Quantum Electronics
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Module No.# 05
Lecture No.# 32
Quantum States of EM Field
(Contd.)

We continue with our discussion on quantum states of electromagnetic field. We started to discuss coherent states in the last lecture, towards the end. You have any questions from the last lecture?

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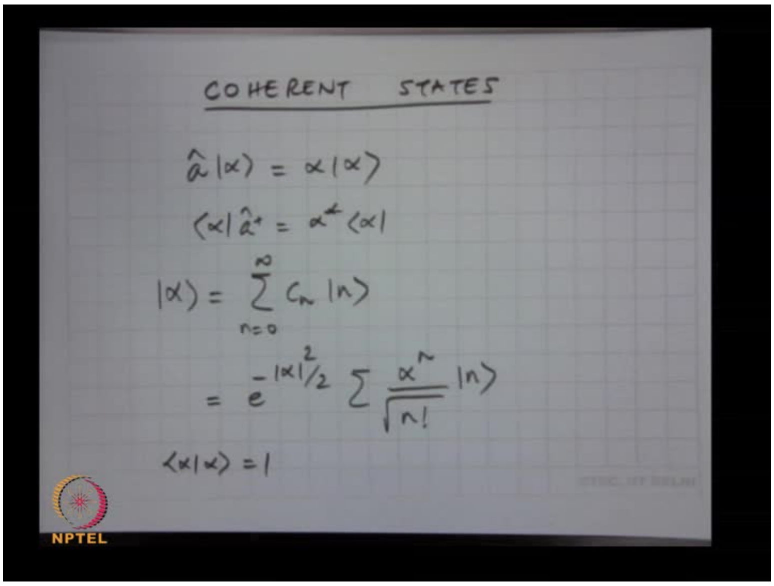

So, let us continue with our discussion on coherent states. Let me recall, we are looking at single mode states that means, states in which the electromagnetic field corresponding to one mode is exited. This particular coherent state is a state, which is defined as the eigen state of the annihilation operator a.

Now, because a is not annihilation operator, alpha can be complex, the eigen values alpha can be complex. So, this equation implies alpha a dagger is equal to alpha star. In the last class, we wrote this alpha as a superposition of the n ket states. Because, n ket states from a complete set of function, complete set of states, you can expand any state in terms of these n ket states and c n of the expansion coefficients.

We had calculated the values of c n in the last class. We have shown that this is a exponential minus mod alpha square by 2 sigma alpha raise power n by square root of n factorial get n. Alpha satisfies of course the normalization condition, alpha alpha is equal to 1.

So, this is a state, which is a superposition of the various n ket state. n ket state is a state, which is an eigen state of the Hamiltonian operator. To measure the energy of a n ket state, you will always precisely get the same value every time. This is a superposition of n ket state, let me try to calculate what is the probability of observing n photons in the state.

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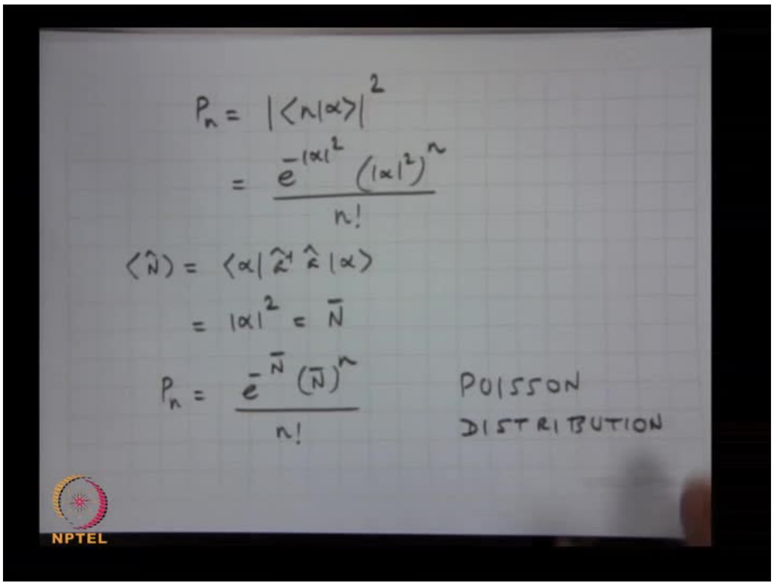

p n is equal to mod n alpha; if you use this expression for alpha, you can show that this is minus mod alpha square; mod alpha square this per n by n factorial.

Now, I can relate alpha to the expectation value of the number of photons. So, let me calculate the expectation value of n operator, this is alpha a dagger a alpha. a alpha is alpha alpha, alpha a dagger is alpha star alpha, because alpha is normalized, so this is mod alpha. This is the expectation value, which I just wrote as the average number of photons n bar. So, the probability observing and photons is exponential minus n bar n bar raise power by n factorial, this is the Poisson distribution.

The probability observing n photons is given by this equation. For example, the probability observing 0 photon is exponential minus n bar, n is equal to 0. So, probability observing 10 photons is exponential minus n bar, n bar raise power 10 by 10 factorial. So, this is a state of superposition of the n ket states. What you have calculated if the probability observing a certain number of photons in that state?

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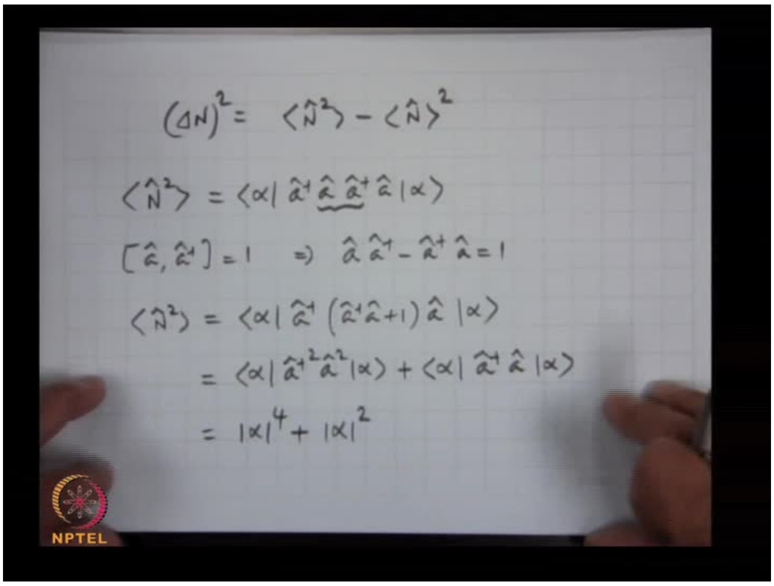

Now, let me calculate what is the variance in the number of photons in that state? So, remember, variance is given by n square expectation value minus n expectation valve whole square. So, for this, I need to calculate n square expectation value, so n square expectation value alpha a dagger a a dagger a alpha.

Now, I have this relation a a dagger is equal to 1. So, this implies a a dagger minus a dagger a is equal to 1. So, I replace this a a dagger by 1 plus a dagger a and I get n square expectation value is alpha a dagger into a dagger a plus 1 into a alpha.

So, this is equal to alpha a dagger square a square alpha plus alpha a dagger a alpha. a operating twice on alpha, gives me alpha square, a dagger operating twice on alpha bra, gets me alpha star square. So, this is mod alpha raise power 4 plus, this is mod alpha square that expectation value of n square; expectation value of n is mod alpha square.

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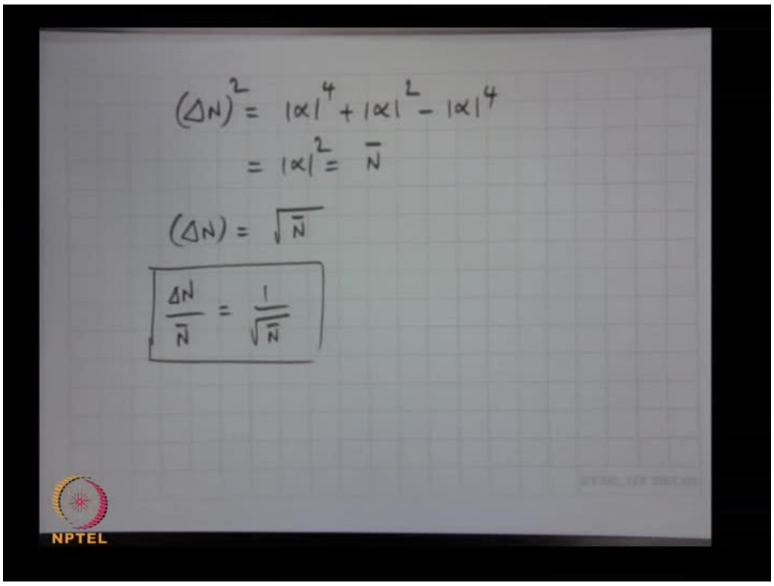

So, I can substitute these two expressions in this equation, to find out what is the variance in the number of photons in this coherent state. That comes out to be, so delta n square is equal to mod alpha 4 plus mod alpha square minus mod alpha 4, which is equal to mod alpha square, which is equal to n bar. So, the variance in the number of photons in this state is equal to the average number of photons in that state. The uncertainty in the number of photons is square root of n bar. So, this means that the fractional uncertainty in the number of photons is given by 1 by square root of n bar.

So, if you have an electromagnetic field generated in the single mod coherent state, is fractional, uncertainty in the number of photons in that state decreases as n bar increases. That means, as alpha increases, as mod alpha increases, the fractional uncertainty in the number of photons in that state decreases, is given by, decreases as 1 by square root of n bar; this is a characteristic feature of Poisson distribution.

Now, to relate this coherent sate to a classical electromagnetic field, let us try to calculate what are the expectation values of electric field in this state?

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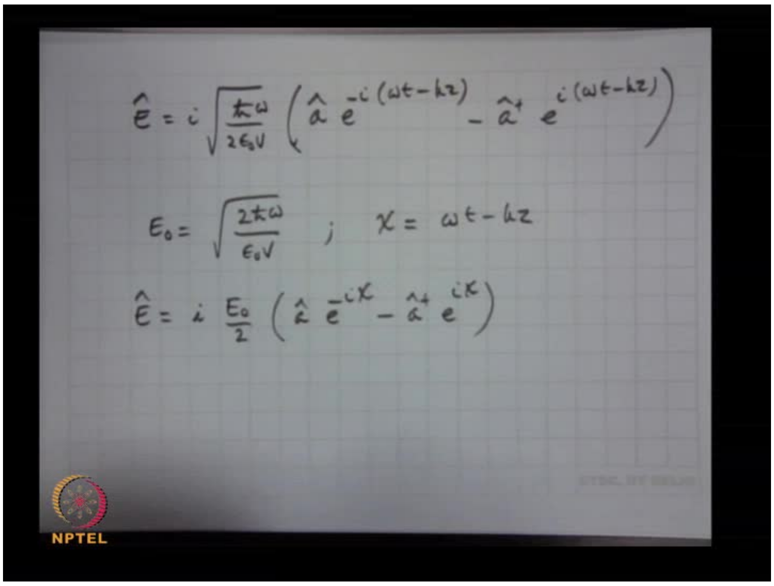

Recall, the electrical field operator is given by i times square root of h cross by h cross omega by 2 epsilon 0 V into a exponential minus i omega t minus k z minus a dagger exponential i omega t minus k z. We are assuming the snood, the coherent state to be in a mod, which is propagating along this z direction. It is linearly polarized, so I am not writing any vector here, so it is a scalar electric field. Now, let me define two quantities E naught, which is 2 h cross omega by epsilon 0 E V and chi, which is omega t minus k z.

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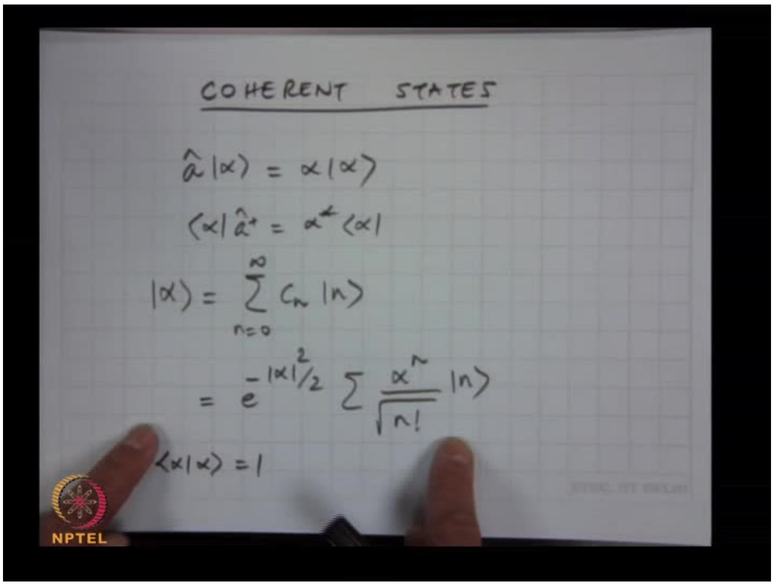

So, in terms of these two expressions, the electric field operator becomes I times E naught by 2 a exponential minus i chi minus a dagger exponential i chi. Please note that this electric field is a time dependent operator, we are using the Heisenberg picture in this electric field operator, is a time dependent operator. The state is time independent, given by this expression for this.

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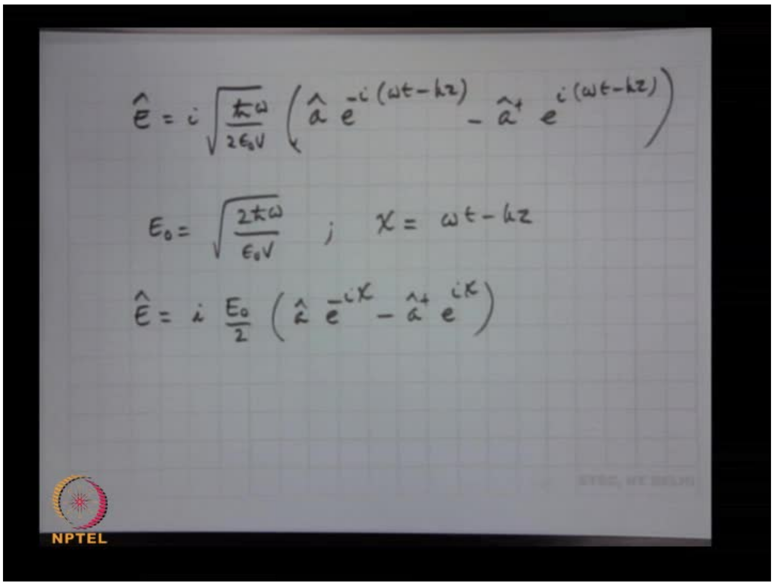

You can do the same calculations in the Schrodinger picture, all expectations values that we are calculating will exactly have the same values in the Schrodinger picture. So, this is an electric field operator, chi contents time, z depends here. So, let me calculate what is the expectation value of the electric field in the coherent state?

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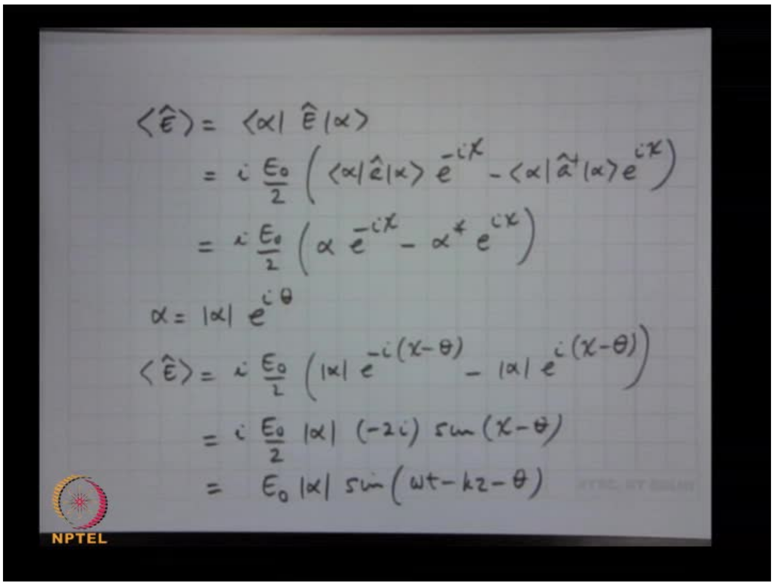

Expectation value of E is equal to alpha E alpha, which is equal to i E naught by 2 alpha a alpha exponential minus i chi minus alpha a dagger alpha exponential i chi, which is equal to i E naught by 2. What is this? This is alpha into alpha alpha, which is 1 minus i chi minus alpha a dagger is alpha star alpha, so this is alpha star exponential i chi.

So, alpha is complex, so let me write alpha is equal to mod alpha exponential i theta. So, expectation value of E becomes i E naught by 2mod alpha e to the power minus i i minus theta minus mod alpha e the power i i minus theta, which is equal to i E naught by 2 mod alpha. This reaming factor within this bracket is minus 2 i sin chi minus theta.

So, this two cancels off, i into minus i is 1, so E naught mod alpha sin omega t minus kz minus theta. So, notice is that the expectation value in the electric field varies a sin omega t minus kz minus theta, exactly like a classical electromagnetic field.

The amplitude of this is given by E naught times mod alpha. If you recall, when we looked at the n ket state, the electric field expectation value of n ket state was equal to 0. So, the n ket state is an eigen state of the Hamiltonian operator, it as a fixed number of photons exactly a well-defined number of photons. The expectation value electric field was 0 there.

That is not a representation of the classical electromagnetic field. This state alpha, which we are looking at the coherent state, which is an eigen state of the annihilation operator

a. If you prepare a state in this, if you prepare a electromagnetic field in this state, expectation value of the electric field in the state is E naught mod alpha sin omega t minus kz minus theta, varies in exactly the same passion as a cluster at the electromagnetic field.

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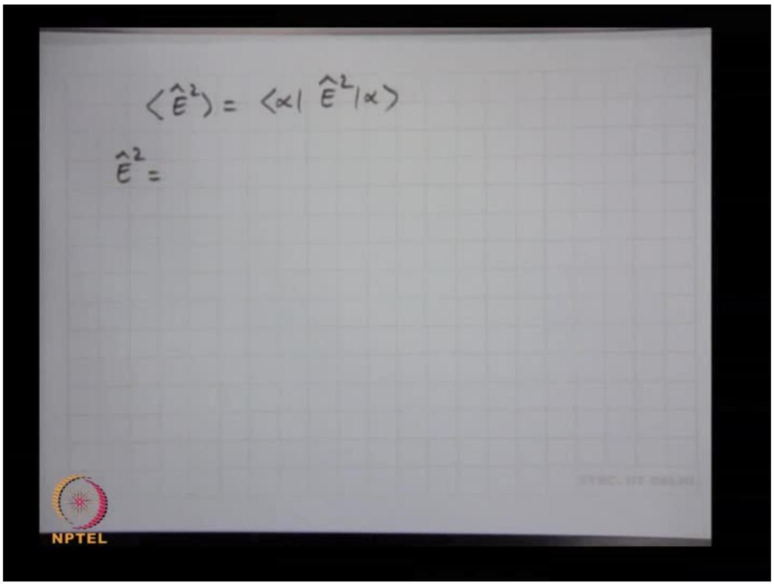

Of course, this is a quantum field; it is not a classical field, because as you can see now, as you will see, the electric field as uncertainties, this is the expectation values of electric field to find out what is the uncertainty in the electric field. We need to calculate expectation value in E square. So, for example, let me calculate, the expectation value of E square. This is alpha, E square alpha, what is E square?

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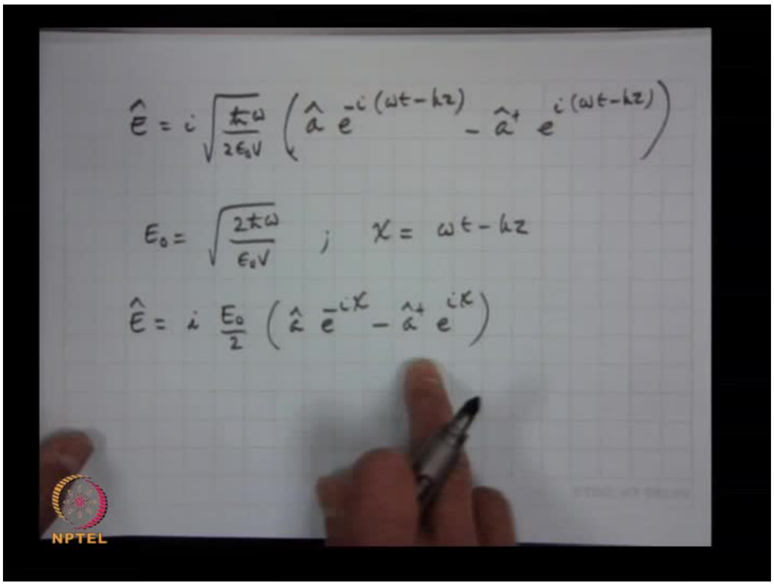

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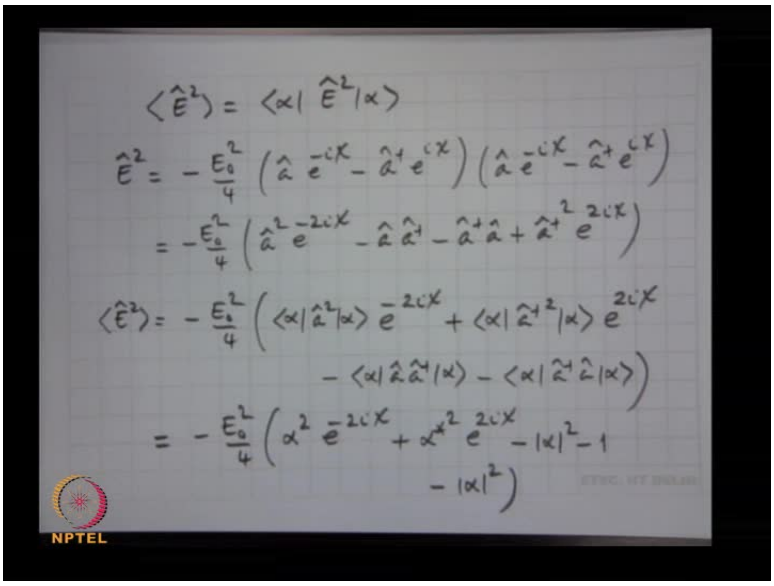

Now E, remember, we had written as - E was written in terms of a and a dagger like this. So, e square is minus e naught square by 4 a exponential minus i chi minus a dagger exponential i chi into a exponential minus i chi minus a dagger exponential i chi. That is equal to minus E naught square by 4 a square exponential minus two i chi minus a a dagger minus a dagger a plus a dagger square exponential 2 i chi.

So, expectation value of E square is minus E naught square by 4 alpha a square alpha exponential minus 2 i chi that we bring it in the front, so plus, alpha a dagger square alpha exponential 2 i chi minus alpha a dagger alpha minus alpha a dagger a. So, this is equal to minus E naught square by 4, so this is nothing but alpha square. So, alpha square exponential minus 2 i chi, this is alpha star square exponential 2 i chi.

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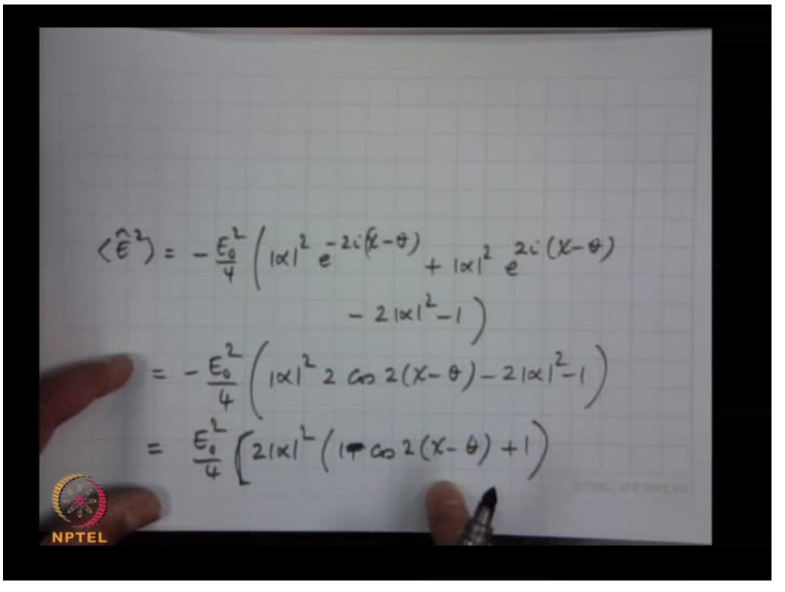

Now, please note that a dagger alpha is not alpha star alpha. a dagger operates on bra alpha, I need to replace a a dagger in terms of a dagger a by using the commutation relation. a a dagger is a dagger a plus 1, so I get minus alpha a dagger a alpha which is small alpha square and its minus 1 minus mod alpha square.

This a dagger a expectation value is small alpha square, a a dagger expectation value is mod alpha square plus 1. Expectation if E square becomes, expectation value E square is minus E naught square by 4. This is mod alpha square exponential minus 2 i chi minus theta, because alpha square is mod alpha square exponential 2 i theta, so I take it is here, plus mod alpha square exponential 2 i chi minus theta minus 2 mod alpha square minus 1, which is equal to minus E naught square by 4.

So, this is mod alpha square into some of these two exponential, is 2 cosin of 2 chi minus theta minus 2 mod alpha square minus 1. Let me take the minus sign inside, so I get E naught square by 4 2 mod alpha square into 1 plus cos 2 chi minus theta plus 1.

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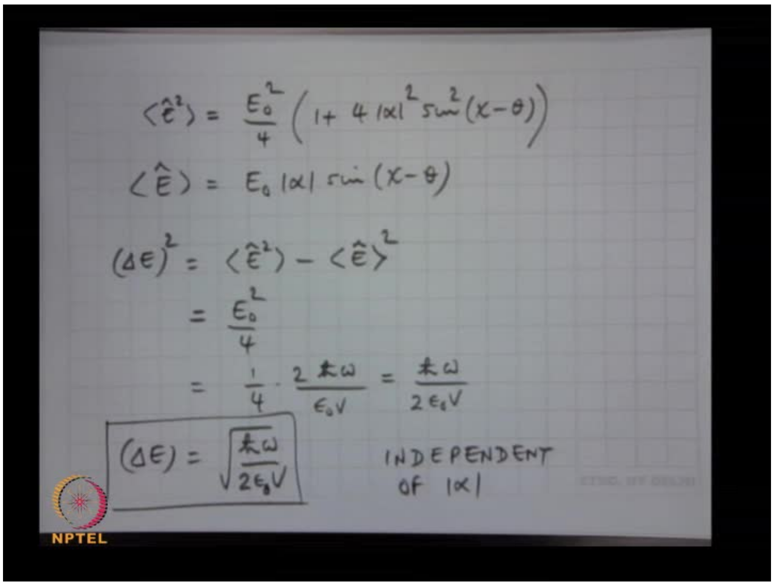

So, there is a minus sign here. This I take the minus sign inside, so this becomes 2 of mod alpha square plus 1 and then, I minus 2 mod alpha square cos 2 chi minus theta. So, what is this value? cos 2 theta is 1 minus 2 sin square theta. So, I can replace this and I get expectation value of E square, is equal to E naught square by 4 into 1 plus 4 mod alpha square sin square chi minus theta.

Remember that expectation value of E, we obtained earlier, was equal to E naught mod alpha sin chi minus theta. So, the variance in the electric field is equal to expectation value of E square minus expectation value of E Whole Square, which is equal to - so expectation value of E square is this, so the second term is E naught square mod alpha square sin square chi minus theta that cancels with this, then I get E naught square by 4. E naught square if E replace, so this is 1 by 4 2 h cross omega by epsilon 0 V; so this is H cross omega.

So, delta E the uncertainty in the electric field is square root of h cross omega by 2 epsilon 0, independent of mod alpha; independent of mod alpha, independent of time, it is a constant. The expectation value of electric field increases as mod alpha increases, the uncertainty in the electric field remains constant.

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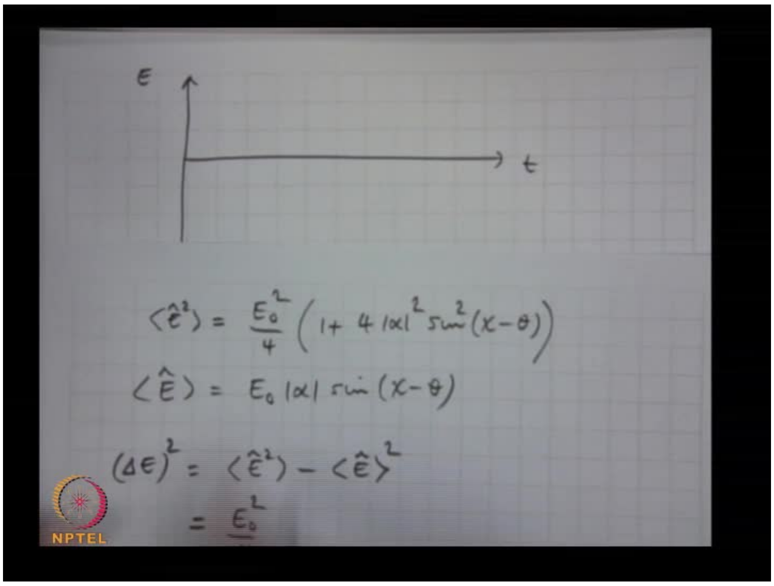

So, I can actually picturize the electric field variation in this form. So, the electric field expectation value - let me draw a curve representing the expectation value in the electric field first.

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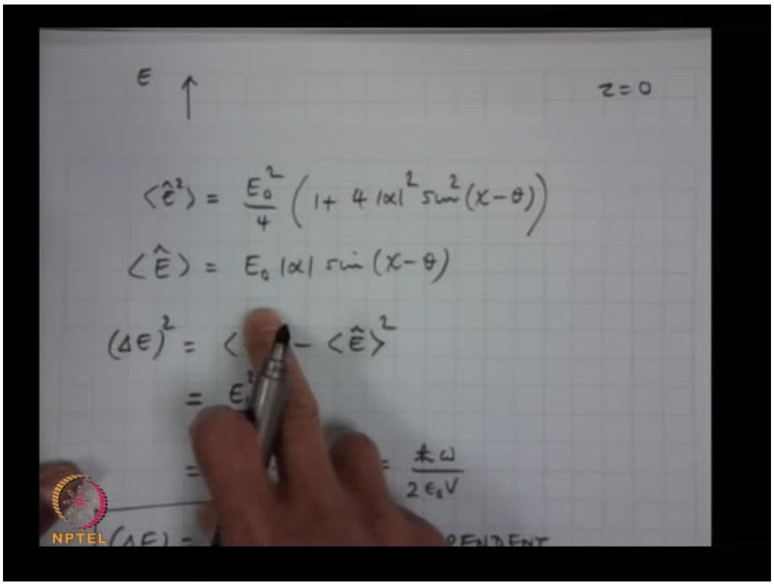

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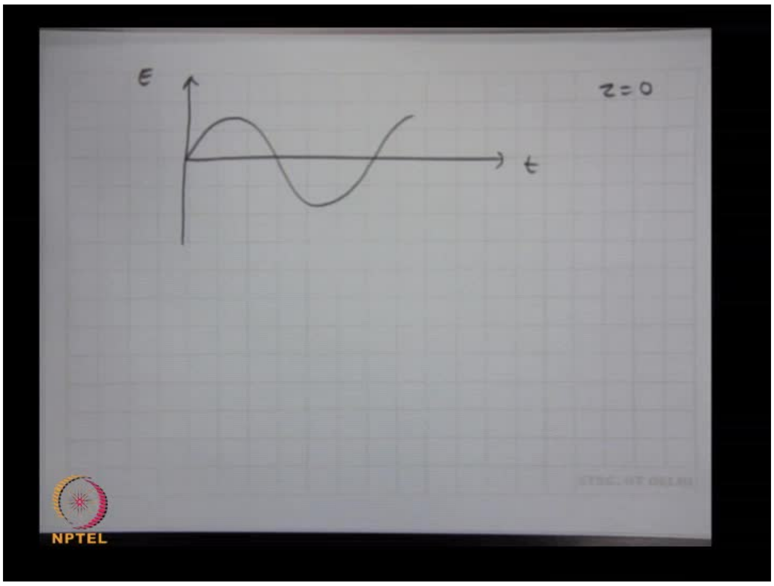

So, this is E naught mod alpha sin omega t minus kz minus 0, so let me plot at z is equal to 0. So, this will go as E naught mod alpha sin omega t minus theta, so it has some variation likes this.

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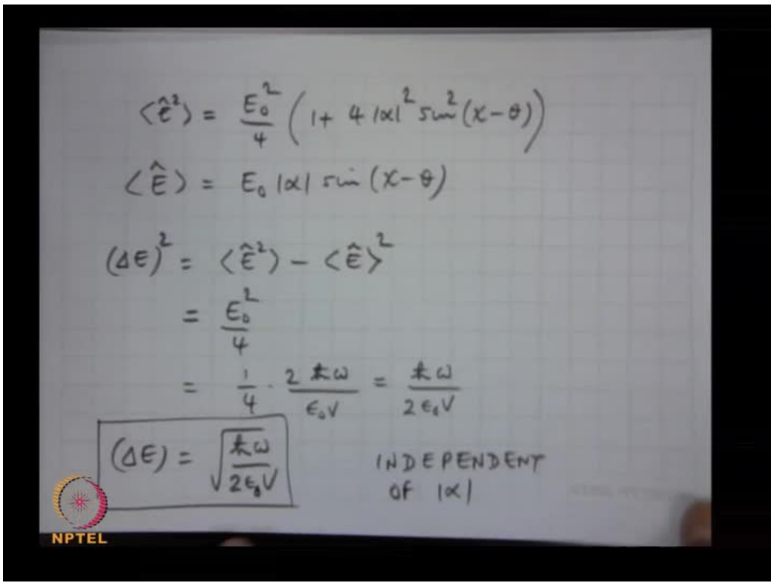

So, let me plot- let me plot for example, if theta is to 0, it goes like this, is the expectation value going like this. There is an uncertainty of square root of h cross omega by 2 epsilon 0 V of the electric field, at heavy instant of time.

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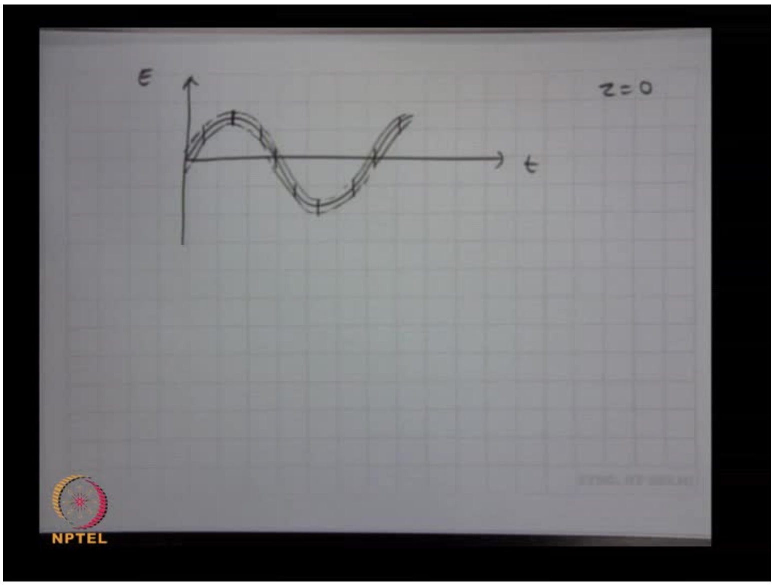

So, let me plot the upper bound, the lower bound here and the lower region. So, this will represent - so this is uncertainty in the electric field at any time, this remains constant, independent of time. So, this is a representation of the coherent state.

Remember, this uncertainty is independent of mod alpha, so if you take a coherent state with a large value of mod alpha, this expectation, this height is quite large, amplitude is quite large, because amplitude is e naught mod alpha and the uncertainty is always square root of h cross omega by 2 epsilon 0V.

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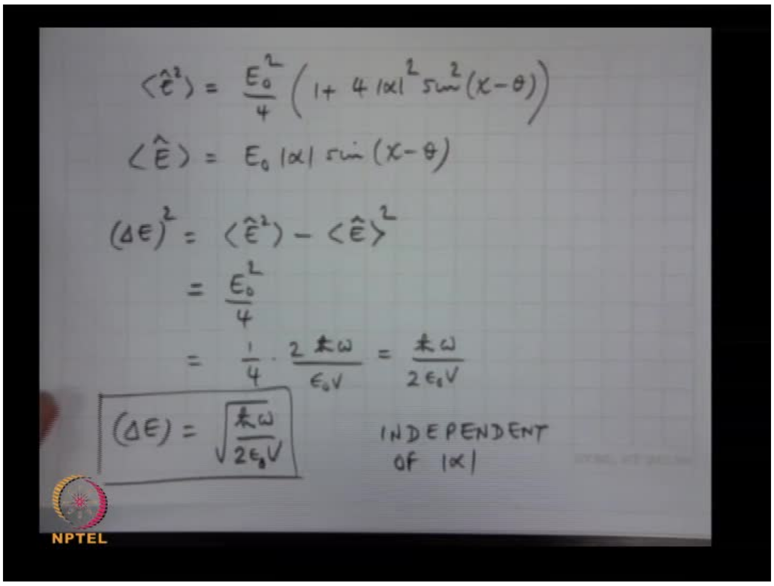

So, this field resembles more and more of a classical field. For very large values of mod alpha, this noise that is represented in the electric field become negligibly small compare to the amplitude itself.

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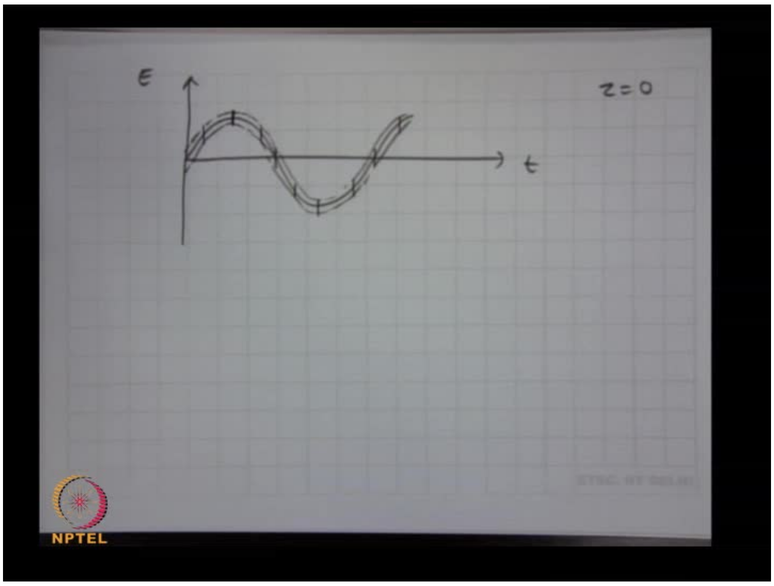

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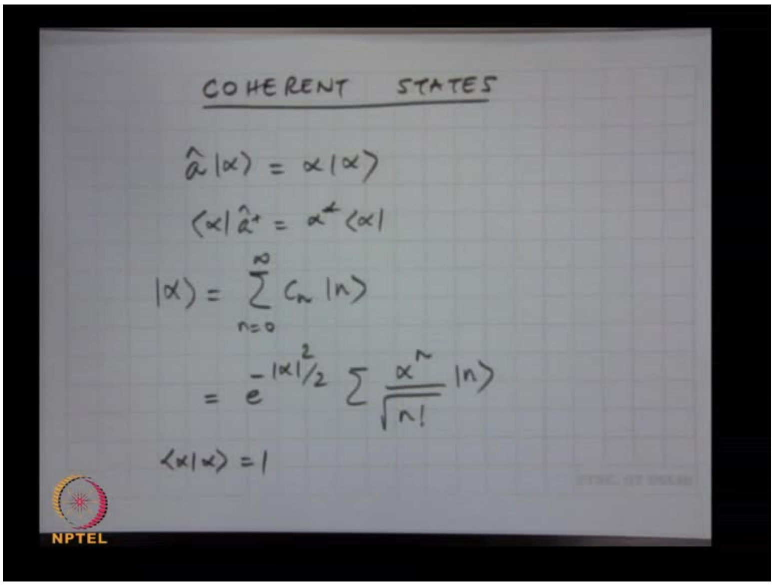

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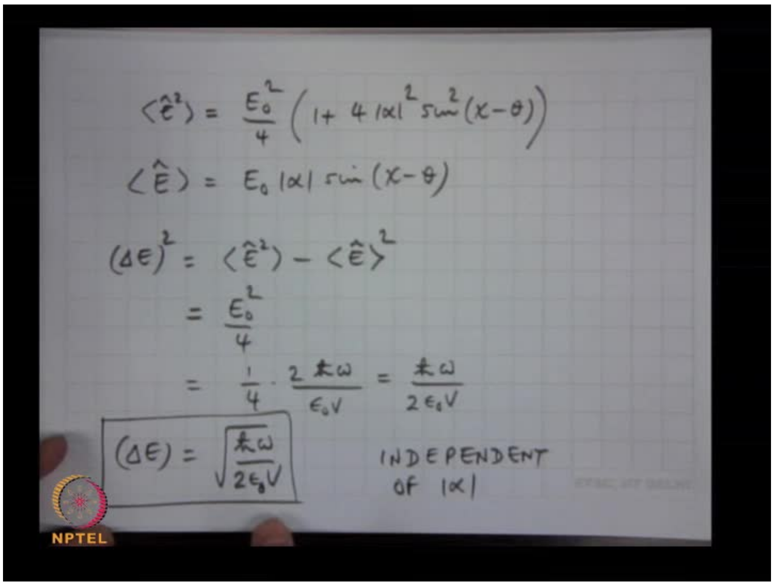

The field tends more and more towards a classical field is always noisy, there is always noise in the electric field. Please remember of this amount, in this coherent state and this state. This noise is actually the same noise that you will find in a state, which is the vacuum state. Because, a vacuum state, if you go back and look at this equation, for the definition of coherent state, a vacuum state correspondence to a alpha is a 0, is equal to 0. So, vacuum state correspondence to alpha is equal to 0, so the expectation value of electric field in the vacuum state becomes 0, because mod alpha is 0, but there is uncertainty electric field, which is the vacuum fluctuation.

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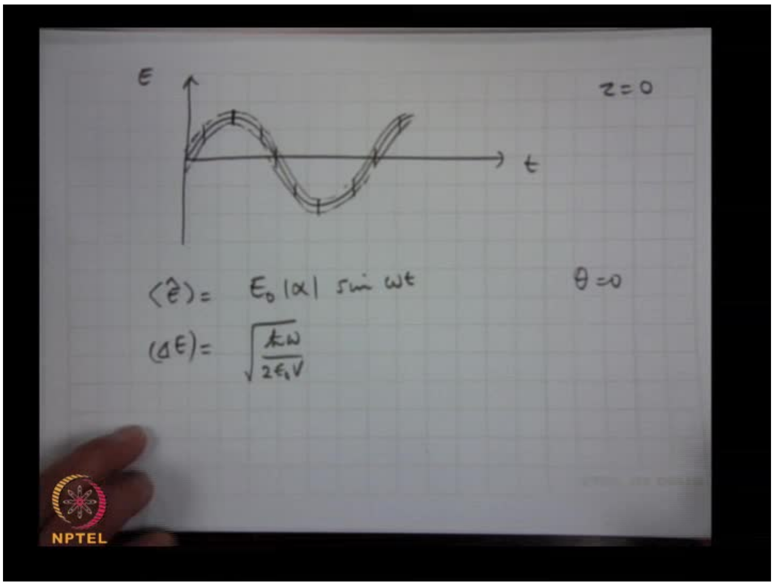

So, this noise that is present in a coherent state is exactly the same amount of noise that is present in the vacuum state. As the amplitude of the coherent state increases, a fractional uncertainty in the electric field decreases, because delta E remains constant, while the amplitude keeps on increasing and finally, you get close and closen to the classical state of electromagnetic field.

So, in this state, expectation value of electric field is E naught mod alpha sin omega t. This is by assuming theta is equal to 0 and delta E is equal to square root of h cross omega by 2 epsilon 0.

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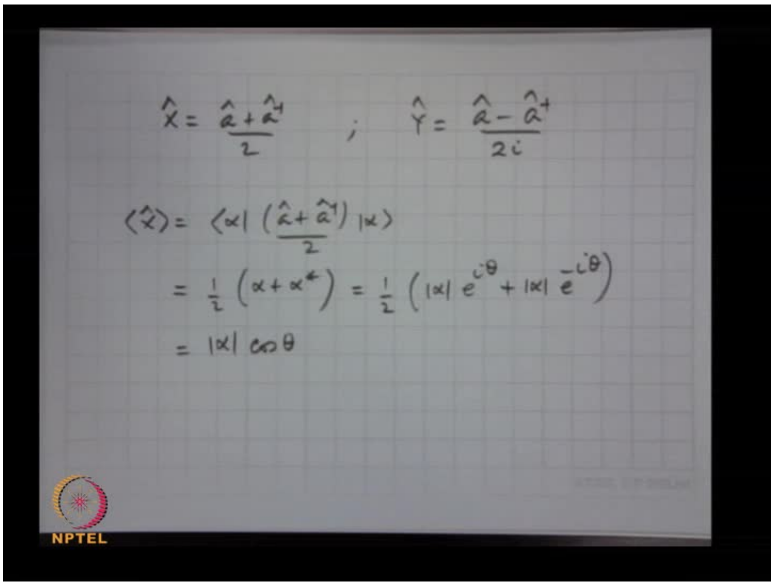

Now, there is another pictorial representation that is used to represent these states, which we discussed earlier. That is in terms of the uncertainties and the expectation values of the quadrature operators x and y. please remember, we defined a quadrature operators x, is a plus a dagger by 2, another quadrature operator y by a minus a dagger by 2 i.

So, let us calculate what the expectation value is and what are the uncertainties in the two quadrature operators when the field is in coherent state? So, expectation value of x is equal to alpha a plus a dagger by 2 alpha, which is equal to half of alpha plus alpha star, which is equal to half of mod alpha e to the power of i theta plus mod alpha e to the power minus i theta, which is equal to mod alpha cos theta.

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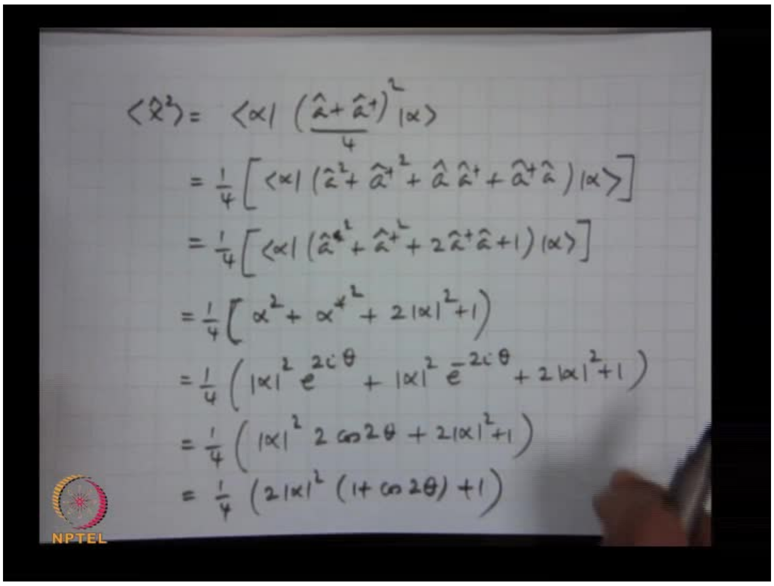

That is the expectation value of x, what is the expectation value of x square? We need to calculate the variance of the uncertainty in x, so for that I need to calculate expectation value of x square. Which is equal to alpha a plus a dagger whole square by 4 alpha, which is 1 by 4 alpha a square plus a dagger square plus a a dagger plus a dagger a alpha, which is equal to 1 by 4.

Now, I can replace a a dagger by a dagger a plus 1, I get alpha a square plus a dagger square plus 2 a dagger a plus 1 mod alpha, which is equal to 1 by 4 a square expectation value is alpha square, a dagger square expectation value is alpha star square, a dagger a expectation value is mod alpha square plus 1. Which is equal to 1 by 4 mod alpha square e to the power 2 i theta plus mod alpha square e to the power minus 2 i theta plus 2 mod alpha square plus 1. This is equal to 1 by 4 mod alpha square into 2 cosine 2 theta plus 2 mod alpha square plus 1.

So, this is 2 mod alpha square into 1 plus cos 2 theta, which it can be written as in terms of cos theta, so you get 2 mod alpha square into 1 plus cos 2 theta plus 1.

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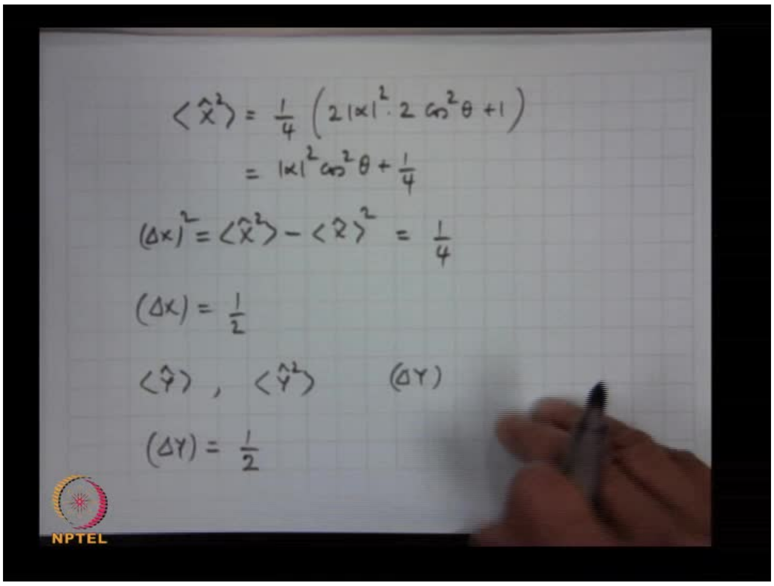

So, cos 2 theta is 2 cos plus theta minus 1, so this becomes expectation value of x square, becomes 1 by 4 2 mod alpha square into 2 cos square theta plus 1, which is equal to mod alpha square cos square theta plus 1 by 4. That is the expectation value of x square, so the variation in x operator is equal to x square average minus x average square. x average we have just now calculated, is mod alpha cos theta, so this becomes 1 by 4.

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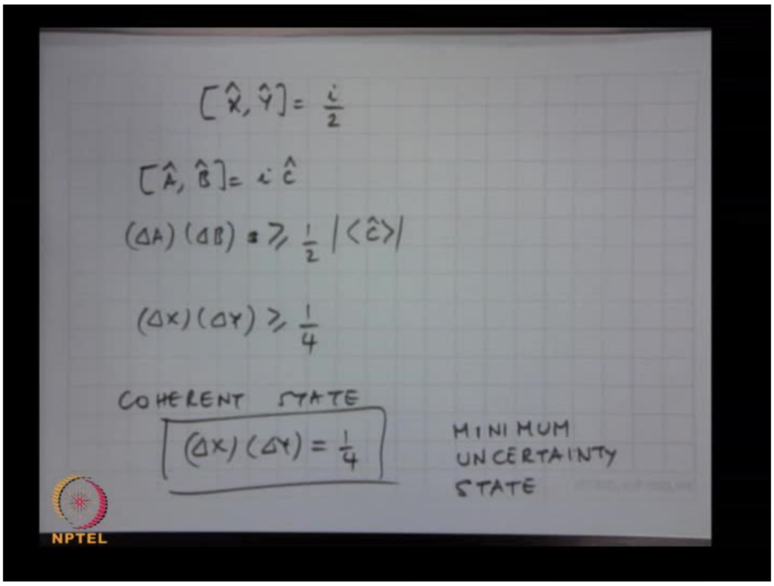

So, the uncertainty in the x coordinator is half, again independent of the denominator alpha. I can similarly calculate the expectation value of y, the expectation value of y square; from the two I can calculate delta y. The expression, I leave it you to calculate, this comes out to be again 1 by 2, delta y is also 1 by 2, delta x is also 1 by 2. Please remember, x and y satisfy commutation relation, which we have discussed earlier, x comma y is equal to i by 2.

Now, when you have a commutation relation between two operators, they have to satisfy a uncertainty relationship between them. In fact, if you have two operators satisfy in this relation A B is equal to i c, then you can show that delta A into delta B the product of the uncertainty is greater than equal to half of mod of expectation value of c.

If you have two operators a and b satisfying the commutation relation a b is equal to i c, then delta a can delta b, must be greater than or equal to half of the modulus of the expectation value of c. Because, x and y satisfy this equation, delta x times delta y, because c is just 1 by 2, this must be greater than or equal to 1 by 4. So, the product of the uncertainties in the x quadrature and the y quadrature must be greater than equal to 1 by 4. Or the coherent state we just now calculated, delta x is half and delta y is half, so for the coherent state, delta x delta y is equal to 1 by 4 and this is called a minimum uncertainty state. This satisfies the minimum value of the product of the uncertainties in x and y, hence it is called the minimum uncertainty state.

This is smallest value of the product of the uncertainties in x and y, you may have states for which this product is greater than 1 by 4. Do you have minimum uncertainty states? But, these states are special states, for which the product of the uncertainties in x and y is actually 1 by 4. Coherent state, which we are discussing, is one of the minimum uncertainty states.

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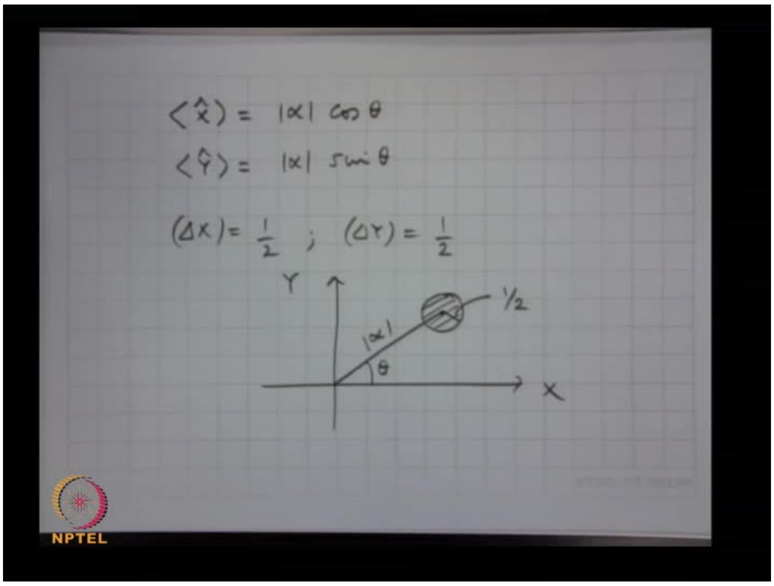

So, what we have got is the following. We calculated the expectation value of x, so what you got is expectation value of x is equal to mod alpha cos theta. I left it as a problem to you to calculate the expectation value of y operator; you can show that this is mod alpha sin theta. We have also seen that delta x is equal to half and delta y is equal to half, so using these quantities I can draw the phase of diagram correspondent to coherent state.

This is x, this is y, so I put a point corresponding to the expectation value of x and y. So, if I join this line, this angle is theta, expectation value of this distance is mod alpha, so expectation value of x is mod alpha cos theta, expectation value of y is mod alpha sin theta. There is an uncertainty around this point and this radius is half.

So, the coherent state is represented in the phase of diagram by a small circle of radius half. The center of the circle is situated at a distance mod alpha from the origin and this line the radial line makes an angle theta with the x axis. So that the expectation value of x is mod alpha cos theta, the expectation value of y is mod alpha sin theta.

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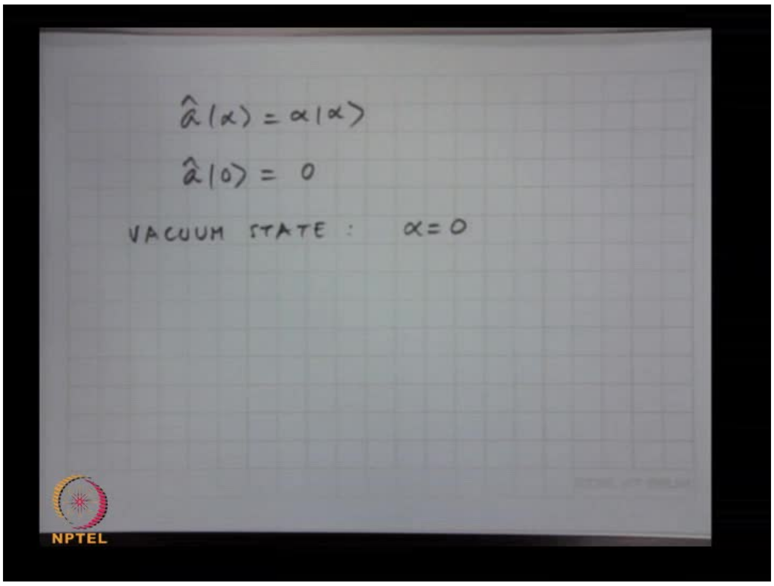

So, if you look at a state which happens to be the vacuum state, please note that in the coherent state, we have these equations. A vacuum state is defined by, so vacuum state is corresponds to alpha is equal to 0.

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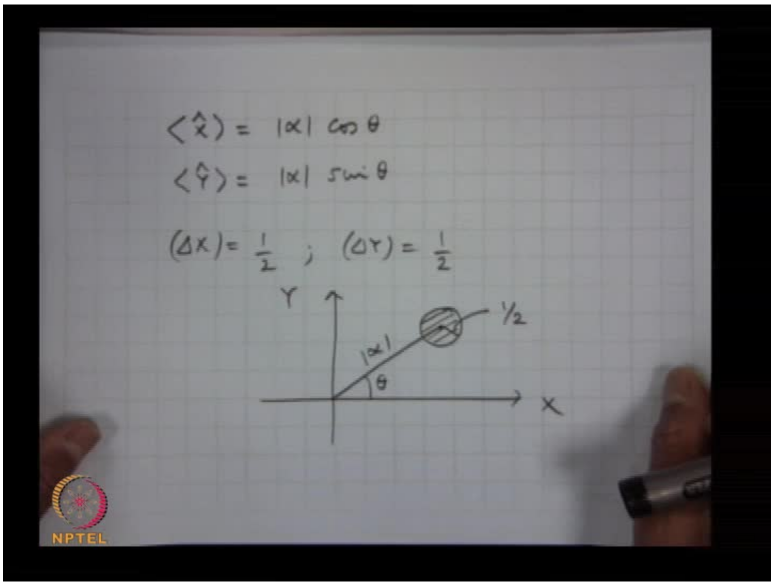

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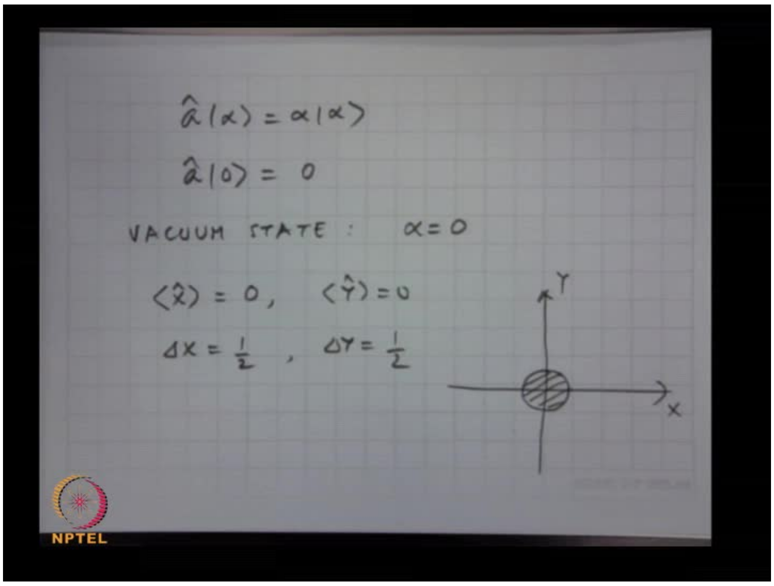

So, the same phase I saw, I can actually use these expression that we have got to obtain the parameters of vacuum state. I get for a vacuum state x average is equal to 0, y average is equal to 0, delta x is equal to half and delta y is equal to half. So, in the phase of diagram vacuum state, will be represented by a circle of radius half. On the origin, because the expectation value of x and y are 0, there is uncertainty of half in both x and y quadratures.

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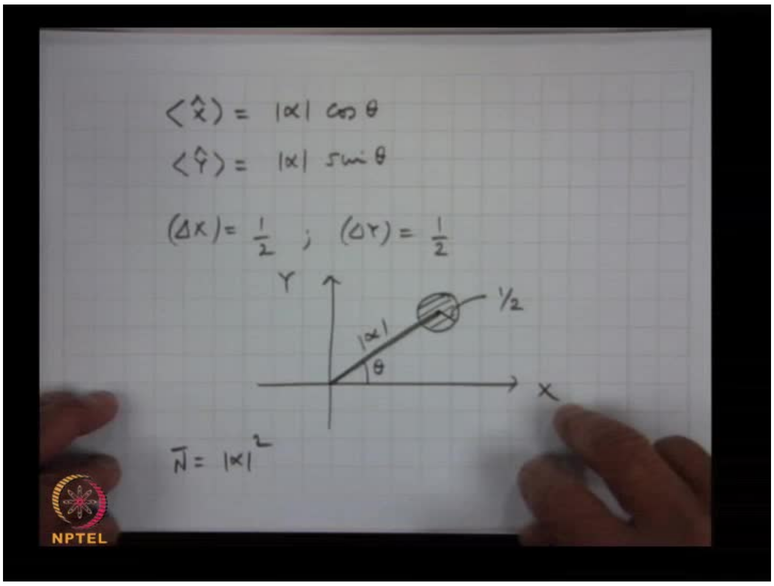

So, for a coherent state, this is mod alpha. Remember, mod alpha is related to the average number of photons; average number photons is equal to mod alpha square.

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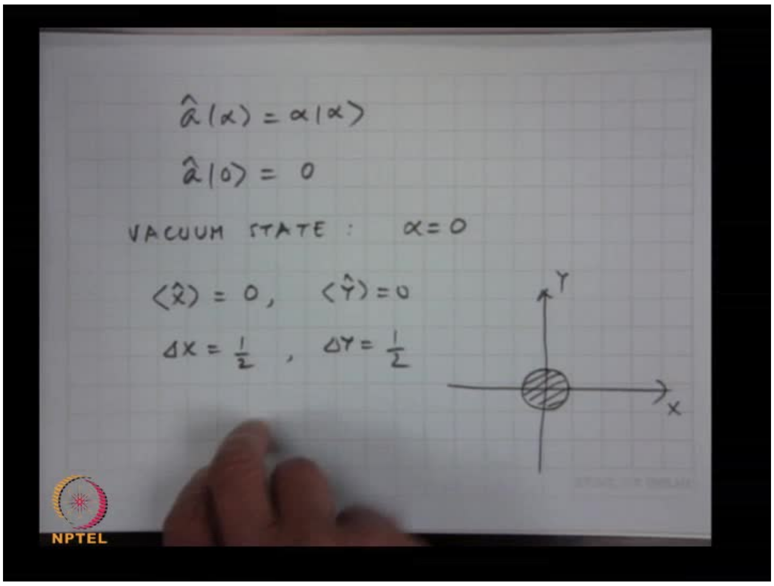

So, this distance square is this expectation value of the number of photons in that state, which is n bar. So, if you increase the value of alpha in a coherent state, this distance will increase of course, the radius of this circle remains constant, because that is independent of mod alpha. If you look at a vacuum state, this correspondence to a state in which expectation values of x and y are 0 and the noise uncertainty in x and y is still half.

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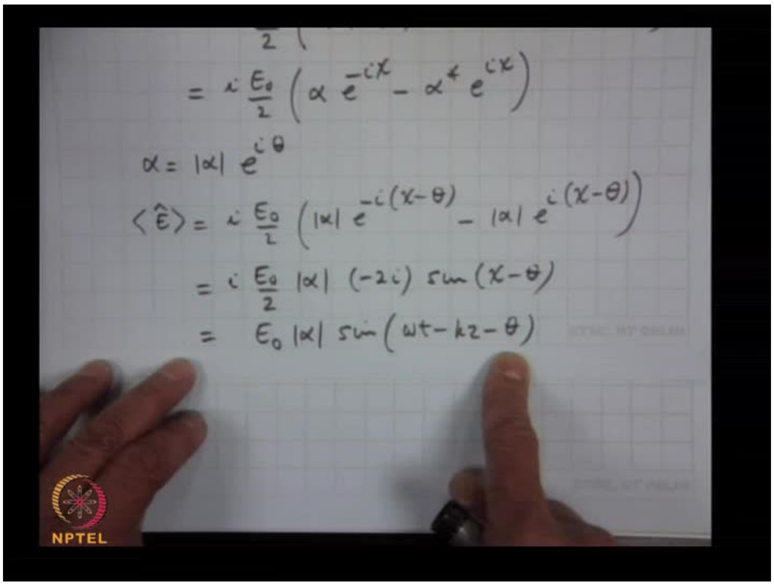

So, coherent states are states which are close to the classical electromagnetic field, because as we have seen here, the expectation value of the electric field in this state, in the coherent state, varies as exactly like in a classical electromagnetic field. Expectation value of electric field goes as E naught mod alpha sin omega t minus kz minus theta.

A laser, which is a very well stabilized laser, the light coming out of that kind of laser is a coherent state. It does not have precise number of photons, there is an uncertainty in the number of photons and there is an expectation value you can calculate. You can calculate the variance in the number photons; you can calculate the expectation value of the electric field, the variance in the electric field and all these you can calculate for the coherent state; this is a state, which is closest to a classical electromagnetic field.

So, do you have any questions in the discussion that we have had in coherent states? Please note that we are still discussing single mod states of electromagnetic field, in which, the excitation is only in one of the modes of the electromagnetic wave.

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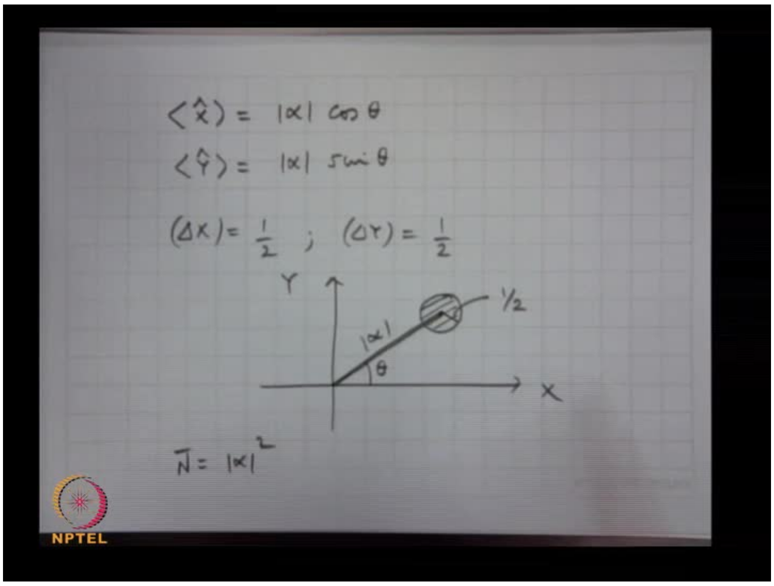

We will consider another kind of states, later on squeeze states, before we move on to multi-mode states. So, in this states, in the coherent state, the uncertainty in the x and y quadratures are equal - equal to the minimum values.

There can be states in which this product is still the product of delta x delta y, which as the minimum value is 1 by 4. The product is still 1 by 4, but delta x or delta y, one of them could be less than half, at the expense of the other one being more than half.

Do you have reduced uncertainties in one quadrature, at the expense of increase uncertainties in other quadrature, which leads to state, which are called squeeze states? Which are very interesting, they are completely non classical states and we are finding a lot of the applications. So that is the discussion on coherent states, from next class, we will look at squeeze states and continue with of states of single modes of electromagnetic field, before we move on to multi-mode states; thank you very much.