Plasma Physics

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Lecture No. # 05

RF Conductivity of Plasma (contd.)

Well friends. Last time we discussed the response of plasma to radio frequency electric field, in the absence of any other field like a magnetic field or anything else. Today, I like to extend that study to electric field that has space dependence, and also study the response of plasma to an electric field in the presence of a DC magnetic field. Because most plasmas, if you have to confine them from expansion stops them from expansion, you employ a static magnetic field. And hence, a realistic plasma response would have to be investigated in the presence of a DC magnetic field, but before I go over to discuss the response of plasma in the presence of a DC magnetic field.

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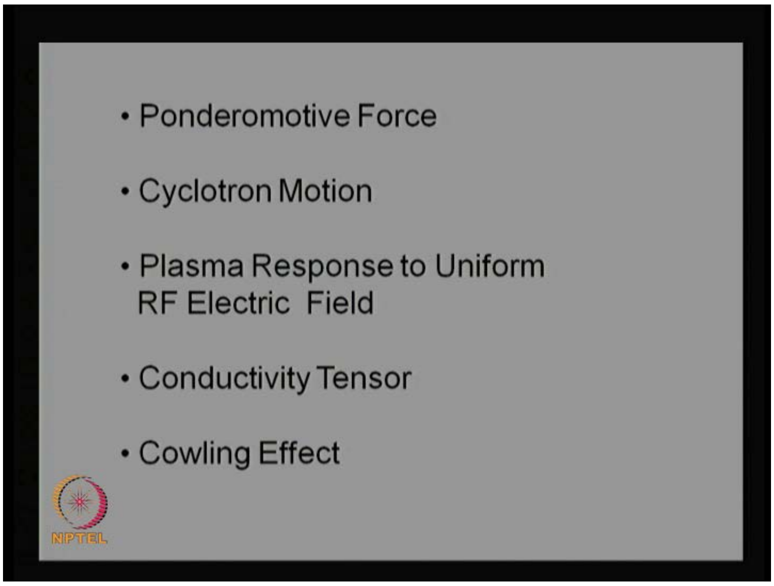

Let me consider the problem of plasma response to a RF field, which depends on a space. And then I will go over to discuss the electron motion in a magnetic field alone, and then plasma response to uniform radio frequency electric field. I will reduce an expression for conductivity tensor. And Then I will discuss an important effect called Cowling effect.

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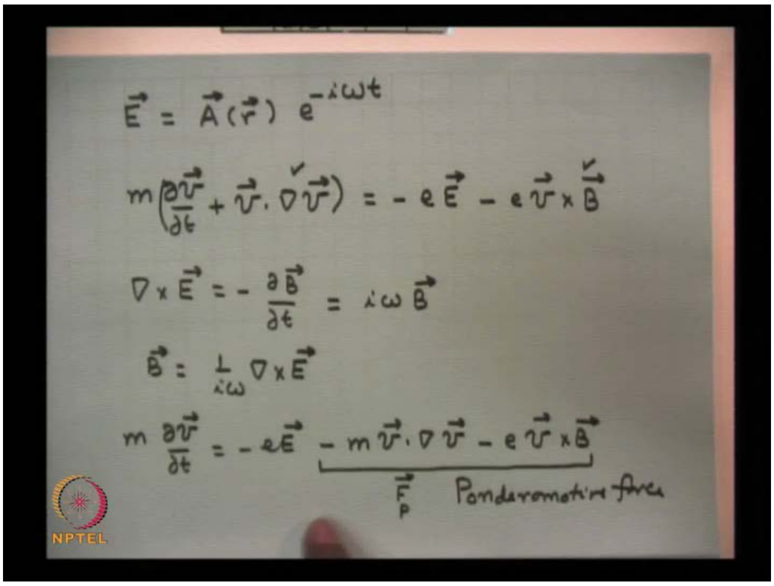

So, in order to understand the response of plasma to an electric field, which varies, whose amplitude varies in position, say r in general and time dependence of this form. This is very important issue, understand, that the electric field gives rise to velocity to particles, velocity means their position changes. So, when position changes r will change it time. So, you cannot take this to be constant.

So, there is an inherent or hidden dependence of an on time. As explicitly, it does not depend on time, But because the position of a particle that, say this electric field varies from time to time due to the velocity it acquires, due to the RF field, this A has hidden time dependence.

So, this is a serious problem and I will have to employ the technique of iteration to solve the equation of motion. Let me write down the equation of motion, for the sake of simplicity, I will ignore collisions; I will ignore pressure term anything.

Just consider, simple response of plasma to an electric field of this form, the equation of motion is m delta v by delta t plus v dot del v is equal to minus e E. However, there is a problem here, if there is time varying electric field with amplitude a function of position, if you take curl of e, it may be 0, and it may not be 0. If it is not 0, then third Maxwell equation will tell that, there is a magnetic field also should be there. And hence, I must add a term here, minus e v cross B. So, this B is the magnetic field associated with a time dependent electric field, whose amplitude depends on r.

Now, this is a little complicated equation. This is the electric force; this is the magnetic force which involves the response v also. My goal is to find v due to e, what is the particle velocity due to the electric field e? Well, first let me calculate, what is B? From third Maxwell equation, curl of E is equal to minus delta B by delta t in mts units. And because electric field in changing in time like this. In the steady state, B will also change with time like this.

So, I can replace this by i omega, delta delta t by minus i omega, already minus is there. So, simply this becomes like this. So, you see here, that B is equal to 1 upon i omega curl of E. If my electric field is small, B is also small and the response v will also be small.

So, how do we solve this equation? The terms that depend on the product of a small quantity are called the small terms. Like, v if I multiply with another v here, this will be product of very small quantities; it will be much smaller than this term. Similarly, v is a small response, small velocity, if my e is not large. And B also will be small, because e is not large. So, this is a product of two small quantities and you can ignore them.

So, we solve this equation iteratively, by first ignoring this term and this term. Rather than ignoring them, let me rewrite this equation like this; m delta v by delta t is equal to minus e E, So, take this term on the right hand side, it becomes; minus m v dot del v minus e v cross B. These two terms are called non-linear terms, because they are the product of small quantities and they have the dimensions of force. So, I can combine them together from here to here and give a name to them, in literature, the name given to them is F p; non-linear force also called ponders motive force.

So, I would like to calculate the value of this iteratively. First of all, I will ignore the ponder motive force, obtain the response of electrons to the electric field alone and then, use the value that I will calculate here, in these terms and evaluate those terms. Let us do that.

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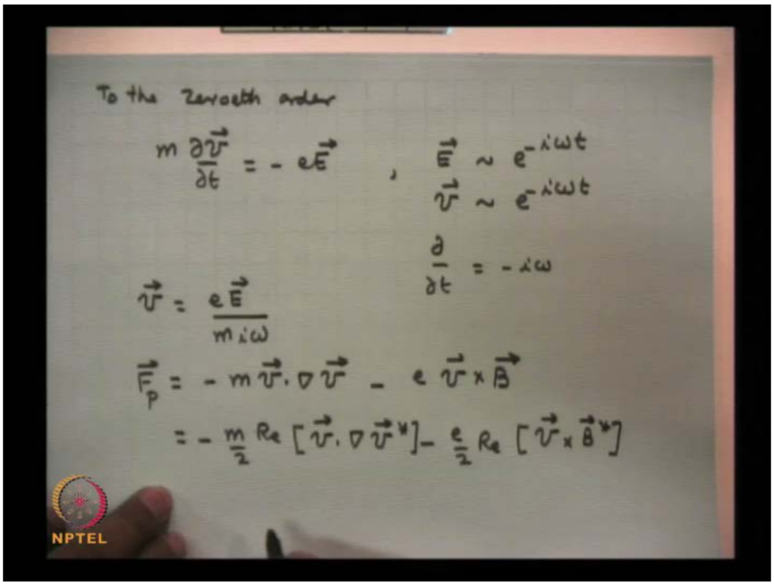

So, through the zeroth order, what do I get, m delta v by delta t is equal to minus e E and because this has simple time dependence of the form, exponential minus i omega t.

So, velocity also, I would expect to have time dependence of the form exponential minus i omega t. Hence, delta t, I will replace by minus i omega as I discussed earlier.

Put this in here, you will get the velocity of the electron is equal to e E upon m i omega. So, now my goal is to calculate the ponder motive force, F p which is equal to minus m v dot del v minus e v cross B. I will have to use the complex number identity to evaluate these products. Because here really, real part of v is to be placed here, a real part of v has to be used here. Similarly, real part of v has to be used there. So, it is a product of real part of v into real part of this quantity. Similarly, real part of v is to be multiplying with the real part of capital B there.

What will I do? I can write this as, minus m real part of, so, real part is implied; I will not write explicitly the symbol Re. what is implied? Or let me write this, half there and v dot del v; I am not interested in time dependent part of ponder motive force, for the moment, I want time independent part of ponder motive force. So, I will take only complex conjugative.

I just explained, in my previous lecture, that when you have to multiply terms, then, there are always two terms; one without complex conjugate, the other one is complex conjugate. The term with complex conjugate, the exponentials cancel out. So, this becomes a time independent term. So, I will write only that.

And then, here it becomes e by 2 real part of v cross B star. So, this is time average ponder motive force otherwise I should write two more terms, I will not write them.

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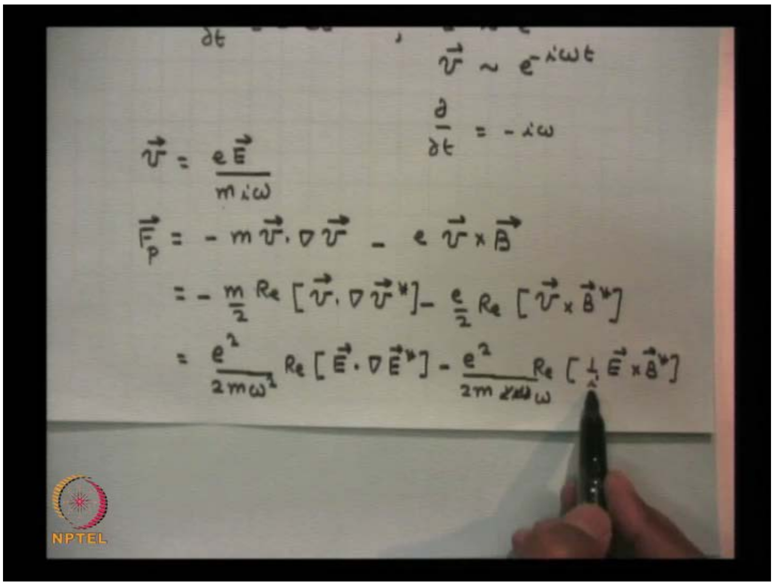

Let me evaluate this number, this expression, this force. What you get here, v if I substitute from here, and takes complex conjugate there, this becomes simply equal to e square, one e is there. So, e square m square comes in the denominator, one m will cancel out with this, so e square upon 2 m omega square will be common, and will be real part of E dot del E star minus if I put the value of v again here, it becomes e square upon 2 m.

Now, i omega is there, but B also has... So, let me just remind you well, I think let me put this, to in two steps; i omega is because of this. I think I should cancel this here, because real part I have to take. So, this becomes e square upon 2 m omega outside ,and this becomes 1 upon i into E cross B star. So, I just substituted the value of v from this expression in here, and taken the constant terms outside, imaginary term I should keep inside, this is what you get.

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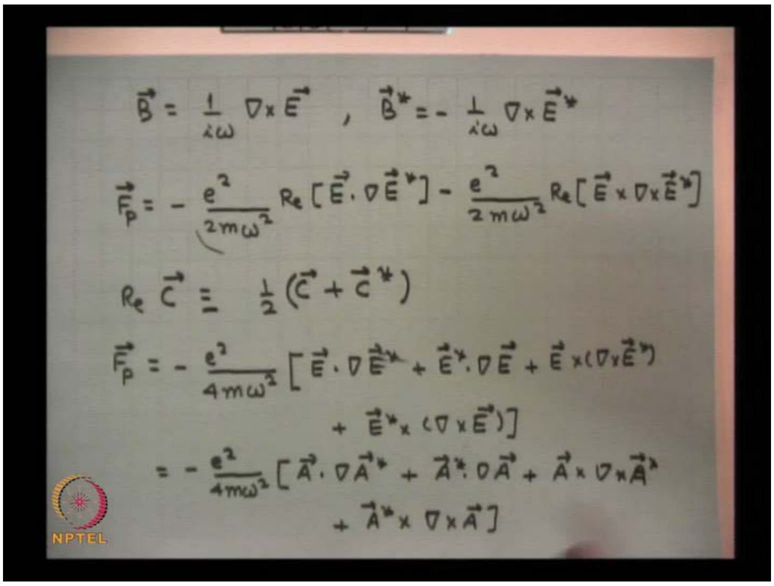

Let me, remind you that B, I had written as, 1 upon i omega curl of E. So, use there; B star will be minus 1 upon i omega into curl of E star, complex conjugate of e.

So, if I use this in the expression for F p, becomes, equal to, I forgot the first term was a negative sign also, in the previous case it was actually had a negative term, minus e square upon twice m omega square and real part of E dot del E star minus e square upon twice m omega square and you will get real part of E cross curl of E star. This factor is common in both the terms. And you always know that, real part of any complex number C is the same thing as, half C plus C star. If there is any complex number, if you add its complex conjugate with it, the imaginary parts will cancel out and this is always true.

So, now this is a complex number in general, what I can do? I can write this expression as, this expression plus it is complex conjugate divided by 2. So, F p becomes minus e square upon 4 m omega square and this becomes E dot dell E star plus complex conjugate of this quantity, which is E star dot grade E, just reverses the stars.

Similarly here, this is plus E cross curl of E star plus E star cross curl of E. This is a very general expression for ponder motive force due to a RF field, which does not depend, whose amplitude does not depend on time, but depends on space and this could be arbitrary displace dependence. So, this is the general expression.

Now, put this, e in terms of A, then, this can be written simply as, minus e square upon 4 m omega e square, all the exponential factors will cancel out, because they are complex conjugate terms and they will be simply, A dot del A star plus A star dot del A plus A cross del A star here plus A star cross curl of A. This is a vector quantity, it has x component, y component and z component. I evaluate only one component, F p x and then, I will generalize to. Similarly, I will write down y, z component. So, let me write down the x component, for a general case. F p x will be how much? Let us see.

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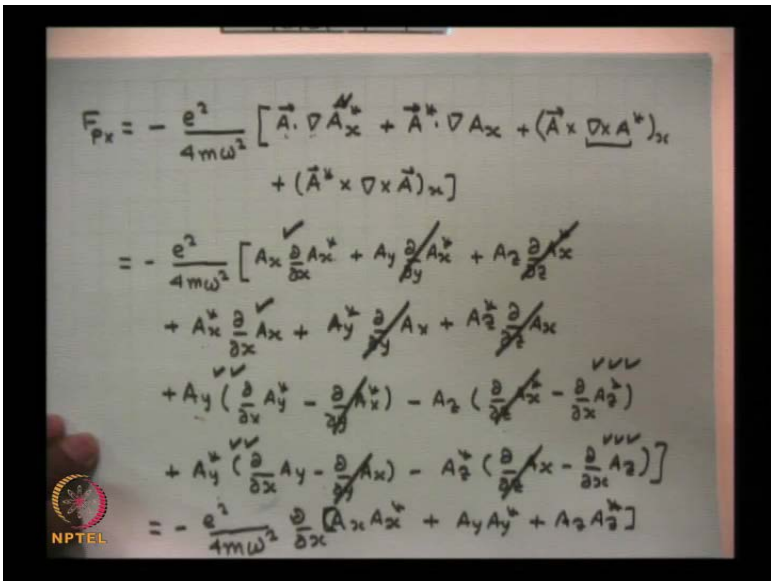

F p x would be minus e square upon 4 m omega square out. Now, the first term was A dot del A star. So, I am writing x component, this will be A dot del of A star, x component, I should write here, forgot this sign here. Because A dot del is a scalar quantity. So, when I had written a star, I should write like this, plus second term was A star dot del, which is a scalar quantity, so total quantity has to be written and A was there, so, put A x there.

Similarly, A cross curl of A star, I want x component. Let me write this simply, A cross curl of A star, I want it is x component. Then, I want x components quantity, A star cross curl of A, x component of this quantity I want.

Let me simplify this, this is equal to minus e square upon 4 m omega square, A dot del has three terms; this is A x delta delta x of A x star plus A y delta delta y of A x star plus

A z delta delta z of A x star. A dot del is a scalar product. So, it has A x delta delta x plus A y delta delta y plus A z delta delta z and all the three components have to be added, but this they all operate over A x star.

Similarly, second term will give me, plus A x star delta delta x of A x plus A y star delta delta y of A x plus A z star delta delta z of A x. Then, the x component of this product which is little complicated, but it is not that bad.

So, I treat this as one vector, this is another vector; A cross B kind of thing. So, I will get x component of this quantity means, y here z there minus z here y there. So, this will be, x component will be, A y multiplied by z component of this quantity, z component means, x y minus y x. So, delta delta x of A y star minus delta delta y of A x star, this is A y into z component of this quantity, minus z component of this and y component of this. So, y component of this quantity would be delta delta z of A x star minus delta delta x of A z star. I am very explicit on this, because this is something, which is very useful, very important in non-linear plasma theory.

So, the derivation of this, I am really doing step by step, so, that you have some feeling for this, plus I have to write down the x component of this quantity, which is very similar to this one, only the stars are interchanged.

So, this becomes A y star delta delta x of A y minus delta delta y of A x minus A z star delta delta z of A x minus delta delta x of A z like this. I have been very explicit, but this is useful.

Now, you may note here, that only two terms survive, others cancel out. Not to me, let this, most of the terms cancel out. For instance, this term A y delta delta y, you will find A y delta delta y here as well, A y delta delta y of A x star; this term will cancel with this term. This A z delta delta z A x star term, A z delta delta z of A x star term, this will cancel out with this. Then this term, this will survive, A y star delta delta y, A y star delta delta y A x, this is negative, this is positive, they will cancel.

Then this, A z star delta delta z, A z star delta delta z of A x, this will cancel out with this one. So, lot of terms have canceled out, what you are left with, this is A x delta delta x of

A x star and this is, these two terms can be combined, become delta delta x of x star A x A x star.

Similarly, this term delta delta x is common, so it becomes, this term can be combined with A y term, this one. So, these two can be combined, so, this can be combined with this term, this can be combined with this term and this term can be combined to the last one. They have same sign. So, this can be combined with this term and when you combine them, you get a simple expression minus e square by 4 m omega square delta delta x of A x A x star plus A y A y star plus A z A z star.

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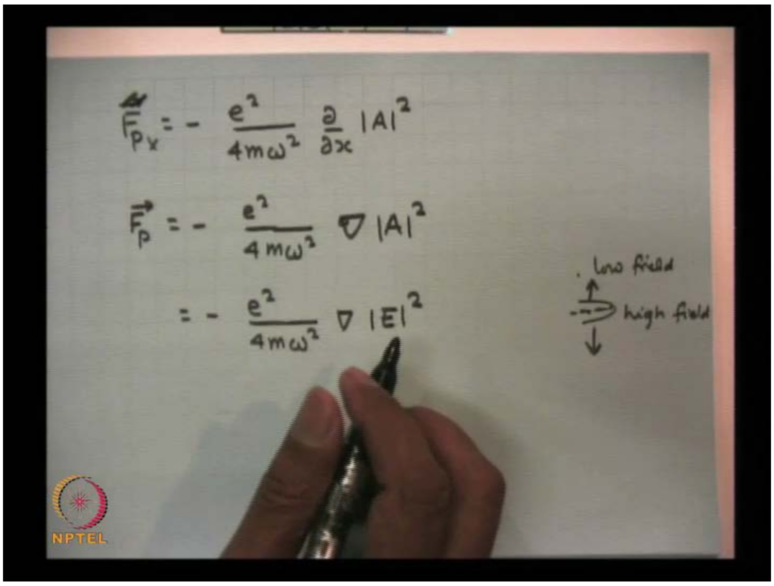

Let me rewrite this expression, this becomes, so, the x component of ponder motive force is simple, minus e square upon 4 m omega square delta delta x of modulus of A square.

So, this is, this is not vector, this is a scalar quantity; x component. Similarly, if you calculate the y component, z component, then, you will write F p vector is equal to minus e square upon 4 m omega square del operating over A square modulus of A square. But modulus of A is the same as modulus of the electric field, so, minus e square upon 4 m omega square gradient of e square.

This quantity is very important in plasma theory, plasma physics. What does it tell? This says that, if any region of space electric field has non-uniform amplitude, suppose the electric field has a larger value here and a smaller value there, then, the gradient of electric field is in this direction, the force will be opposite. Means, this force compels the electron to go from high field region, this is high field region for instance, to low field region.

Electrons will go like this, they will flow out. It is a very important consequence; it has a very important consequence in plasmas, because if you are sending for instance, a high power laser in plasma, the intensity of the laser is large on the axis and small outside. So, when the laser goes into plasma, the electric field is large on the laser axis and less outside. So, laser repairs the electrons from the central region or actual region to outside, creating a plasma channel. This non-linear force has been recognized, has been measured and has been observed in numerous experiments and has been found to play very important role in laser guiding and many other non-linear phenomena.

Here, I have calculated only the time average part of this ponder motive force. If you had calculated the time dependent part of ponder motive force, this is responsible for harmonic generation, which is another important non-linear effect.

So, in our response of plasma to radio frequency field, we have learnt that the radio frequency response at low fields can be described in terms of a quantity called plasma conductivity. But when the fields are strong and when they become time space dependent, in that case, an additional response term of force x on the particles, which is called ponder motive force. What is the response of plasma to this force? We shall discuss later on, when we discuss the phenomenon of shell focusing, may be a few lectures later. So, we conclude our discussion on radio conductivity or radio frequency response of plasma in the absence of a magnetic field.

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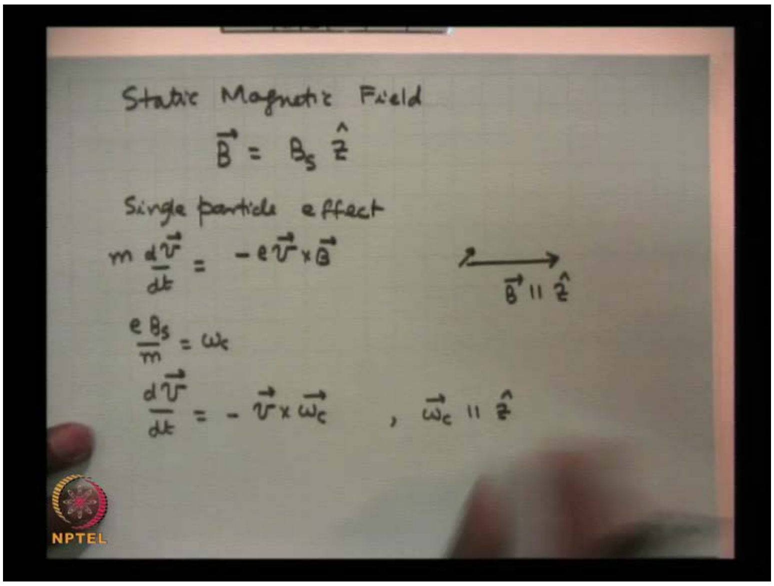

Now, I would like to take you to the plasma response in the presence of a static magnetic field. A static magnetic field in the easiest form can be written as, B static which is independent of position and time. And suppose the direction, I choose to be in the z direction. First of all, I would like to see, what an individual electron will do in the presence of a magnetic field. And then, later on, I will discuss, if you have millions and trillions of electrons. So, electron is a family, electron is a community, how do they respond to this magnetic field or in the presence of magnetic field, how do they respond to the electric field.

So, first of all, let me do the single particle effect. Well suppose, this is my z axis, where I have applied my dc magnetic field. If I launch an electron in the system, which is moving with some velocity here for instance, you know that the velocity of the particle can be resolved into two components; one along the magnetic field and one perpendicular. The electron will experience a force. The force will be charge of the electron into velocity of the electron cross B, this is the force on the electron. And the equation of motion would be m d v by d t for a single electron, this is the equation of motion.

So, what will happen? The electron, physically let us understand what happens, the electron resolve in two components; one parallel to magnetic field, one perpendicular. The component of velocity which is parallel to magnetic field, the force vanishes. Only because of the perpendicular velocity, this will experience a force. And this will try to; the force will be perpendicular to both the velocity and magnetic field. So, perpendicular to the board and electron gyrates about the line of force.

And that motion can be easily understood from this equation. If I divide this equation by m and I define a quantity e B s upon m is equal to omega c for instance. So, this equation becomes d v by d t is equal to minus v cross omega c. Well, omega c is a vector, parallel to z axis. Because static magnetic field in the z direction.

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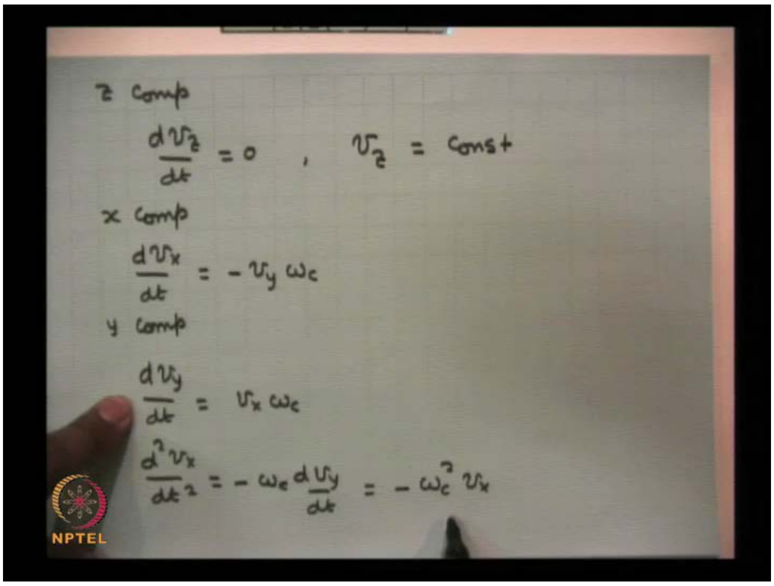

So, I want to solve this equation, this simple equation. But this has very rich physics. First of all, I will write down the z component of this equation, it becomes, d v z by d t is equal to 0, which means that the particle velocity in the z direction will be constant.

Secondly, I write down the x component, that would give me, d v x by d t is equal to v cross omega c; if you write down the x component will be, minus v y omega c. And then, write down the y component the equation of motion, d v y by d t is equal to v x omega c. The x and y components of velocity equations are coupled. So, what you should do, take the time derivative of the first equation, you will get d 2 v x by d t square is equal to minus, omega c is already there, d v y by d t. Now, use of value of d v y by d t from here, it becomes, minus omega c square v x. And a general solution of this equation can be written. Let me, write down that solution.

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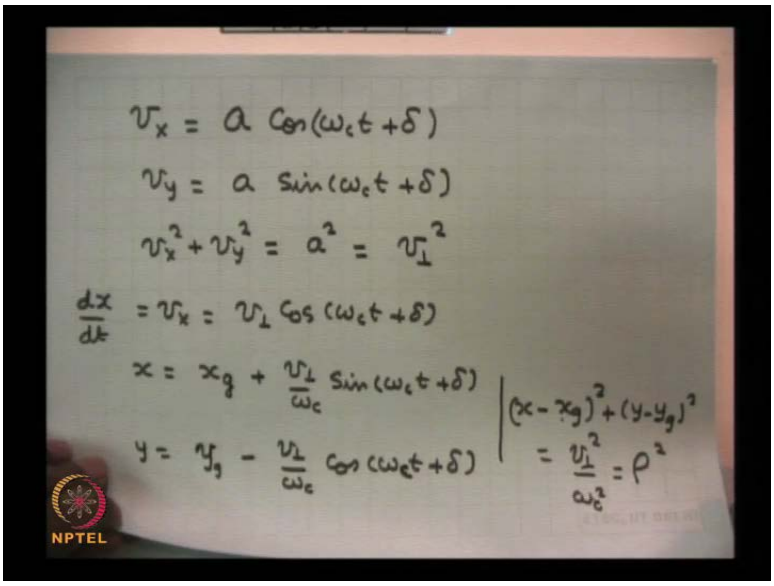

That solution is, v x is equal to some constant, let me call this constant as, a cos omega c t plus another constant, delta. A second order differential equation has two constants. So, this is one constant, this is another constant. And if, v x is like this and if you use the x component equation of motion, then, v y can be obtained to be equal to a sin omega c t plus delta.

So, I have substitute the value of v x in the x component of equation of motion, I get the value of v y. So, these two I obtained. And you may note one thing in here, that v x square plus v y square, if I add, they give me a square. So, a is a constant, means v x square plus v y square is a constant; this is usually given a symbol v perp square.

So, rather than writing a, I will write down a symbol v perp. Then, this equation becomes, v x is equal to v perp cos of omega c t plus delta, but v x is the same thing as d x by d t. So, I can integrate this equation to obtain x as a function of time, because v perp is a constant and this time dependence is simple.

So, x turns out to be some constant, let me call this constant as, x g plus v perp upon omega c sin of omega c t plus delta. And similarly, if you solve this equation for v y, you will get, y is equal to y g; a constant y g minus v perp upon omega c cos omega c t plus delta. This is a very general expression. But the beauty here is, that if I take, x minus x g square and add y minus y g square to it, you will find any constant. So, these two equations give me, x minus x g whole square plus y minus y g whole square is equal v perp square upon omega c square, this quantity is given a symbol called rho or normal radius. This is the equation of a circle whose center is at x g, y g and whose radius is rho, which is v perp upon omega c.

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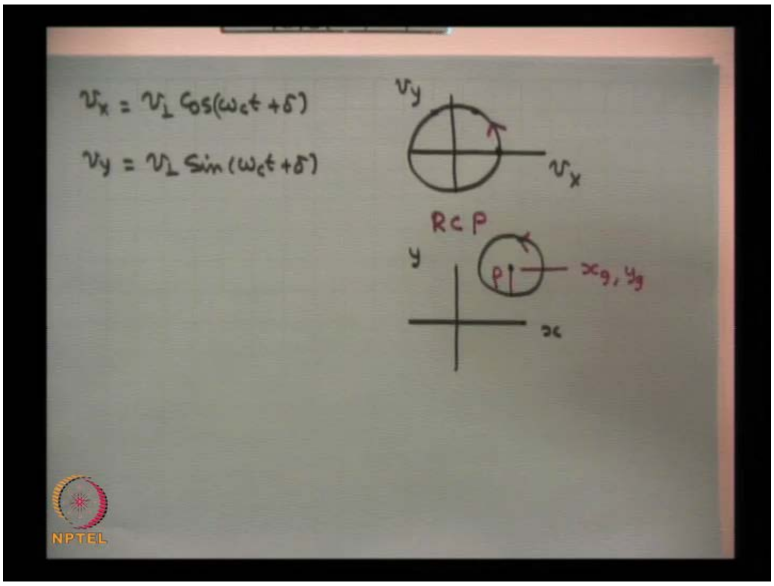

And so, the electron really rotates on a circle, if you draw a graph of, suppose I plot here, v x and v y here and v x I have taken is equal to v perp cos omega c t plus delta and v y is equal v perp sin of omega c t plus delta.

Now, if you plot for the sake of specificness, suppose I choose delta equal to 0, really does not matter, what delta you choose? Delta equal to 0, time t equal to 0, v x will be v perp and v y will be 0, so, you are somewhere here.

At a later time, this quantity will decrease, this will increase and this electron will rotate like this. So, this is the way the electron will move, the electron was initially here, it went from here to here, then here and everywhere. So, this is rotating, in a sense that if I take a right handed screw and rotate along the path of this like this, if I rotate it like this, then, this will advance in the direction of z axis, this electron motion is known as Right Circularly Polarized motion. Right circularly polarized or it is rotating in the right handed sense.

Similarly, if I plot, x here particle coordinate x there and particle coordinate y there, then, around a point x g, y g, the electron will rotate over a circle and the coordinates are like this. This coordinate are x g, y g and this radius is r, rho rather and electron is rotating like this.

So, what is happening that, electrons rotate about a line of force, they can move along the line of force with velocity v z, but they cannot live the line of force. So, whenever there is a magnetic field in the plasma, the electrons in the absence of any other force, I am considering no collisions, no temperature variation, nothing except a dc magnetic field, which is uniform.

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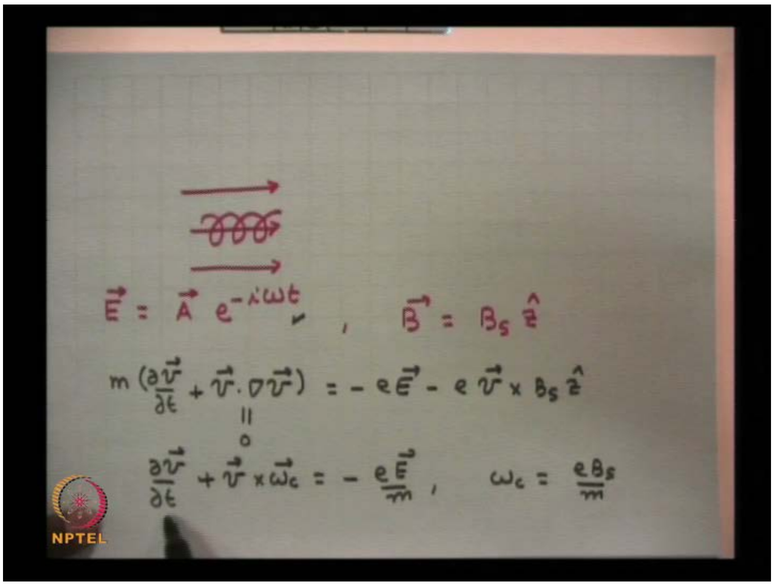

So, if you have lines of force like this, then, the electrons move along the line of force like this, but they do not move perpendicular to the line of force.

So, anyway the electrons are confined to the lines of force and magnetic field provides a confinement to the plasma. Obviously, there is problem that what would you do, either you find some sort of reflection of electrons along the lines of force by some mechanism or do something or close the line of force, only then, you can confine the plasma, that is a separate issue. So, but we have really found something very significant, that a single electron in the absence of any other force will not move perpendicular to line of force.

Now, I would like to examine the effect of a dc magnetic field, RF electric field on the electrons in the presence of a magnetic field.

So, my issue is, I have a plasma in which I am applying a electric field of this form whose amplitude is constant in time, the field is independent of position coordinate also. And I would like to examine the response of the plasma, when there is a static magnetic field also, which is equal to B s in the z direction.

So, the electric field of the RF can have a component along z axis and it can also have components perpendicular to B. Let us see, what is the response? Because we are ignoring any space dependence. So, the equation of motion, then, in this case would be, m delta v by delta t plus v dot del v is equal to minus e E minus e v cross B s, which is in the z direction. This is my equation of motion, I take this term to be 0, because v does not depend on x y z.

So then, if I divide this equation by m, my equation becomes, delta v by delta t, I will take this from the left hand side, so it becomes, plus v cross omega c is equal to minus e E upon m. Remember, omega c I am defining as, e B s upon m. The same omega c was appearing in the equation of motion in a pure electric magnetic field and this is known as the electron cyclone frequency, because the electron rotates with this frequency around the line of force.

So, I have to solve this equation. Because my time dependence of the electric field is like this, in the steady state, I expect the v also to be having same time dependence. Hence, I will replace delta delta t operator by minus i omega and let us see, what do I get.

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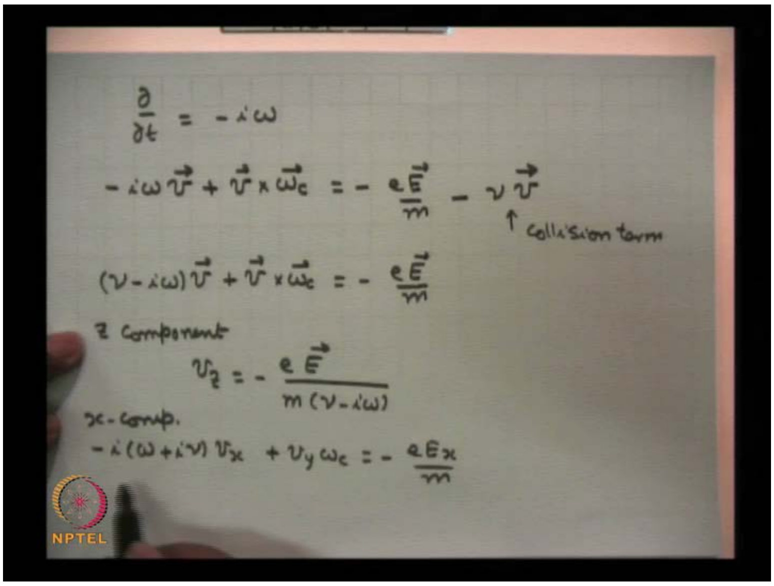

So, by replacing delta delta t by minus i omega, my equation of motion becomes, minus i omega v plus v cross omega c is equal to minus e E upon m. If there were collision, then I should use a collision term also which will be minus nu v is the collision term, I can add. And this can be combined easily with this v term here. So, it becomes this is the collision term, in many cases you can ignore this, but for the sake of completeness one can include this here. So, you get, nu minus i omega into v plus v cross omega c is equal to minus e E upon m.

I will write down different components of this equation. Let me begin with the simplest one, which is z component, because this term vanishes. So, v z turns out to be, e E upon m nu minus i omega with a negative sign. Then, write down this x component of equation, I can write down this x component is equal to minus I, actually take minus i out, omega plus i nu inside into v x, this will become v y omega c is equal to minus e E x upon m, this is my x component equation.

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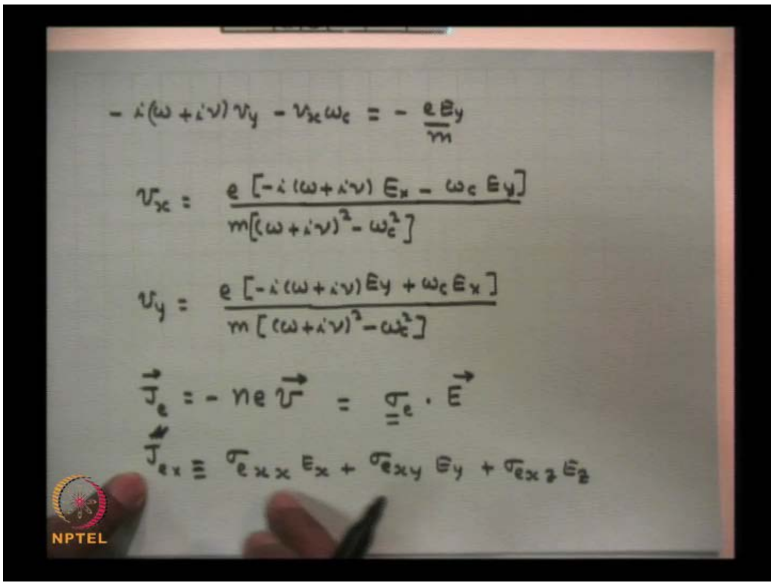

Similarly, I will write down the y component, which turns out to be, equal to minus i omega plus i nu into v y minus v x into omega c is equal to minus e E y upon m. These equations are coupled algebraic equations for v x and v y, which can be solved to obtain v x and v y. And I will simply write the value of v x and v y separately. This turns out to be, is equal to e upon m omega plus i nu whole square minus omega c square multiplied by minus i omega plus i nu into E x minus omega c E y, this sort of expression you get.

And v y, can simply be written, v x is equal to this expression, I have correctly written and v y is the same similar expression, e upon m omega plus i nu whole square minus omega c square into minus i omega plus i nu into E y plus omega c E x. Once, we know the velocities of the particles, drift velocities of the electrons, you can write down the current density. Current density due to electrons, I will write as minus n e v. You may note one thing, v x involves not only E x, it involves E y also. V y involves not only E y, it involves E x also. Means, one component of J will involve not only the electric field that component, but other components of electric field also. So, J x will involve J y besides E x, J y will involve E x besides E y and so on.

So, in general, if I substitute this v in this one, I can write down, sigma e electron conductivity dot E. It is a symbolic representation; sigma is a tenser called conductivity tenser. And this implies that, if I have to write down, J e electron current density x component, then, I must write down the x component here. Now, this tenser has two indices, sigma e so, I should write down the first index, is the same as of x of J and other is running. So, like this is, J x x E x plus sigma e x y E y plus sigma e x z E z. This is the meaning of the y definition. So, in tenser representation, J x is a scalar quantity, so I should not put a vector sign there. Like this similarly, you can write down J e y. So, if you compare the coefficients of E x in x component of current density, you will write down the value of sigma e x x, sigma e x y, sigma e x z and if you evaluate these quantities, you get this expression.

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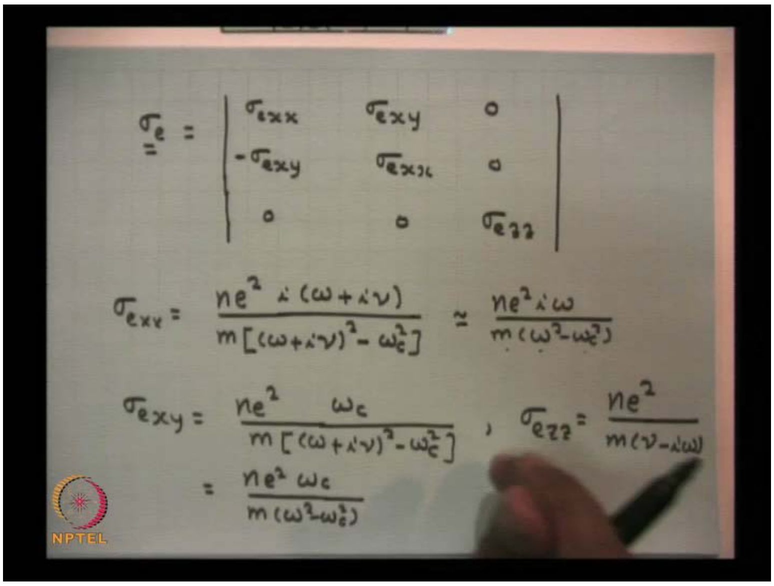

The expression is, sigma is a tenser electron conductivity, which is equal to, the following components are finite, sigma e x x, this is sigma e x y, sigma e x z is 0. The sigma e y x component is equal to sigma e x y with a negative sign. The sigma e y y is equal to sigma e x x. This is 0, this is 0, this is 0 and this is sigma e z z. Where these values are, sigma e x x turns out to be equal to n e square upon m and here is, omega plus i nu whole square minus omega c square multiplied by i omega plus i nu. This is correct expression.

And sigma e x y is equal to n e square upon m omega plus i nu whole square minus omega c square into omega c. And sigma e z z is equal to n e square upon m nu minus i omega. So, this is a tenser and the tenser has non-diagonal terms, finite and they are of opposite sign. The diagonal terms are, two of them are equal, but the third one is not equal. So, this is a characteristic, this anisotropy is a characteristic of a magnetic field or a magnetize plasma.

You may note one thing more here, if you forgot collisions, because collisions are usually a very small quantity. If you forgot collisions, then, this can be written as, n e square i omega upon m omega square minus omega c square. There is a cyclotron resonance effect here, when the frequency of the electric field is closed to the cyclotron frequency of electrons, in that case, this quantity could be very huge. And similarly, this quantity, n e square omega c upon m omega square minus omega c square could also be huge. So, this is one important consequence of applying a magnetic field, so magnetic field not only confines the plasma, it can enhance the conductivity immensely, in the vicinity of cyclotron frequency.

Secondly, if omega is less then omega c, mush less than omega c, then, sigma e x x goes as m into omega c square. In the case of ions, if you write ion conductivity, these expressions are valid. Just replace mass of the electrons, by mass of the ion, and omega c by ion cyclotron frequency, which is very small quantity. And it is possible that ion conductivity could be comparable or even bigger than the electron conductivity that is a very important consequence of magnetic field. And this, presence of the cyclotron resonance effect at electro cyclotron frequency, as well as, at ion cyclotron frequency gives rise to very important wave phenomena, that are useful in exploring space plasmas, as well as, laboratory plasmas. It has tremendous consequence on wave propagation, there are lots of waves in a magnetize plasma, that are possible because of this cyclone effects.

So, plasma response is very strongly influenced by the presence of magnetic field and ion dynamics cannot be ignored in general. Obviously, at high frequencies, when omega is comparable to omega c, much bigger than the ion cyclotron frequency, you can ignore the ion motion, but at lower frequencies you cannot. So, we have to study little more carefully the consequence of magnetic field in plasmas.

This is one thing, the derivation of conductivity tenser. It will be very useful, if you can go step by step and appreciate this. Because when we study the structures, which have non-uniform magnetic field, then, the particles dynamics becomes a little more complicated, but the understanding of this tenser will help us in understanding more complex geometries.