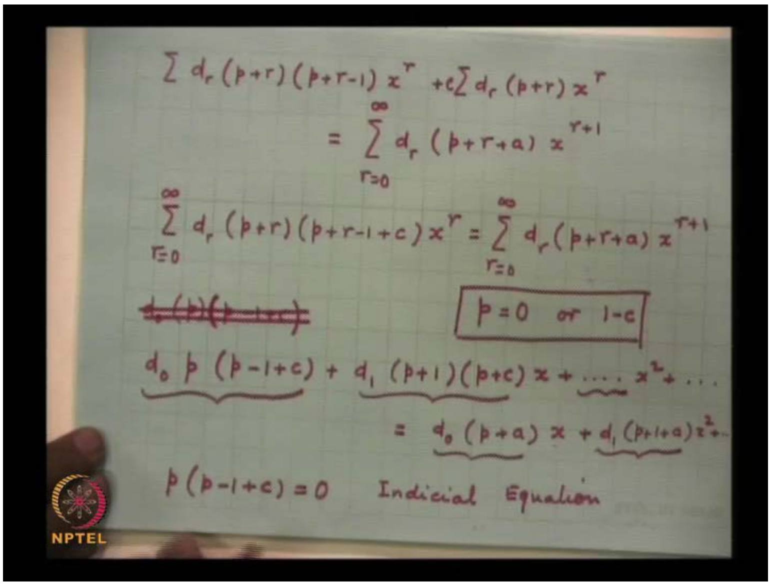

So, please see this, I am rewriting this equation and I will obtain from d r p plus r p plus r minus 1 plus c x to the power of r, this is equal to from r equal to 1 to infinity, d r minus 1 p plus r minus 1 plus a x to the power of r . So, now, I have equated that and so therefore, d r by d r minus 1 will be equal to p plus r minus 1 plus a divided by p plus r into p plus r, most of you may have done this before. So, I take this d r minus 1 here. So, this is known as the recurrence relation, and initially I had written out that the roots of the indicial equations are p can be either 0 or 1 minus c. Let me take the first value p equal to 0. So, you will have d r is equal to p equal to 0. So, therefore, this will be r minus 1 plus a divided by r into r minus 1 plus c d r minus 1.

So, you will have, please see this when you put r equal to 1. So, you have d 1 r equal to 1. So, 1 minus 1 is 0. So, a r is 1. So, 1 times r is 1. So, this is 0 times c into d 0 d 2. So, r is 2. So, 2 minus 1 is 1. So, this is a plus 1 divided by 2 into r is 2 minus 1 is plus 1. So, c plus 1 d 1 and d 1 is a into 1 into c into d zero. So, this comes out to be if I put a here a into a plus 1 divided by c into c plus 1 into 1 over 2 factorial 1 into 2 into d 0. Let me write down 1 more term and that will be something like this. So, I found out d 1.

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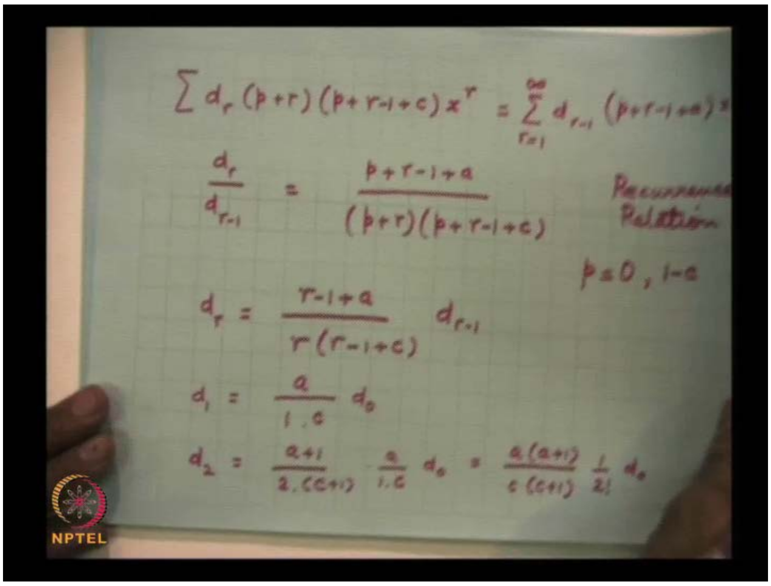

So, let me write down one more term d 3. So, r is 3. So, 3 minus 1 is 2. So, this is a plus 2 divided by 3 and 3 minus 1 is 2. So, 2 plus c into d 2. So, into d 2 and d 2 is given by this. So, you will get a into a plus 1 into a plus 2 divided by c into c plus 1 into c plus 2, and then it is multiplied 2 factorial multiplied by 3 is 3 factorial into d zero. So, you obtain the infinite series that is like this, f. You write this down as f 1 a c x this is 1 of the solutions which I written down that as d r x to the power of p plus r p is 0, I have assumed p equal to zero. So, this will become d r d r x to the power of r. So, you will have if I take d 0 x to the power of 0 plus d 1 x plus d 2 x square and so on. So, if i assume d 0 outside.

So, I get 1 plus x d 1 by d 0. So, that is a by c x by 1 factorial plus a into a plus 1 into c by c plus 1 x by 2 factorial plus a into a plus 1 into a plus 2 into c into c plus 1 x square. And, this becomes x cube by factorial 3, this infinite series when d 0 i assume to be equal to 1 is known as the confluent hypergeometric series.

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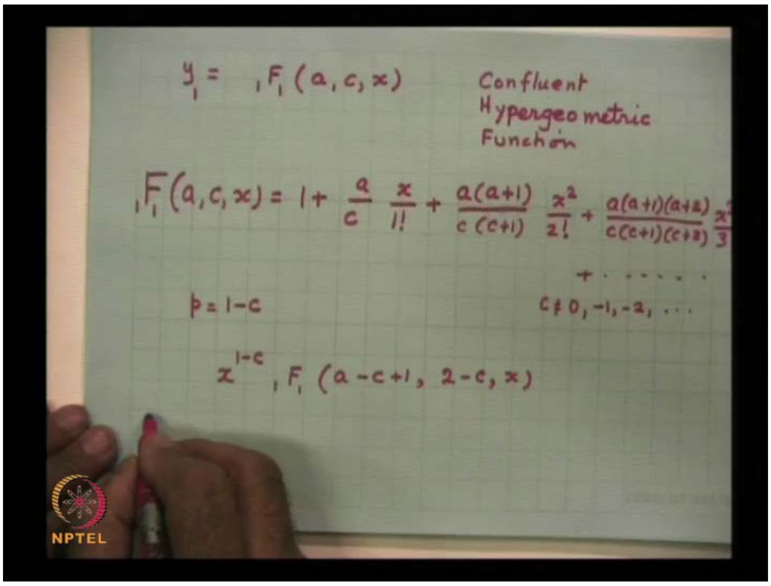

So, I have 1 of the solutions of the differential equation is y is equal to f 1 1 that is y 1 of x y 1 of x is f 1 one a c x, this is known as the confluent hypergeometric function. This is an extremely important function, and it is very easy to remember. So, f 1 a comma c comma x is equal to 1 plus a comma c x factorial 1 plus a into a plus 1 by c into c plus 1 x square by factorial 2 plus a into a plus 1 into a plus 2 c into c plus 1 into c plus 2 x cube by factorial 3 plus, this is an infinite series. Obviously, c cannot be 0 or a negative integer, the function will blow up this is 1 solution and the other solution as you can see is p was equal to 1 minus c and the other solution will be other solution will be x to the power of 1 minus c.

I leave that as an exercise f 1 1 a minus c plus 1 2 minus c comma x and if, actually, if c is a positive integer. Say let us suppose c is equal to 3, then this becomes negative. So, that will blow up. So, the general solution of the differential equation is given by the following.

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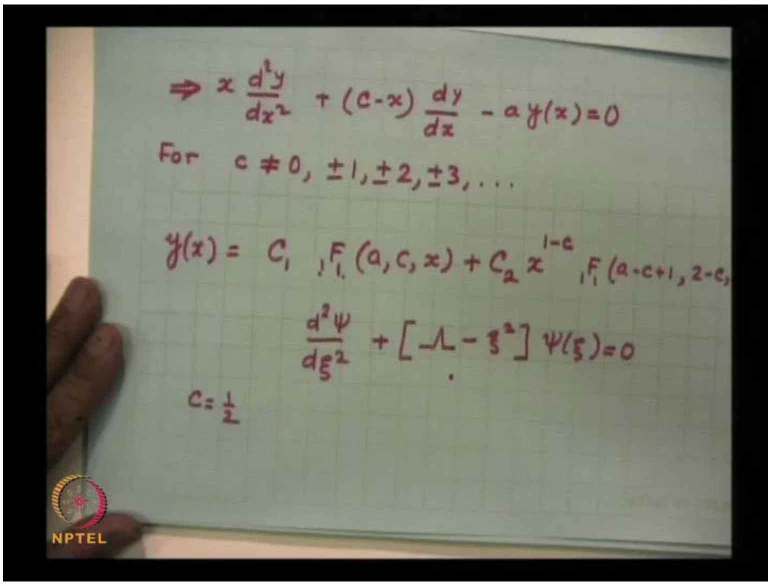

So, we started out with the equation x d 2 y by d x square plus c minus x d y by d x plus minus a y of x, for this you must remember for c not equal to 0 plus minus 1 plus minus 2 plus minus 3. The general solution of this equation is c 1 f 1 one a c x plus c 2 x to the power of 1 minus c f 1 1,there is a superscript, there is a subscript on both on the left as well as on the right of f. So, this will be f 1 1 a minus c plus 1 into comma 2 minus c into x.

What we will do in our next lecture is that, you remember that, we had the equation like this d 2 psi by d xi square. The Schrodinger equation was equal was given by lambda minus xi square, we will transform this equation into an equation which is hyper geometric equation. And, we will find that the values are c is equal to half small c become equal to half. We will transform this equation into a differential equation of this type, and we will obtain the solutions of for the linear harmonic oscillator problem. So, with this, we end today's lecture, and we will request that that you go through the solution of the confluent hyper geometric equation. And that is given in any book on mathematical physics, and go through that before you come to the, in any case the analysis that I have given is complete by itself.

And, next time I will talk on the on the behavior of the series for large values of x and we will obtain the solutions of the harmonic oscillator problem. Thank you.