Quantum Electronics
Prof. K. Thyagarajan
Department of Physics
Indian Institute of Technology, Delhi

Module No. # 05
Lecture No. # 31
Quantum States of EM Field (Contd.)

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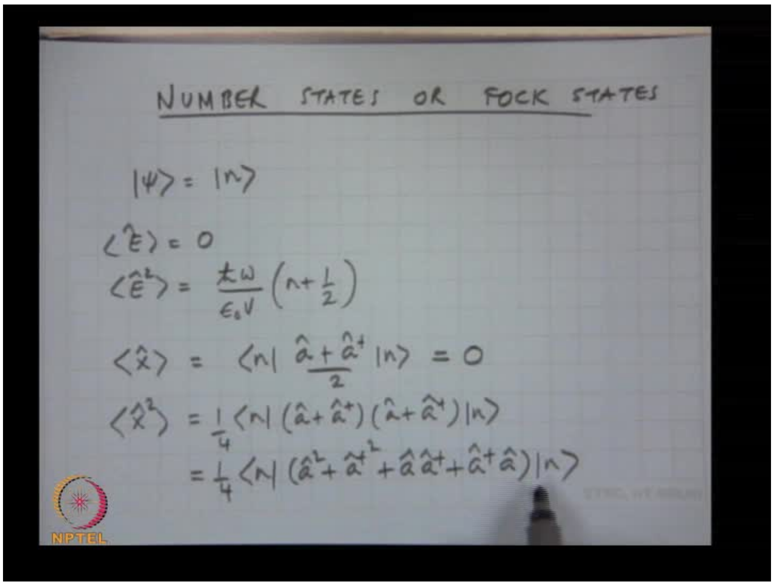

What we are looking at is quantum states of electromagnetic field so, yesterday remember we started looking at number states or fock states.

So, if i consider a now as i mentioned we are looking at one mole of radiation which means one propagation direction, one frequency and one polarization direction. So, if i propagate on z axis and if i have a x polarized beam and i have a at a certain frequency so that is one mole. So, we are looking at just a single mole and when i write for example, if i write n this is this means that the state is excited to be n th eigen state of that single mole.

So, each radiation mod is like a harmonic oscillator and the harmonic oscillator states are n ket states, n ket state has an energy n plus half h cross omega and so, we are looking at one of the harmonic oscillator which is one of the radiation fields modes and in that mode i am looking at a state in which the excitation is in the n ket states.

So, yesterday if we calculated the expectation value of electric field, that zero for the state and towards the end we also calculated expectation value e square and that is h cross omega by epsilon zero v into n plus half.

We use the operators for the electric field and calculate essentially psi e psi and psi e square psi. So, for this state the expectation value of e is zero, expectation value e square is h cross omega by epsilon zero v into n plus half. Actually, you can also calculate expectation value of x and by the quadrature operators, what will be the expectation value of x; remember this is n, a plus a dagger by 2 n, which will be zero. Because, a acting on n is you see n minus one a dagger [act/acting] acting on n gives you n plus 1 which is orthogonal to n state, x square n so, 1 by four a plus a dagger, into a plus a dagger of n so, this is one by 4 n, a square plus a dagger square, plus a a dagger plus a dagger a ok.

Now, can you find out and tell me what will be the value of x square expectation value. a square will give you zero a dagger square will give you zero a a dagger can be written as a dagger a plus 1.

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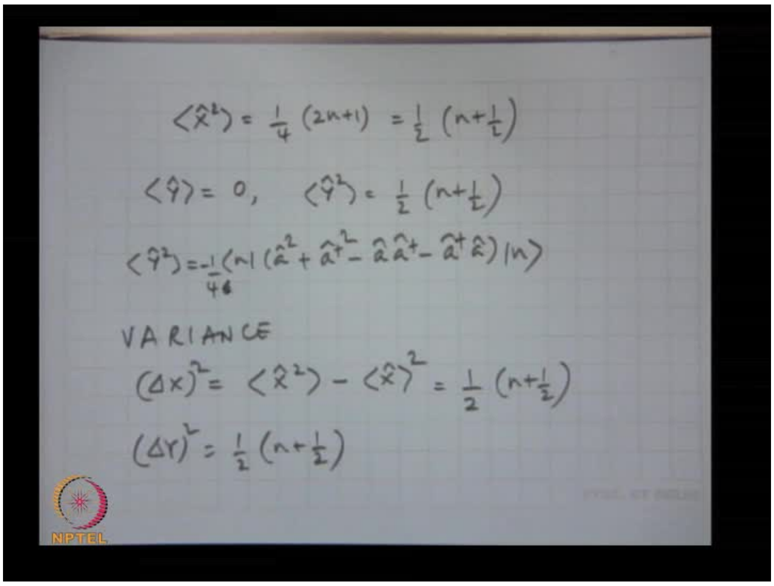

So, I will have two a dagger a plus 1 and a dagger a acting on n gives you n so, I have expectation value of x square will be 1 by 1 of 2 n plus one which is equal to half of n plus half yeah. [noise] no x is the is a plus a dagger by 2 defined, the quadrature operators are defined as a plus a dagger by 2 and a minus a dagger by 2 y the time and space dependence is separately sitting in the electric field expression ok.

Similarly, you can show that expectation value of y is zero and expectation value of y square is half n plus half no. Expectation value of y y square will be n minus 1 by 4 I, a square plus a dagger square minus a a dagger minus a dagger a of n 4 sorry, minus 1 by 4 right there are minus sine coming from here ok.

So, this is a a dagger a dagger a plus one so, you will get essentially 2 n plus 1 here, with the minus sign and you will get essentially half of n plus half. So, we define variance of a quantity for example, delta x square, a x average square, x square average, minus x average square and because, x average is zero and x square average is half n plus half so the variance in x, is half of n plus half and similarly, variance in y is also half of n plus half.

So, variance is something like, it is because of noise, fluctuations. Because, if value is not well defined so, when you make measurements of the observable corresponding to x ,you do not get a precise value every time. So, you take an average, average is 1 quantity and then variance, how much is the variation around the average. So, this is the strategical quantity that tells you, how much is the fluctuation the larger the variance the larger is the fluctuation of the of the quantity. In fact let me calculate for example, he variance in the number operator in this state what do you expect.

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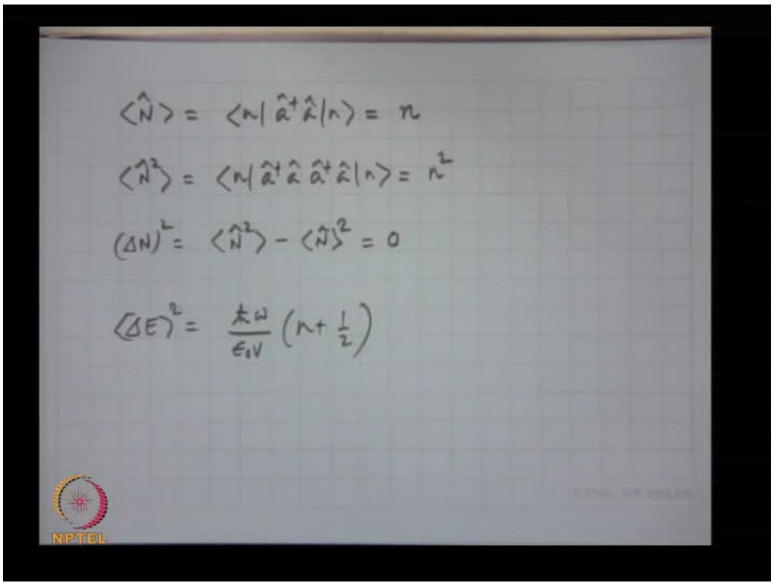

So, expectation value of number operator will be n, what is the number operator; a dagger a n, which is equal to n, a dagger a acting on n is n, n ket so this is n what about n square n a dagger a, a dagger a n. So, delta n square is n square minus n average square. Because ,this is a state with a well-defined number of photons, the number of photons is exactly n and so there is no variance variance is zero. What is the probability of finding n photons in the state, one suppose i take a ket n ket ten state the probability of finding ten photons is one the probability of [fact/finding] finding three photons is zero probability of finding any other number of photons is zero because the ket is ket n state.

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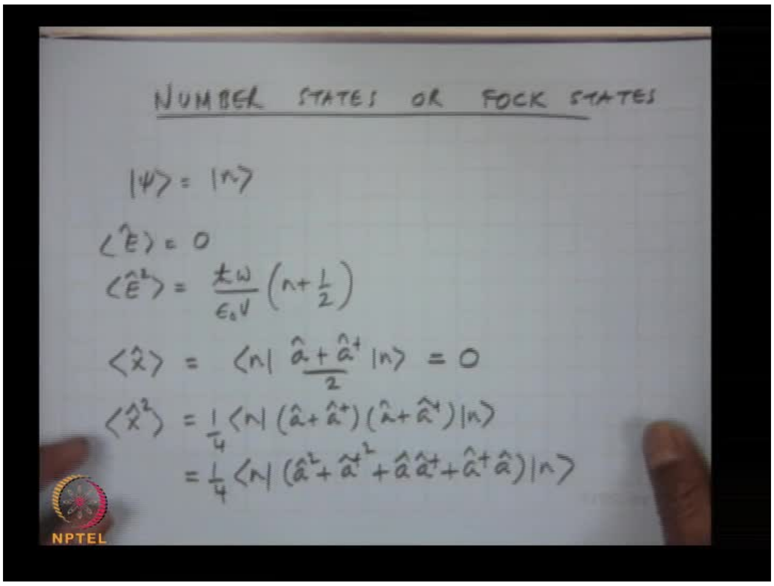

So, there is no variation fluctuation in the photon number because, I am taking a state which is a well-defined ket n state. What I find is the electric fields expectation value is zero, there is a variation in the electric field in fact I can calculate what is the variance in the electric field so delta e square is equal to h cross omega by epsilon zero v into n plus half. e square average minus e average square e average is zero so delta e square is h cross omega by epsilon zero v n plus half.

So, this is a state; in which the number of photons is precisely defined, the average value of electric field is zero, the variance in the electric field is given by this expression and the variance in the two quadrature operators are given by half of n plus half. Please remember, we are looking at a state in which only one mode is excited, all other states have zero excitation single mode state, I am looking at I am having just a state in which there is excitation is in only one mod and all the others are in the vacuum state yes mohith.

In this particular n ket state, we have an energy which is n plus half h cross omega and expectation value of this is zero this it comes out to be zero.

No, expectation value energy let me calculate, how do I calculated expectation value of energy, expectation value of energy is how much, n plus half h cross omega. Because, expectation value of energy is expectation value n plus half into h cross omega because energy is n plus half h cross omega n vect n ket n sorry n operator ok.

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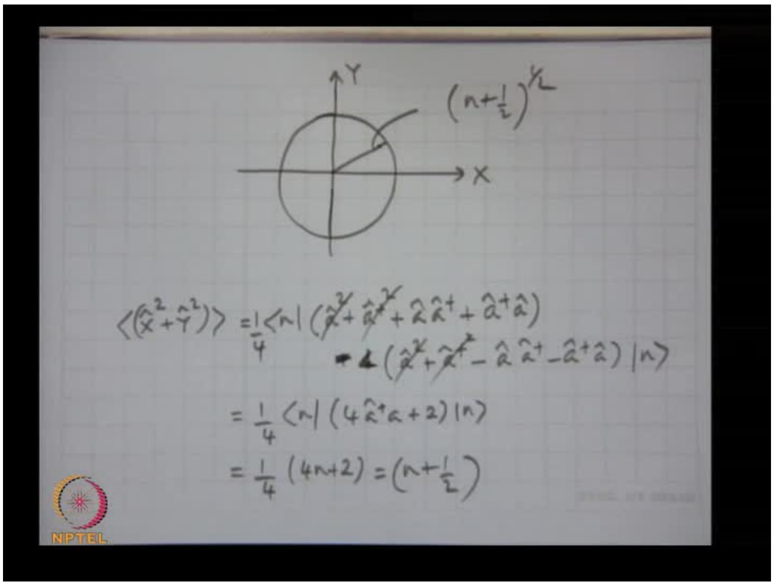

Ok so, now there are some pictorial representations which are used to sort of describe this. Now, we must take those pictures little more carefully because, they are not true representations of these quantum states but for example, there is a a [pict/picture] picture in terms of plotting. How the expectation values of the two quadrature operators, this is the picture of representation of the expectation values of the two quadrature operators ok.

Now, let me calculate what is the because it is a plot in terms of x and y, let me calculate the expectation value of x square plus y square first ok so let me go back here. So, let me calculate x square plus y square. So, this is n, a square plus a dagger square plus a a dagger plus a dagger a that is x square operator plus y square operators. So, that is minus one by so, ok one by four comes out with a minus sign so this a minus a dagger square. So, a square minus plus a dagger square minus a a dagger minus a dagger a, this goes off and I get 1 by 4 n. So, this a a dagger a dagger a plus 1, this a dagger a , a dagger a , a dagger a, a dagger a there is a minus sign there is a minus sign here. So, I will get four a dagger a plus 2 which is equal to 1 by 4, 4 n plus two which is equal to n plus r.

Actually, x square plus y square, is this operator x square plus y square is nothing but, a dagger a plus half. So, the hamiltonian is h cross omega into x square plus y square ok. Because, x square x square plus y square operator is 4 a dagger a plus half. If I take the four inside, x square plus y square operator is a dagger a plus half and hamiltonian is h cross omega into a a dagger a plus half so this is nothing but, so hamiltonian is related to x square plus y square x square plus y square is is an operator which is related to Hamiltonian. Hamiltonian is h cross omega into x square plus y square is operators.

So, this is a diff[erent]- this is a state in which x square plus y square is precisely defined. Because, a dagger a is precisely defined, the the variance of a dagger a operator which is number operator is zero. So, x square plus y square is precisely defined, x is not precisely defined because, expectation value of x is zero expectation value of y is zero but x square plus y square is well defined. So, I can draw a figure like this, the value of x and y which i will get will lie on a circle of radius square root of this.

So, if you prepare nonsemble of identical states in the n ket states, each one measures the value of x square plus y square all of them will get n plus half. Because, if you calculate the variance in x square plus y square it will be zero because, x square plus y square is nothing but, related to a dagger a and I know that a dagger operator has no variance zero variance.

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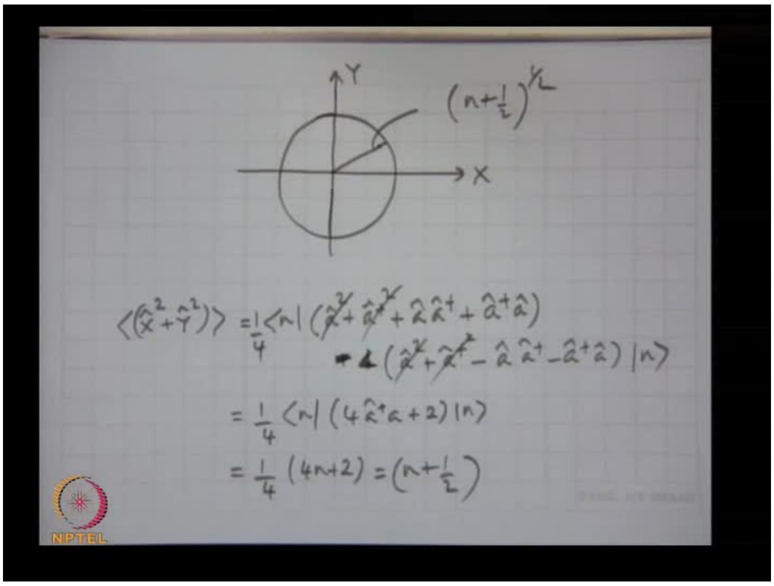

So, if I take a nonsemble of n ket states, each individual measurement will give me precisely the value n plus half but, the value of x if I measure, I will get different values and on a average it will be zero. Similarly, for y, so you can see here different points on the circle actually correspond to different sets of observation and you you will get you will get some sets of x and y such that x square plus y square is precisely n plus half but, average value of x zero average value of y zero.

So, this is one way of representing this particular quantum state n ket state. The smallest circle you can obtain is corresponding to n is equal to zero, which is the vacuum state and that is the circle of radius half raise per half one by root two. The variance in the electric field or in the quadrature operators keeps on increasing as you go to higher and higher fock states. Because, it is proportional to n plus half (( )) the variance in e square is the variance in e is proportional to n plus half the variance in x and y are both proportional to n plus half here and here.

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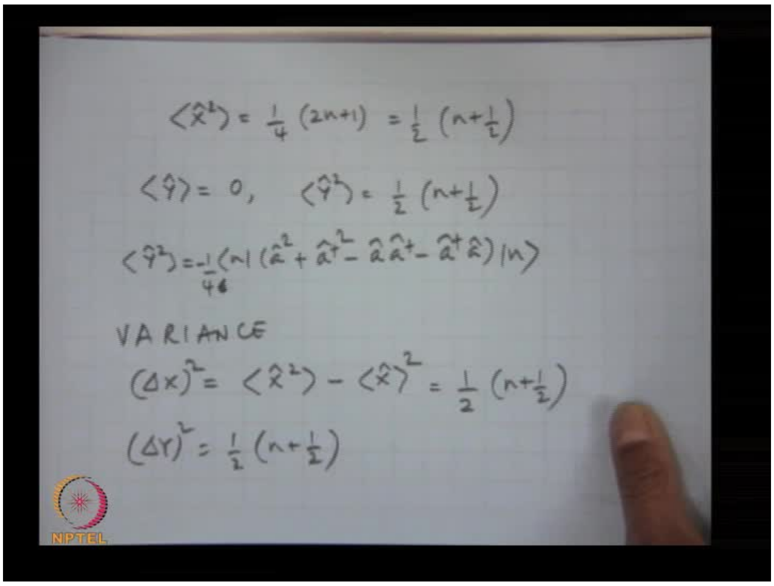

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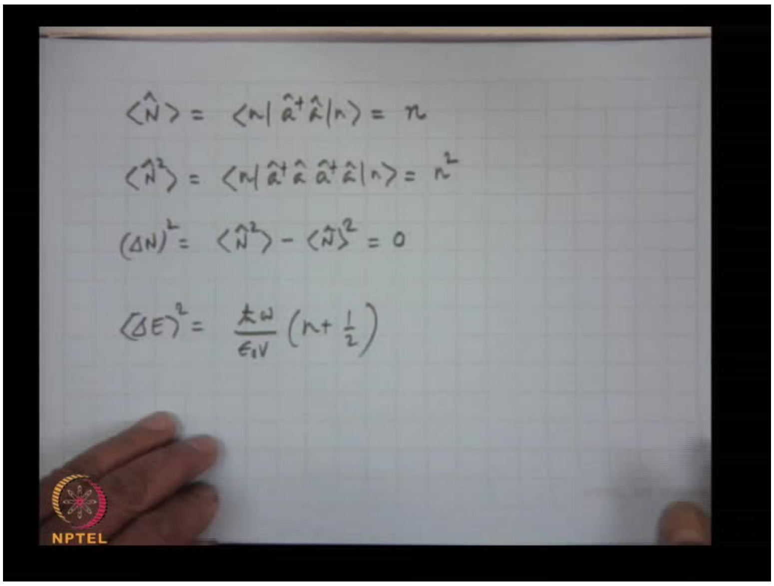

So, the lowest variance you can get, is in the vacuum state and that corresponds to variance in electric field which is h cross omega by two epsilon zero v that is, if I use this equation for with n is equal to zero for the vacuum state. I have the minimum uncertain the minimum variance in electric field is h cross omega by two epsilon zero v for the n ket states.

All higher order n ket states states having larger number of photons precisely defined photon number states will have larger and larger variance. Now, there is another way also of looking at this fact, that the average electric field is coming out to be zero, there is another picture that I can draw.

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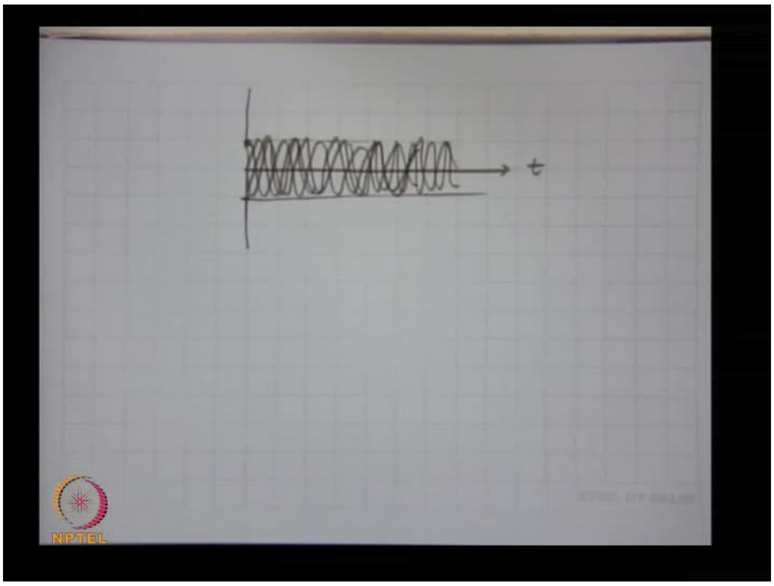

If I were to position myself at a certain value of z and measure the electric field. The because average value of electric field is zero what I will actually get is each person each observation will be different phases. This is this is filled, this one pictorial representation of the fact that each observation gives me for example, at t is equal to zero if I measure I can find anywhere from here to here. So, in that sense the state has no well-defined phase. If I take an identical set of n ket states and I measure the electric field of individual of the states of the nonsemble, the average electric field will be zero because each person phase is different. But all of them I have if they measure the number of photons they will all get exactly n. If they measure x square plus y square observable they will get exactly n plus half ok.

So, because this is a a interesting representation because, we will when we go to other states like coherent states and squeezed states, we will get back to this kind of a representation and try to draw similar pictures corresponding to that to get a better understanding of what the various variances and expectation values could be ok. So, this is a these are n ket states and as I was mentioning yesterday, it is very difficult to produce states, single mode states with well-defined number of photons. What people can produce is multimode states, which I will come to little later which means it is not a single mode which is I have to consider I have to consider many modes together. But, still having one photon but, this ket n n ket states, form a very good representation because they form a complete set of states you can represent any state as a linear superposition of the n ket states.

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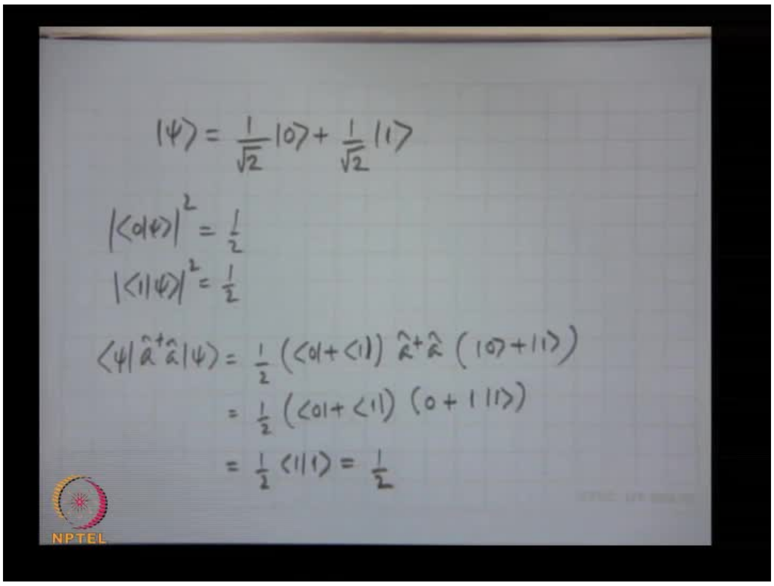

Ok so, now let me go one step ahead and let me take another state which is still a single mod state but, in a superposition of two n kets. So, for example, I want to look at a state, this is a state again I am not writing any subscript, this is one particular mode corresponding to one particular frequency, one polarization, one propagation direction it could be propagating like this, along z direction polarization is vertical and the frequency omega. But the state is not a well-defined n ket state it is a linear super position of two n ket states ok. Now, what is the state, what is the probability of finding no photons ok because the probability of finding no photons is which is half.

The probability of finding one photon is also half and the probability of finding any other number of photons is zero. [bec/because] Because if you take n psi with n not equal to zero or one, you will get zero. So, this is the state in which it is a linear superposition of a zero ket state and one ket state.

So please note, I have an nonsemble of identical states in this and each person each observer measures the number of photons in that state some people will get value zero some people will get a value one with equal probabilities because i have chosen one by root two one by root two.

I could have chosen alpha and beta and with the condition that mode alpha square mode beta square must be equal to one normalization condition ok.

Now, let us look at some interesting let us let us look at some interesting aspect of this. So, let me calculate, what is the expectation value of the number of photons in this state. So, this is psi a dagger a psi ok, what will what do what you expect, half because there is half probability of getting no photons, half probability of getting one photon so, this is the average will be half ok So, you can actually calculate so, this is 1 by 2 of zero plus 1 a dagger a into zero plus 1 so this is bra psi here 1 by with the one by root 2, this ket psi with a 1 by root 2 and then there is a a dagger a and so, this is half, zero plus 1 a dagger a on zero is zero plus a dagger a on 1 1 that is all. So, this is half so zero ket on 1 is zero and so I will have just 1 1 right so, that is the expectation value of the photon number.

Please note expectation value need not be an integer or half integer tell me anything it depends on the values of alpha and beta. But, if you measure the number of photons in any superposition state you will only get integer values because, any state will be written as a superposition of various n ket states with different probabilities

So, if you calculate if you find out how many photons are there in that state, you will always get an integer number. But, the expectation value of the number of photons average number photons can be anything ok.

So, this is an interesting state this is superposition state and you can form actually in even in an single mode state you can form so many different kinds of superpositions.

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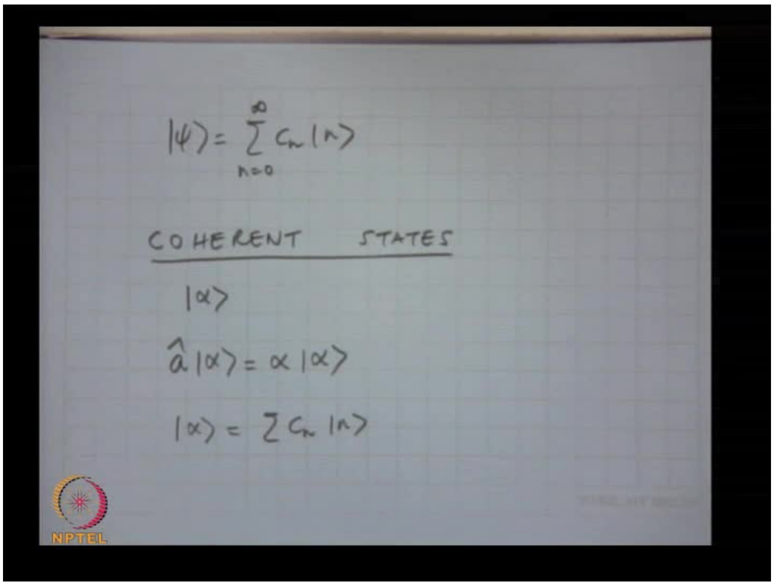

Because, actually this is an the most general single mod state as we wrote in the first day is essentially ok ok.

Now, I want to look at states which are which in element looks like classical states, which means I want to look at a quantum state of a single mode sta[te]- a single mod radiation state. Because, I know if I have an electromagnetic field in a single mode I expect electric field to vary as e is equal to e zero sign omega t minus k z and electromagnetic [fie/field] field a linearly polarized electromagnetic field for propagating in free space along this z direction at frequency omega will have a certain amplitude e naught sign omega t [mi/minus] minus k z plus phi some some phase. And I know that, in certain limits the quantum states quantum analysis quantum mechanics should give me the classical analysis also.

So, I must look at states, those states which will then give me a representation of the classical states which quantum mechanical representation of the classical states and there are some states called the coherent states which we will discuss now.

Incidentally, this these states were proposed by Roy Glauber in way back in the early nineteen sixties and he got the Nobel prize in 2005 for his work in quantum optics area essentially.

Quantum theory of optical coherence, he has developed entire theory optical coherence and he introduced this coherent states. That time and these states as I will show you our states in certain limit will look like classical states and these are the states that a well stabilized laser produces. The output from a well stabilized laser a quantum description of that is a coherent state ok.

So, let me introduce the coherent states. So, these states are so let me look at some states like this, I look at some states which I call alpha ket and these kets are eigen states of this operator.

Now, please note a is not a hermitian operator so, alpha need not be real alpha can be complex. Because, that is a is a non hermitian operator and a alpha is equal to alpha alpha so this is an eigen value equation for the operator a and I want to find out this alpha ket states in terms of the number representation, the fock state representation.

So, let me write let alpha be equal to I want to find out values of c n.[noise] no,[noise] no, I want to I want to find out what is those were set of c n's which satisfies this equation.

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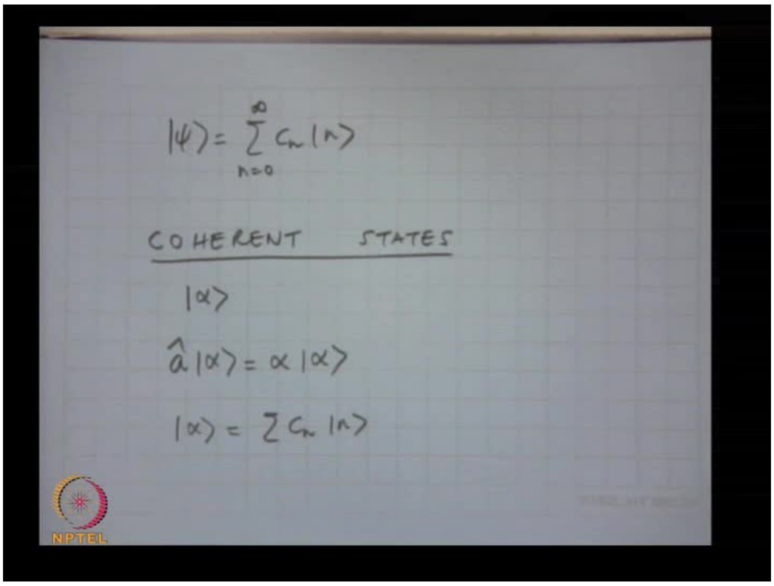

I am I am expanding alpha alpha is some state which I say it is an eigen state of the annihilation operator. I want to express alpha state in terms of [at/ket] ket n states expansion and to do this I need to get the expansion c n so that this satisfy this equation [noise].

I will show you that coherent states in one certain limit will look exactly like classical electromagnetic waves. And the they will have minimum noise, the noise of the vacuum state.

So, there are couple of properties of this states which are very important states and it so happens that lasers produce this state the output from a laser is is in a coherent state.

So, what I am doing is I I want to look at states which are eigen states of the annihilation operator and I expand because, n ket states form a complete set I can write any state as a linear superposition of n ket states.

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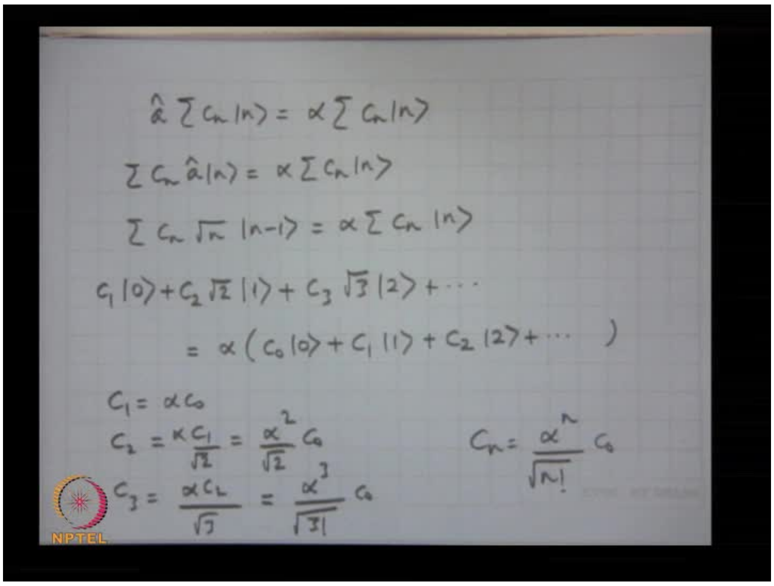

So, I want to substitute this into this equation to find out what are the values of c n. So, if I let me substitute this here a sigma c n n is equal to alpha sigma c n n. So, this is equal to [sig/sigma] sigma c n, a n is equal to alpha sigma c n n.

Now what is a times n, square root of n n minus 1 ok. So, let me expand this sum so, c zero term will be zero because n is equal to zero so, have c 1 zero plus c 2 under root 2 into 1 plus c 3 square root of 3 2 ket plus so on which is equal to alpha times c zero zero ket plus c 1 1 ket plus c 2 2 ket and so on.

In these sums, n goes from zero to infinity. So, I just expanded the sum and written explicit terms now, note if I multiply both sides by bra zero I will get c one is equal to alpha c zero because all other terms will vanish. So, what I am essentially stating is the coefficients this zero ket on both sides must be the same coefficient of n ket state must be the same on both sides. So, c 1 is alpha c zero c 2 will be equal to c 1 by alpha c 1 by root 2 which is equal to alpha square by square root of 2 into c zero. c 3 will be alpha c 2 by square root of 3, which is equal to alpha cube by square root of factorial three into c zero and so on. So, in general c n will be alpha raise to the power n by square root of n factorial into c zero.

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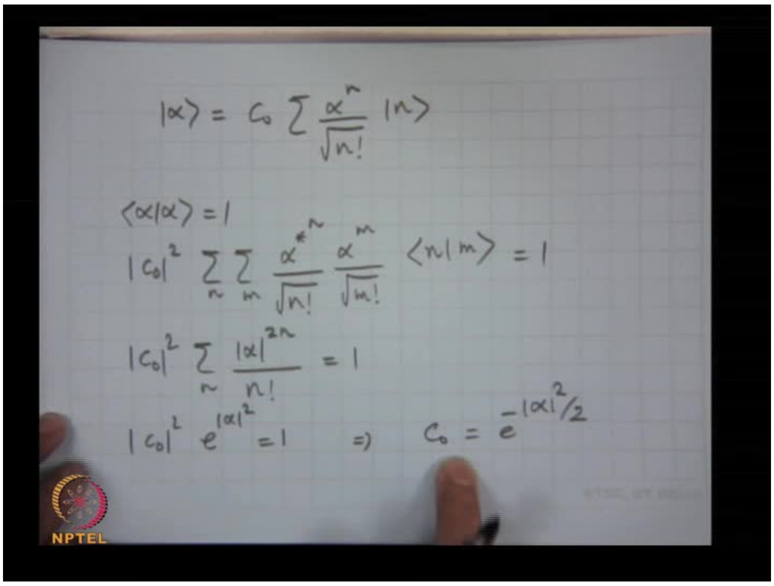

So, just from recurrence relations and then finding out the value for c n. So, I can use this c n in the expansion for the alpha ket and what I find is ket alpha is equal to c zero sigma alpha raise to the power n by square root of n factorial into n.

Now, what should this alpha satisfy, normalization so I should have alpha alpha is equal to 1 so this implies, mode c zero square sigma n sigma m alpha star raise to power n by square root of n factorial alpha raise to power m by square root of m factorial into n m.

Please note there is a product and I need to use two different indices for the summation. So, bra alpha is written as sigma n is equal to 1, bra alpha is written as sigma n alpha star raise to power n by square root of n factorial bra n into c zero star. Ket n is given by this and so, if I multiply this so this is what this is delta n m so this is mod c zero square. So, one of the sums goes off so, m must be equal to n so I get sigma n alpha star raise to power n into alpha raise to power n is mode alpha raise to the power 2 n by n factorial is equal to 1.

What is this?. Sigma n mode alpha raise to power 2 n by n factorial exponential exponential mode alpha square exponential x is a sigma n x raise to power n by n factorial which is equal to 1. So, this implies apart from a phase factor c zero exponent is equal to minus apart from phase factor. I can always another phase factor sitting here. So, that is x value for c zero so I can substitute the value of c zero here to get a complete expression for the alpha ket.

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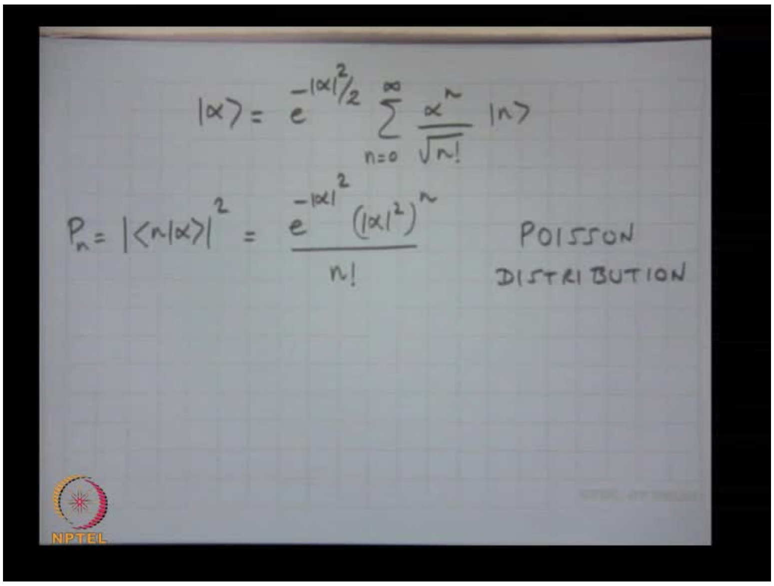

This is a linear super position state of infinite number of n ket states. The example, we looked at just before this was a state in which there were only 2 n ket states which was superposed, zero ket state and one ket state. This is a state which has contributions from all n ket states from zero to infinity and each one has an amplitude given by exponential minus mod alpha square by 2 alpha raise to the power n by n factorial.

Now, for this state I can calculate various quantities. For example, let me calculate what is the probability of finding n photons in this state?. Can you tell me, what is the probability. The probability of finding n photons in this state is mode n alpha square p n. How much is that so, in fact this n and this n is a summed over so this I could write n here ok. So, only one of the terms will survive so, I will have exponential minus mode alpha square, mode alpha square raise to the power n by factorial ok. n ket n alpha mode square so, one will be alpha star raise to the power n into alpha raise to the power so mode alpha square raise to the power n divided by n factorial.

What is this distribution, it is a poisson distribution. So, if you prepare if you prepare, an nonsemble of identical alpha ket states with a certain value of alpha then, and if you do experiments on each one those nonsemble to find the number of photons in that state, each person will give get a different number of photons and that will be distribution which is the poisson distribution.

And what I will show you I think we will have a quiz now, what I will show in the next class is that this particular state we will calculate the expectation value of the electric field in this state and I will show you, that this electric field expectation value behaves very similar to the expectation to the electric field of a classical electromagnetic field.

But, it is different from a classical electromagnetic field. Because, a classical field you can precisely define the electric field amplitude and phase. Here you will find, that in general it is not possible this particular state has actually what is called as a minimum uncertainty in that state ok.

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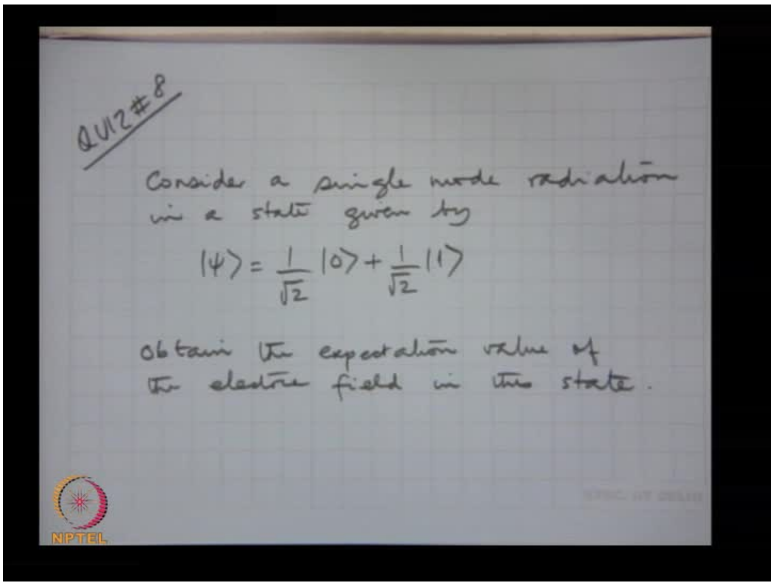

This is the state we were discussing in the class, we did not calculate what is the expectation value of the electric field in the state. Remember in the n ket state we found the expectation value is zero. So, I want I want to find out what will be the expectation value of electric field in the state.