Quantum Electronics

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Module No. # 05
Lecture No. # 26
Review of Quantum Mechanics (Contd.)

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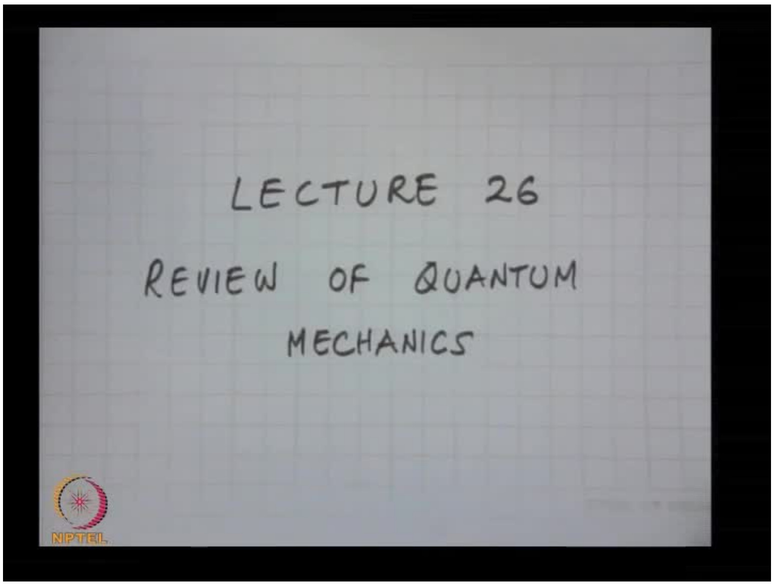

Today, I want to just continue with the review of quantum mechanics, introduce the Schrodinger and the Heisenberg pictures before, we start quantize electromagnetic fields.

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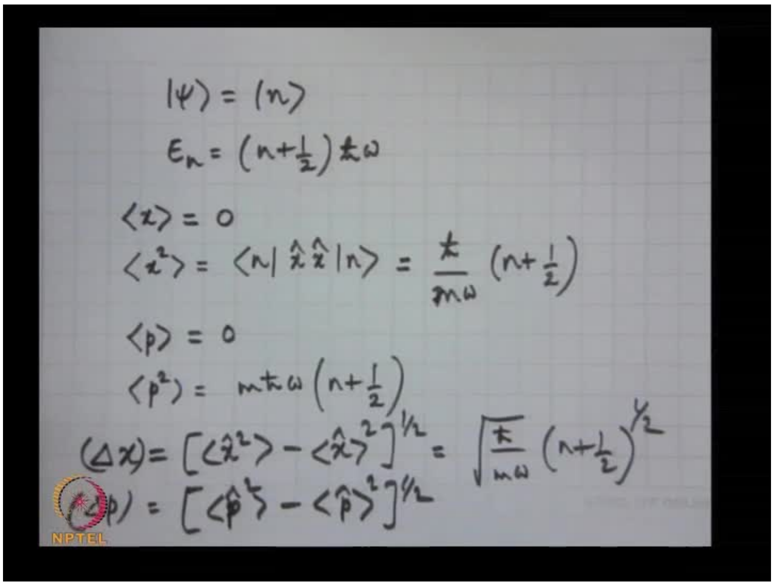

So, yesterday when we finished, we were looking at some problems for example; suppose I take a harmonic oscillator in one of the eigenstates of the energy and ket. So, the energy of this state is n plus half h cross omega.

Can you calculate the expectation value of x in this state. We have done this yesterday is it okay, so this is 0 expectation value of x square, So, I have to calculate n x, x n. i substitute x cap in terms of a and a dagger with this be 0, this will not be 0. Let me give you the value x square expectation value is h cross by m omega into n plus half, expectation value of x is 0. Expectation value of x square is so much similarly, expectation value of P is 0 and expectation value of P square is m h cross omega into n plus half.

So, as I mentioned yesterday, the energy eigenstates have well-defined energy and the phase of the oscillator is completely undetermined and that is why when you take an expectation value from an ensemble of systems, you will get a value of 0.

I will recall that we defined uncertainty in the x as x square average minus x average square raise to the half. So, because, x average is 0 this is simply h cross by m omega into n plus half raised power half. The uncertainty in the position of the oscillator in the nth eigenstate is square root of h cross by m omega n plus half raised power half.

Similarly, I can calculate delta P, the uncertainty in the momentum P square minus P square. That square root of this quantity because, expectation value P is 0.

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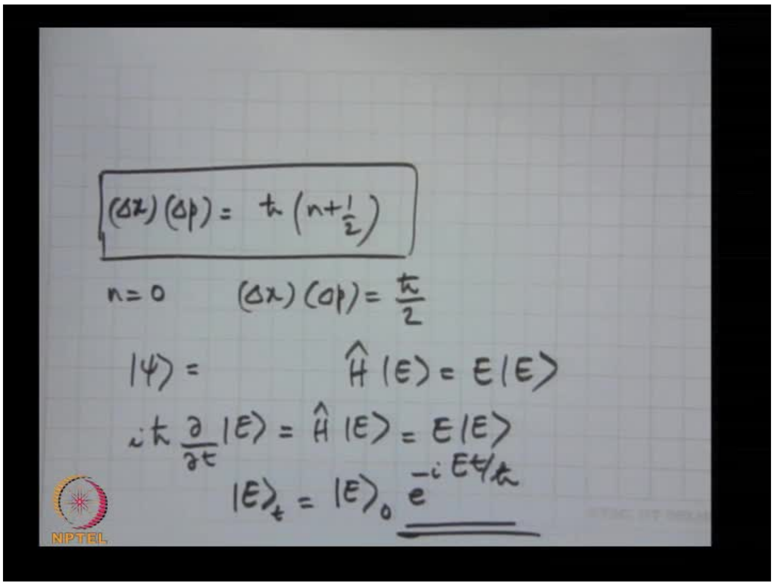

So, I defined the product of uncertainties delta x times delta P, I can calculate from here and what I find is delta x times delta P is equal to h cross into n plus half. I So, I substitute the values of delta x and delta p and I find the uncertainty product in position momentum of the linear harmonic oscillator is h cross n plus half.

And the lowest uncertainty product appears for n is equal to 0 state and for this state delta x delta P is equal to h cross by 2. All higher order higher excided states have a larger value of uncertainty product. Now I need to understand what is the time evolution of the system in one of the energy Eigen states for example; So, suppose I take a state which happens to be in an energy eigenstate.

So, let me take a state for example; H E is equal to E E. So, for any system suppose I take ket E is the energy eigenstate, So H E is equal to E times E, E is the energy Eigen energy eigenvalue. Remember the Schrodinger equation is i h cross Del by Del, Del t of E of the state is equal to H times E. If I want to understand how the system generated in Eigen ket E evolves with time this Eigen ket must satisfy this equation.

For any state for any state psi will satisfy these equation, but if the state is an energy eigenstate H times E is simply E times E and I can then integrate this equation and I get E as function of time is a function of time equal to E at 0 into exponential minus i E t by h cross. So, Energy eigenstate evolve with time according to this equation, there is only a phase change. The phase of the oscillator change with time and the time dependence is exponential minus i E t by h cross.

Sir excuse me.

Yeah?

Sir if e is not an eigenstate

Yes

I have generated state of a system that is supervision of different eigenstate.

Yeah.

Can we equation into different parts corresponding to each eigenstate.

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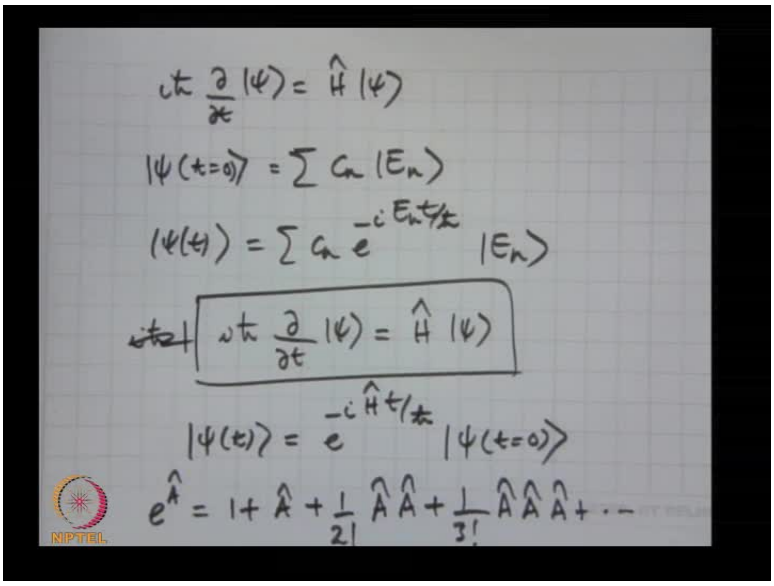

No, not this equation the equation is still this. So, I will have i h cross del by del t of psi is equal to h psi .What I can do is psi at t is equal to 0, I can write as sigma C n, E n where, E n's are the eigenkets.

So, then I can substitute here and then integrate but because, I know that each eigenstate energy eigenstate evolve with time as exponential minus i E n t by h cross, so I can write this as is equal to sigma C n exponential minus i, E n t by h cross into E n.

Sir it is like essentially by taking this equation the first equation you have written and saying that each eigenstate is an independent variable in this and split.

Yeah they are all independent.

Each eigenstates are all orthonormal to each other. Essentially, they do not mix among themselves each one evolves independently and with a time dependence exponential minus i, E n t by h cross okay.

Now before I look at the time evolution of superposition states under two different pictures. Let me introduce this two pictures one is the Schrodinger picture and one is the Heisenberg picture.

As I mentioned to you yesterday, in the Schrodinger picture the Eigen the kets evolve with time and the operators are independent of time. In the Heisenberg picture the kets are fixed in time and the operators evolve with time.

Let me look at let me start with an example; I have the Schrodinger equation. So, this equation i h cross Del by Del t of psi is equal to h psi this is a Schrodinger equation and this is an equation of how the ket evolves with time and so this is a Schrodinger picture. In this picture the operators like position operator and momentum operator they are all the independent of time. This is a Hamilton in operator, so if I am in conservative systems where the total energy is conserved h is also independent of time.

So, we will only look at systems where the Hamilton in his independent of time. So, this H is also independent of time, but psi evolves with time. Now as I said what is the important is that, the expectation value of any operator must be the same whether I look at in the Heisenberg picture or in the Schrodinger picture.

Now if this if H is independent of time, I can write a formal solution of this equation like this psi as a function of time is equal to exponential minus i H t by h cross into psi at t is equal to 0. If I have an exponential of an operator this is 1 plus A plus 1 by 2 factorial A A plus 1 by 3 factorial A A A etcetera.

With single operator, there is no problem because, A and A commute but, if I had for example; exponential a into exponential b, I have to be little careful, I will give you a formula for that, but so that will be I defined this exponential of the operator. You can verify that this is a solution by differentiating both sides and substituting into this equation.

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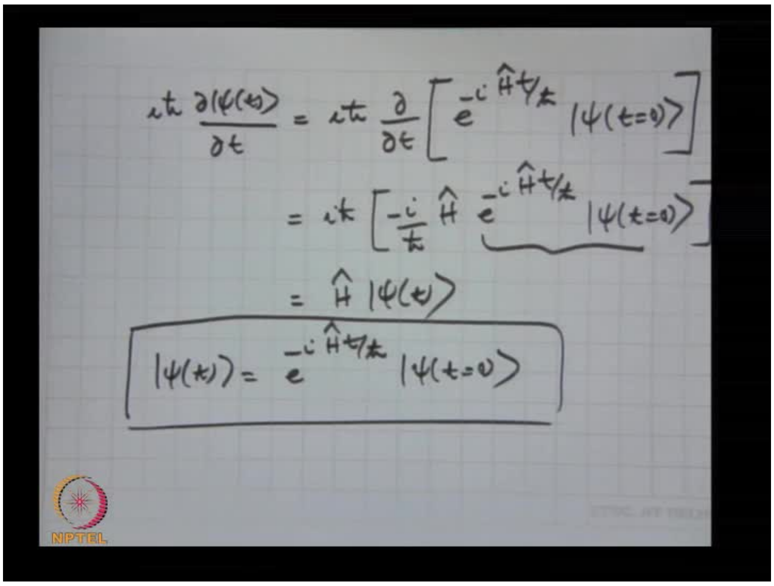

If you differentiate both sides for example; if I calculate i h cross del psi by del t so del psi by del t this will be equal to i h cross del by del t of exponential minus i H t by h cross psi at t is equal to 0. And if I had written this exponential operator as a series in differentiate, I will get essentially minus i by h cross H exponential minus i H t by h cross psi at t is equal to 0.

You can expand this exponential in terms of a series differentiate and you will find that differentiate at one which essentially gives you the same as we have differentiating just like normal quality. So, this is equal to i into minus i is 1 h cross cancels off and I get H into this remaining is nothing but, psi of t.

So, this solution which I wrote this formal solution I wrote is the solution of the Schrodinger equation and this equation represents the way the state evolves with time exponential minus i H t by h cross that is the Schrodinger picture. Now for example; in this picture if I able to calculate the expectation value of an operator a as a function of time, I will have this A. A is independent of time, but this expectation value can change with time because, ket psi changes with time. Now I want to another picture where I want to take away the time dependence from the ket into the operator.

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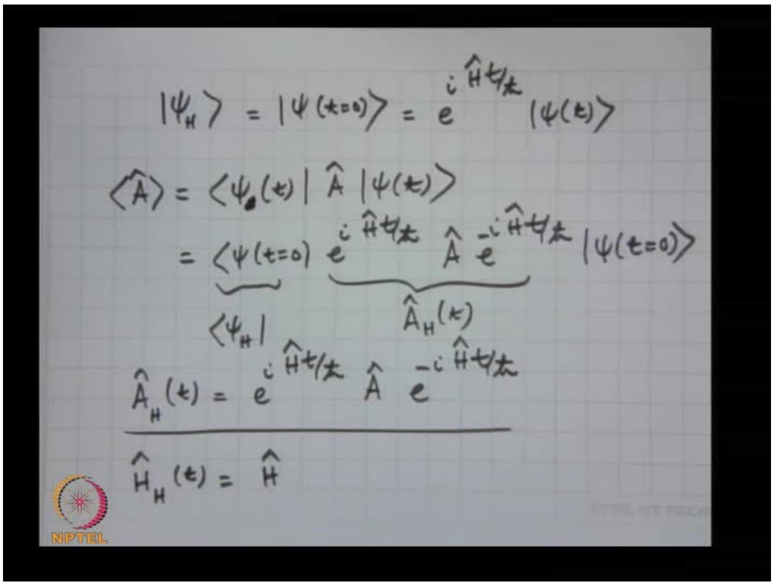

So, let me define the ket corresponding to the Heisenberg picture, I just put a subscript h here as psi at t is equal to 0 that has not changed with time anymore and this equation (Refer Slide Time: 10:34) I can invert this formally and write this as exponential i H t by h cross into psi of t. I have just taken this I have multiplied by an operator exponential i H t by h cross on both sides and then because, the operator H is the same this becomes is equal to 1 identity and I get this equation just gets reversed into psi of H, if psi of t is equal to 0 is equal to this thing and by definition this is independent of time we need now this product is a independent of time. This operator operating on this will always be at of psi at t is equal to 0 okay.

(Refer Slide Time: 13:01) Now the expectation value of an operator is remember psi s, so no subscript means a Schrodinger picture into A into psi of t. So, now I have this equation for psi of t, which I use in this equation. So, what is what is bra psi of t this is ket psi of t, exponential plus i h t by h cross.

Please remember H is a Hermitian operator. So, this will be equal to psi at t is equal to 0, exponential i H t by h cross A, exponential i H t by h cross psi at t is equal to 0. I have just substituted fresh and for the evolution of the ket with time into this equation, I have just substituted for psi of t from this equation essentially I am taking it back here and replacing in terms of psi at t is equal to 0 minus here.

No. There is a minus here.

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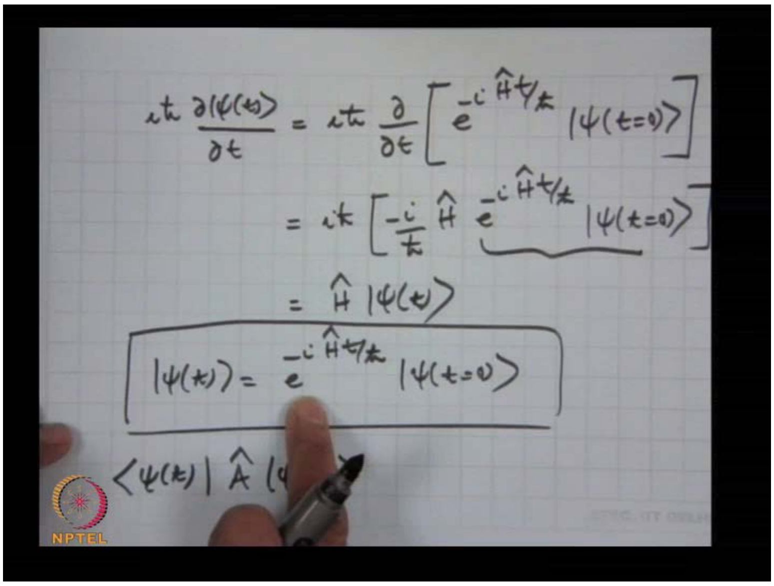

So, this this is psi of t is equal to 0 is a function of psi of t. So, I am replacing psi of t as a function psi of t is equal to 0 right. So, there is a minus sign here there is a plus sign here.

(Refer Slide Time: 13:01) So, I define this as the operator in the Heisenberg picture and this is nothing, but psi of H. So, the operator in the Heisenberg picture is equal to exponential i H t by h cross operator in the Schrodinger picture minus i H t by h cross.

This is independent of time. What is a function of time. And please note that H may not commute with A, so I cannot interchange A and exponential i H t by h cross. If A commutes with x H then for example; what will be the Hamiltonian in the Heisenberg picture. This will be H and H commutes with exponential i H t by h cross, so this is simply be H, so this independent of time. In the Heisenberg picture the Hamiltonian in the Heisenberg picture and the Hamiltonian in the Schrodinger picture are the same because, if I replace A by H and if I expand the exponential, this H commutes with all the H's anywhere. So, I can actually interchange this H and exponential and I get unity that means the Hamiltonian is the same whether, you are looking at the Schrodinger picture or the Heisenberg picture.

Sir (( )) why does Hamiltonian commute with the exponential terms.

Because this is also H only this is an H operator there is an H operator here, H operator always commutes with the H operator. So, I can if you expand this exponential you will have H and that H and all these H will commute anywhere. So, I can take this H out and reform back and the exponential then I will actually that means I can interchange these two because, this and all this function commute with each other okay.

Now, I need to calculate what is the time evolution, I need I can calculate an equation, disturbing the time of evolution of this operator in the Heisenberg picture so for this I differentiate this equation.

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So, I differentiate this with respect to time, and let me so let me calculate d a h by d t. (Refer Slide Time: 13:01). Let me assume in our analysis here that A has no time dependence in the Schrodinger picture they could also be operators which are depending on time in the Schrodinger picture itself, this is called an explicit dependence on time, but we will now look at that. Let me assume in the Schrodinger picture, this operator i is independent of time. For example; position operator, momentum operator they will all be independent of time, so when I differentiate this I do not have to differentiate A with respect to time okay.

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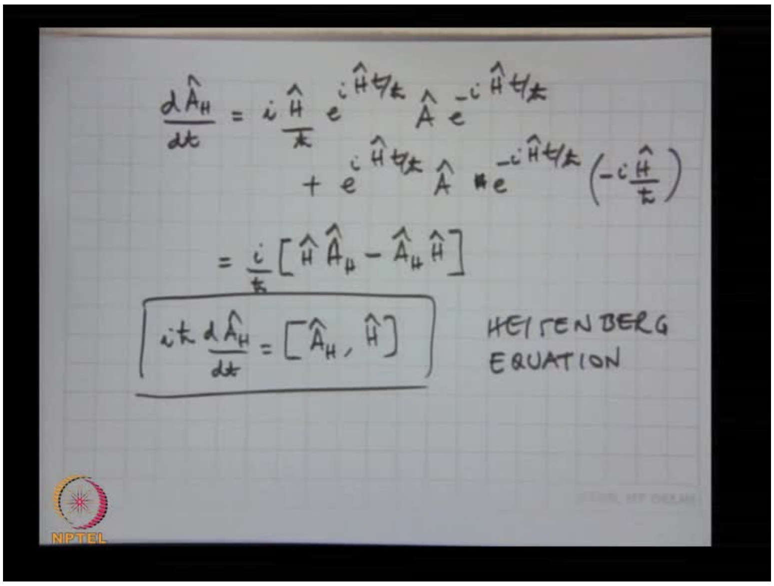

So, when I differentiate this equation what do I get. I differentiate first exponential, so I get i H by h cross into exponential i H t by h cross A exponential minus i H t by h cross plus exponential i H t by h cross A H okay.

(Refer Slide Time: 13:01) See when I differentiate the second exponential, I will have a i H minus i H by h cross because, that commutes with this.

(Refer Slide Time: 18:59) I can draw the exponential first and then the factor which comes out a differentiation afterwards. So, I Just write the differential of this comes and the differential of this comes and what is this, this is nothing but, A H of t this is also A H of t, so this is nothing but, i by h cross A H minus A H.

So, if I take the i h cross on the other side, I get i h cross d a H by d t is equal to commutator A h H this, is called the Heisenberg equation of motion. So, there is no i h cross Del psi by Del t is equal to H psi in the Heisenberg picture because, psi is independent of time. I would have to solve this equation to get how the operators vary with time. From the operator variation with time, I can always calculate the expectation value because, psi does not change with time at all.

So, in the Schrodinger picture, I calculate how psi varies with time to calculate the expectation value. In the Heisenberg picture, I calculate how the operators vary with time to calculate expectation values and other quantities.

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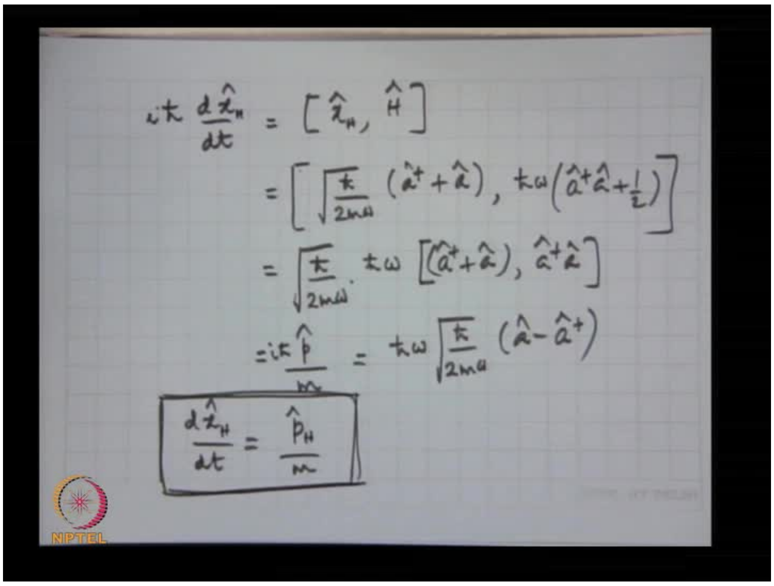

And actually in this particular picture is closed to classical mechanics for example; let us go back to the harmonic oscillator and let me calculate i h cross d x by d t, x is an operator. In the Schrodinger picture x operator is a constant. Please note x operator is a constant, P operator is a constant. In the Heisenberg picture x operator becomes a function of time okay. So, let me put x H, so this is equal to x H comma H.

So, let me substitute from the values of x, if you go back we had written this h cross by 2 m omega a dagger plus a comma H operator is h cross omega a dagger a plus half this is the x operator. All these operators they suppose be function of time, a dagger is a function of time, now a is a function of time everything is seems to be function of time, but let me substitute the x operator and the H operator here. So, if I expand this, I will get h cross by 2 m omega comes out and h cross omega comes out h cross omega comes out what will happen to this factor half. That will not contribute because, a dagger plus a commutator with a dagger a, and a dagger plus a commutator with half. I can open the commutator into two commutators a dagger plus a commutator with a dagger a and a dagger a commutate with half and because, half is a number a dagger a commutate with half just disappears. So, I will get a dagger plus a commutator with a dagger a and I will leave it you to show you can use the commutation relations between a dagger and a and what you will get is essentially sorry into i h cross okay.

So, let me (( )) give on this what you get is actually h cross omega under root h cross by 2 m omega into a minus a dagger. This is what you get and this final leads to the fact that x H by d t is equal to P h. This is the same as a classical equation of motion d x by d t is equal to P by m. Now it is an operator form because, these are all now operators similarly, I leave it for you to calculate d p, H by d t and you will get at as minus omega square into x cap. The Heisenberg picture is very close related to the classical picture.

Sir that P you have written,

Yeah, just wait wait ah. We will repeat it. I think I have to do a little careful analysis by expressing x h in terms of the in terms of the Hamiltonian in terms of the going from the Schrodinger to Heisenberg picture and then calculate, but I think I will repeat it again, but this equation is what you will finally get as the Heisenberg equation of motion of the x operator and the P operators.

So, I think I will have to be little careful in this analysis, I will leave it as a problem why do not you people tried out also and we will resolve it in the next class okay.

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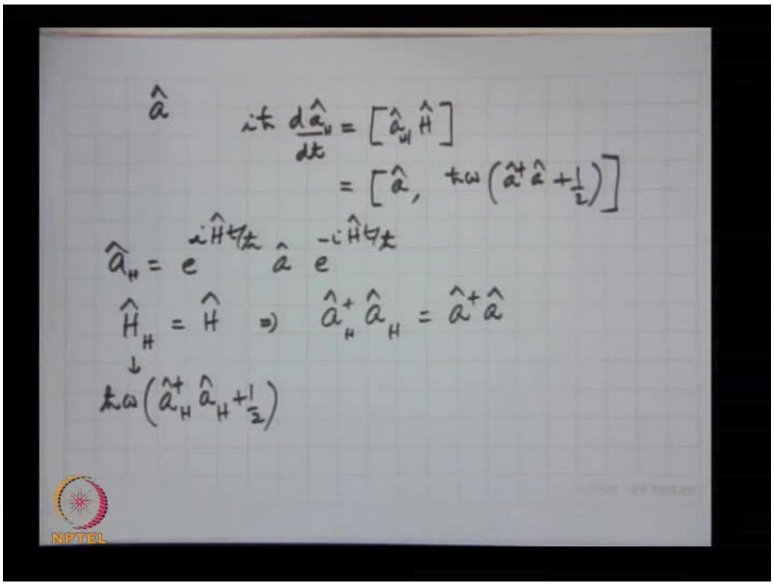

So, now let me calculate. For example; the evolution of this operator a, i h cross d a by d t this is a h cross omega a dagger a plus half (( )) a h operator will be exponential minus i.

No no no I think I have to go back to the fact that the H operator in the Heisenberg picture is the same as the H operator in the Schrodinger picture. So, this implies that a dagger h, a H is equal to a dagger a because, H H I will write as h cross omega a dagger H, a H plus half and similarly, for H operator.

I mean this equation you have written i h cross Del a by Del d.

Yes.

That should be this in the Heisenberg picture this equation of

This one.

In the other equation

Here

Yeah.

Yeah I am just coming back to that. I am just trying to figure it out wait. So, a H is exponential i H t by h cross a H is the same, I have to work out the

Yeah.

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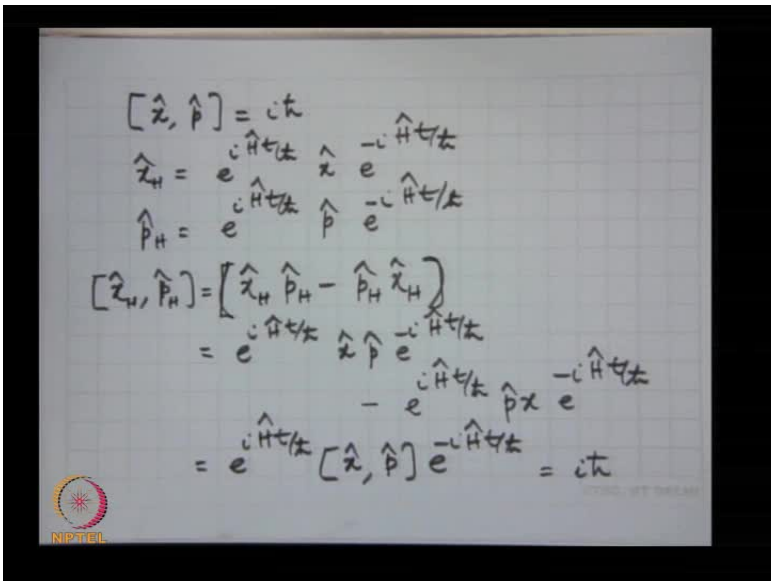

Actually, we have to work out the commutation relations in the Heisenberg picture okay. The commutation relations are the same.

So, let me let me go back. For example; we have commutation relations between say x and P is equal to i h cross right, this is the Schrodinger picture.

Now let me go to the x Heisenberg pictures. X Heisenberg is actually exponential i H t by h cross, x exponential minus i H t by h cross and similarly, P h is equal to exponential i H P by h cross P exponential minus i H t by h cross. So, x h, P h is equal to the commutator of these two, so this into this so will be x H, P h minus P h, x H let me put the bracket okay.

So, if I substitute this, I get x H into P h will be exponential i H t by h cross, x into P because, these two operators will cancel each other and I will get exponential minus i, H t by h cross minus exponential i, H t by h cross into P x into exponential minus i, H t by h cross, which is equal to exponential i, H t by h cross commutator of x and P exponential minus i H t by h cross is equal to (( )) because, x P is i h cross that is a number and I get the operators x H and P h satisfy the same commutation relations as x and P. The commutation relations have not changed.

(Refer Slide Time: 21:40) So, actually in this equation, the commutation relation between x H and H is the same as the commutation relation between x and H okay. Because, the commutation relation in the Heisenberg picture and the commutation relations in the corresponding Schrodinger picture are the same that is why this analysis is still all right because, I am using the commutation relations of the Schrodinger picture to analyze the problem. Now what I will do I will leave this problem to you to write the x H in the Heisenberg picture and analyses problem and finally you will get d x h by d t is equal to P h by m.

Sir

Yeah.

We will be obtain to consider the commutator as an operator and then multiply the exponential on both the sides.

Which one,

That should be commutators

Ok.

This as an operation.

That's an operator yeah.

Then multiply the equations on both the sides and then evaluate it.

Yeah.

Even if it is in.

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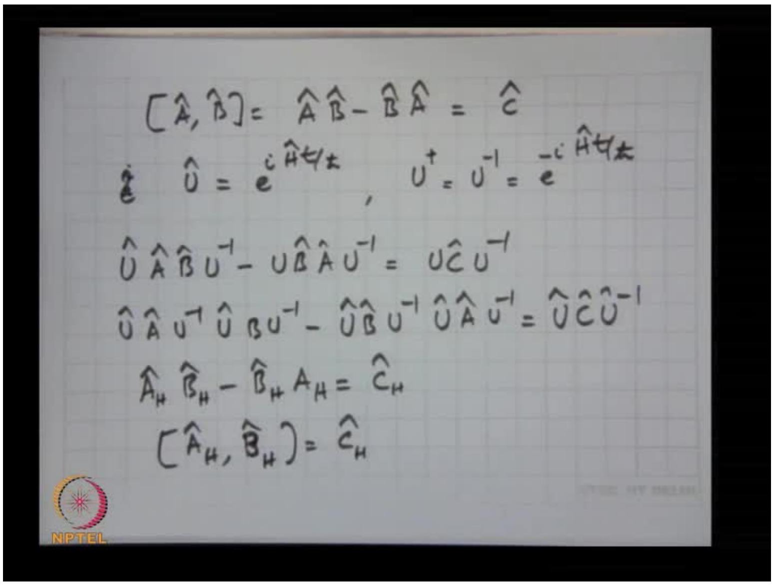

What you are saying is if I have a commutated relation A B this is A B minus B A. So, if I yeah but, this this commutator will remained a same.

If it is operator still multiply the equations on both the sides and it will give me the omega operator in the.

Yeah, this is suppose let me assume C. I will have exponential i. Let me call this U, U operator as exponential i H t by h cross. Remember U dagger is U inverse is equal to exponential minus i H t by h cross. U dagger will have H dagger and H dagger is H and i changes to minus 1 and this is a same as universal operator, I have U A B, U inverse minus U B A, U inverse is equal to U C U inverse.

Yeah.

Yeah, this is U A, U inverse U B, U inverse minus U B U inverse U A, U inverse is equal to U C, U inverse all these are operators. This is A h, B h minus B h, A h is equal to C h. This is A in the Heisenberg picture satisfy the same commutated relation.

Yeah.

Sir in our case B h is already Hamiltonian which is already equal to H in the Heisenberg equation.

I am sorry.

Sir in our case B h is H so,

Yeah that is one particular pair, but I could have any general pair of operators A and A dagger for example; a and a dagger. So, a and a dagger will satisfy the same commutation ration of one even in the Heisenberg picture.

If it is there is an operator on the other side that must be also express in the Heisenberg picture that is all.

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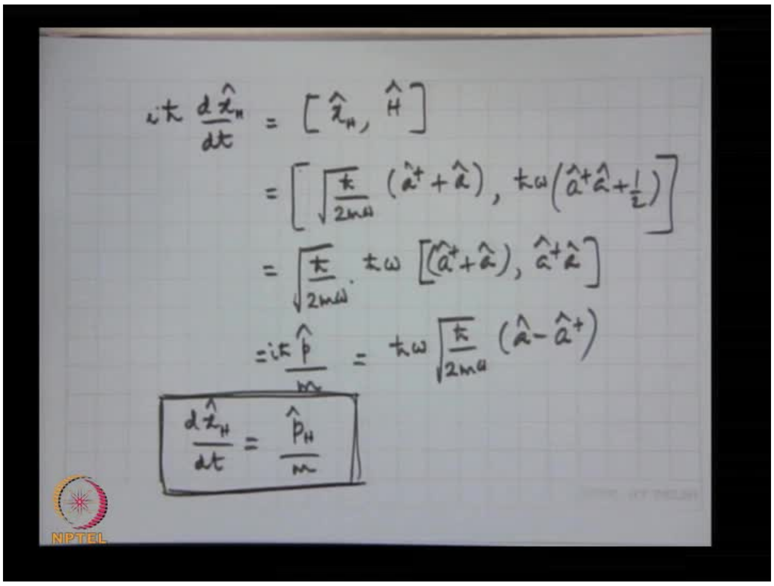

So, let me take as an example. Let me take an harmonic oscillator state which is half of 0 plus half of 1 this is at t is equal to 0. So, what will be psi at t half exponential minus i.

h cross omega at 2

By h cross.

Plus half 1 exponential minus i.

This is energy half h cross omega this is energy 3 by 2 h cross omega.

It should be 1 by root 2

One by root 2 yeah.

Can you calculate what will be x expectation value of the function of time.

So, this will be essentially psi of T. When I am writing, it is the Schrodinger picture because, I am actually evolving the state as a function of time. The state of the system was this combination at t is equal to 0 it has evolved to another combination I will do later time. This is the Schrodinger picture in which, the operators are constants.

So, why do not I leave this to you, you can substitute this use the relationships with destruction and creation operators and what you will get is essentially h cross by 2 m omega cos omega t.

Yeah Sachin problem okay.

So, this is superposition state of two energy eigenstates and the expectation values varies with time. Now what I would like you to do is to please use the Heisenberg picture. So, in the Heisenberg picture x is normal function of time. The Heisenberg picture x will be given by psi of 0, x is a function of time psi of 0 and you should get the same expression.

I can shift the time dependence from the kets to the operators that is a simple exercise let me put the h. Use the Heisenberg picture and show that this expectation value is consistent with expectation value for this, you would need how x change of a time which you will calculated it will be in terms of a and a dagger. Please note a and a dagger also functions of time now, because x itself is a function of time, a and a dagger are functions of time and then you can calculate the expectation value of x and similarly, expectation value of t etcetera.

So, why I did this is is because, when we quantize electromagnetic fields, the Heisenberg picture is much easier to analyze than the Schrodinger picture. In many problems in field theory, Heisenberg picture is much more convenient use than the Schrodinger picture.

So, when I quantize electromagnetic fields, I will have to represents electric field, magnetic field by operators. In the Schrodinger picture, these would be constants independent of time and a state will work as a function of time, but when I go to Heisenberg picture, the electric field operator and magnetic field operators will be functions of time and the state will be fixed and it will be much easier for us to analyze using the Heisenberg picture than the Schrodinger picture.

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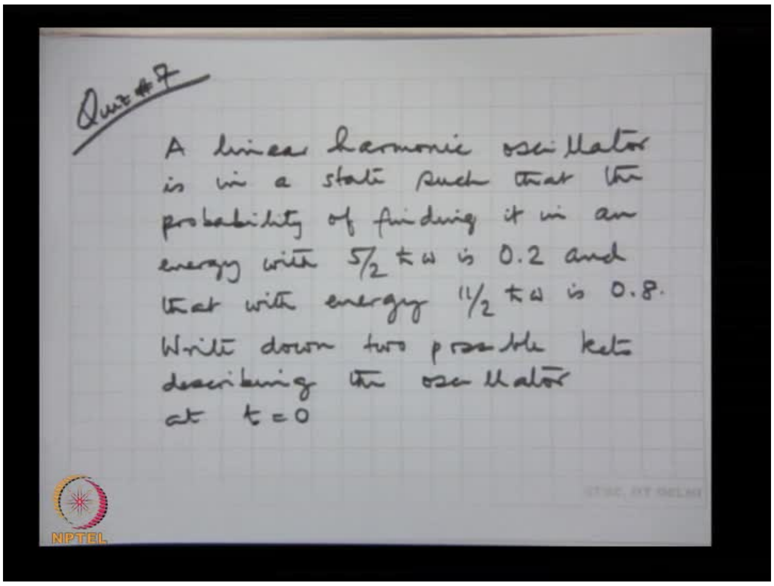

This is a small introduction with harmonic oscillator that I wanted to give before we use this little more extensively when we go through the quantization electromagnetic fields.

Okay very simple question, a linear harmonic oscillator is in some state such that, if I measure it’s energy the probability of finding 5 by 2 half 5 by 2 h cross omega is 0.2 in the property of finding 11 by 2 h cross omega is 0.8. You have to write two possible kets describing this oscillator.