

**Plasma Physics**  
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**Module No. # 01**  
**Lecture No. # 18**  
**Weibel Instability**

Well in our earlier lectures we have studied the generation of waves by electron beams in plasmas, we studied the generation of plasma waves by an electron beam via a process called two stream instability, then we also studied generation of electromagnetic waves via two processes one is called (( )) free electron laser in which the electron beam generates a slow wave by a direct interaction.


And then we discussed the device called free electron laser where an electron beam generates a first wave that moves faster than the beam, but, the presence of a magnetic wiggler provides phase synchronism and that gives rise to growth of waves.

We also discussed the effect of tapering that can lead to efficiency enhancement for radiation generation. However, in plasmas whenever an electron beam travels it is always an issue to examine the instability of the beam or the plasma to filamentation is the beam going to maintain its uniformity of current density or is it going filament or fragment itself into small filaments current filaments. And today I am going to discuss that instability which was discovered by weibel in fifties and given a name weibel instability.

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We shall discuss:

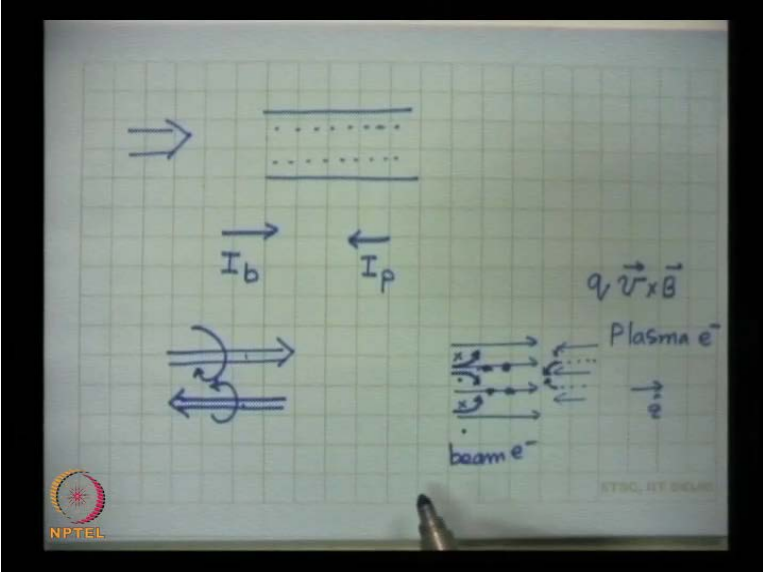
- Counter streaming beam plasma system
- Physical mechanism of growth of magnetic perturbation
- Instability analysis
- Dispersion relation
- Saturation

 Other schemes of self generated magnetic fields

So, today I will discuss what is a counter streaming beam plasma system, then we will discuss the physical mechanism of growth of a magnetic perturbation will carry out a instability analysis, drive a dispersion relation and make some assessment of saturation of the instability and see if time permits are there some other mechanisms of self generated magnetic fields.

Well before I go into the analysis of two stream instability I would like to mention a few things about beam plasma system.

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In a beam plasma system whenever you have suppose a chamber here and launch an electron beam, as the beam enters there then in the beam illuminated region momentarily the space charge becomes negative.

Because if the plasma earlier had equal density of negative charge and positive charge of the ion, but in the additional beam electrons arrive here, then the charge becomes negative. However, in a very short while that negative space charge that is created here will repel the plasma electrons outward it will also repel the beam electrons.

But because the beam electrons are travelling with a large velocity large inertia they have and consequently their departure from this region to outside is negligible, but the beam the plasma electrons which are not moving with any average velocity they move out and you get some sort of a space charge neutralized system this is called space charge neutralization of the beam.

But also happens is that when the beam is launched it creates a magnetic field in the system and as the magnetic field evolves with the arrival of the beam it produces a back e m f, that produces a return current and plasma electrons carry a current in the opposite direction.

So, we often encounter a system in which beam current is in the forward direction and plasma current is in the reverse direction. Obviously, if the two currents are of equal value and they counter each other then the net current in the system is 0 and the beam propagation is termed as propagation under current neutralization. And if you can achieve that situation in which the beam current and plasma current are exactly cancelling each other then there is no magnetic field in the system.

However is it possible that the two currents really do not overlap they may overlap in the beginning, but if there is a small perturbation in the system that makes the two currents deviate from each other.

So, there is no exact overlap then there is a possibility of net current being finite and a magnetic field being generated. I think we are aware of the fact that, if I have a current carrying wire it produces a magnetic field of this sort azimuthally magnetic field around this current carrying wire. And if I have another current carrying wire carrying current in the opposite direction then the magnetic field will be of opposite nature.

So, in the intermediate region the two magnetic fields will add if there are currents in the opposite directions. So, if I consider an electron beam suppose this is my electron beam travelling in this direction and my plasma current is in this direction interpenetrating uniformly interpenetrating and there is no net current in the system suppose I have a system like this.

And if I consider a small magnetic perturbation suppose I have magnetic field which is in one direction and in the opposite direction here in one direction opposite direction perpendicular to the plane of the paper.

So, if there is a current suppose there is a magnetic field produced somehow instantaneously like this then what will happen these beam electrons which are going in this way they will turn around a little bit and here they will turn around this way here, they will turn around this way.

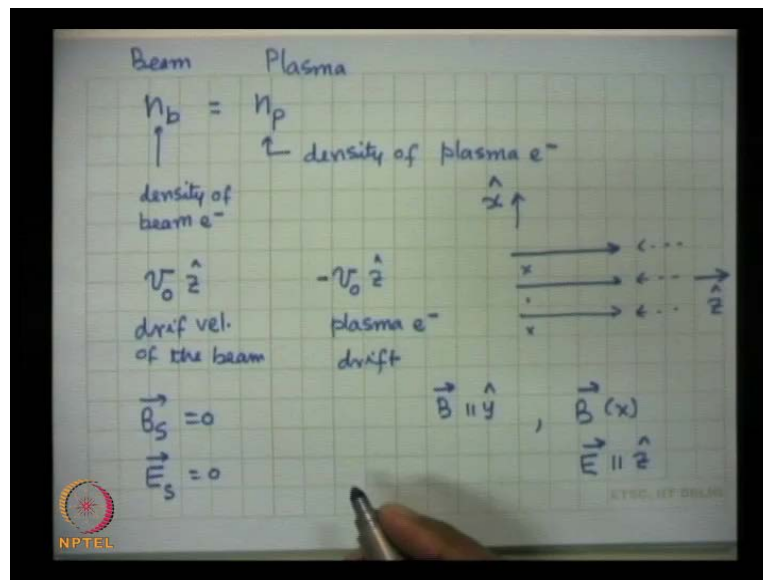
Because the electric magnetic force that acts on them is  $q \mathbf{v} \times \mathbf{B}$ . So, if this velocity is in the z direction suppose this is my z direction and this  $v_0$  this v is in the z direction and magnetic field is perpendicular to the plane of the board of the paper in that case, these electrons will have a tendency to bunch and how about the electrons of the plasma coming from the opposite direction they will have a tendency to bunch in other regions.

Like this electron will try to move in the downward direction and well just like if the electrons were here, this electron because magnetic field is here then this electron will tend to go in the opposite direction this way and this electron which is where here this will tend to go in this direction means the bunching of beam electrons and the we have plasma electrons they will bunch in different regions.

The regions from where the beam electrons are moving out in those regions the plasma electrons will bunch rather I think I have not draw this properly they will bunch in those actually they should they should bunch in these regions the plasma electrons will bunch here and beam electrons will bunch in these regions. This what will happen this is what should happen. And when there is no overlap of the beam current and plasma current then situation becomes very similar to this and in intermediate regions magnetic fields are reinforced.

So, the magnetic perturbation will grow. So, as the magnetic perturbation grows more and more filamentation of plasma current and beam current will take place and that will give rise to growth of the instability. For the generation of magnetic field primarily we acquire a magnetic perturbation net current in the system initially there is no net current, if there is a hundred percent overlap and cancellation of beam current with plasma current, but with the arrival of a small magnetic perturbation the current filaments electron currents in one location and beam current in different location and that gives rise to enhancement of magnetic field and hence the growth of the perturbation this is primarily the mechanism of weibel instability. I would like to consider a situation like this for the sake of instability analysis it is always useful to resolve to a simple geometry.

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I will consider the case, when beam density and plasma density are equal. This is the electron density of plasma electrons density of plasma electrons and this is the density of beam electrons.

So, we will consider a situation for the sake of mathematical simplicity that the two densities are equal and the two particles are moving in opposite direction this is moving with drift velocity  $v_0$  in the positive  $z$  direction and this is beam and this is plasma. So, beam electrons are moving with drift velocity  $v_0$   $z$  velocity of the beam.

And drift velocity of plasma electrons is  $v_0 z$  with a negative sign plasma electron drift this is the equilibrium. I consider there is no d c magnetic field in the system there is no. So, static magnetic field is not there in the system there is no electric field in equilibrium.

So,  $e$  static is also 0. So, this equilibrium I would like to perturb by an electron magnetic wave, now let us physically understand what kind of perturbation we are looking for we are having a stream of counter streaming particles this is the beam stream this is the plasma electron stream I want magnetic field to be perturbation to be perpendicular to the direction of propagation. So, that  $v \times v$  force acts on the particles. So, suppose my perturbation is in this direction, let me for the sake of  $(( ))$  this is my  $z$  direction and let this be my  $x$  direction.

So, I want to consider magnetic field  $b$  of the perturbation to be parallel to  $y$  axis. If you are considering an electromagnetic wave travelling along the  $x$  direction, because  $k$  vector of this wave I want to be  $B$  to be function of  $X$ .

So, though  $b$  itself is parallel to  $y$ , but the value of  $B$  varies with  $X$  means  $B$  depends on  $X$  though it is direction is  $y$ . In such a situation if you examine the Maxwell's equation the electric field has to be perpendicular to  $b$  and  $k$  and hence that should be in the  $z$  direction.

So, perturbed electric field of this perturbation is parallel to  $z$  and we shall discover that other components of electric field are really unconnected to  $z$  component of electric fields and they do not grow.

So, this is the only perturbation. So, it is a simple situation, means consider as if some electromagnetic waves is travelling perpendicular to these waves and we will see what should be the frequency is it 0 or finite or what for a given wave number of this wave.

Now, physically what is happening that the magnetic field is like this? So, these beam electrons will turn around the lines of force will try to move deviate from their own path and they will try to converge somewhere and get rarefied somewhere else.

So, this magnetic field should cause a perturbation or in particle path and hence  $e$  density compression rarefaction somewhere in the process a current is created and beam current

gets filamented in some regions and plasma currents is filamented in a different way, because of the reverse in the velocity of those particles and then density will grow.

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The image shows a handwritten derivation on a grid background. The equations are as follows:

$$\vec{E} = \hat{z} A e^{-i(\omega t - kx)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega \mu_0 \vec{H}$$

$$i\vec{k} \times \vec{E} = i\omega \mu_0 \vec{H} = i\omega \vec{B}$$

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

In the bottom left corner of the grid, there is a small circular logo with the text 'NPTEL' below it.

So, let me consider the perturbation in the system, let the electric field of the perturbation is polarized in the X direction and amplitude is suppose a exponential minus i omega t minus say k X. This is the kind of perturbation I want to consider how about the magnetic field if you examine the Maxwell equation third Maxwell equation curl of E that is equal to minus delta B by delta t, if electric field varies with t and x in such a fashion then (( )) will also vary in the same way. So, that this equation is satisfied at all times and all values of X.

So, if I take the variation of B with time like this I can replace delta delta t by minus i omega and this becomes I omega into B. B is mu 0 H where mu 0 is the free space permeability of the free space permeability for plasmas permeability remains the same as in free space.

Now, this del operator I can replace by i k X cap. So, this becomes i k cross E is equal to I omega mu 0 H or mu 0 B whatever or B.

So, what you get is B you can call this as I omega B also. So, B is equal to k cross E upon omega I would like to allow omega to be complex. So, that the wave amplitude can grow in that case, a is not the amplitude of the wave it is a quantity this is independent of

space and time, but some term because of complex nature of omega will join this term here then that will be called as the amplitude of the wave.

So, I would like to examine under what condition omega acquires a complex character and omega acquires a positive imaginary part. So, there the net amplitude of perturbation grows when amplitude of E grows. Obviously, B should also grow I would like to see this happens.

In order to examine the character of or evolution of this wave we need to obtain the current density because we need to solve the Maxwell's equation forth Maxwell's equation and for that I need to obtain the particle response to this perturbation.

Now, I would like to write down the equation of motion for electrons. I will localize my or rather limit my discussion to non realistic case. So, that this becomes rather clear.

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The image shows a hand-drawn derivation on a grid background. The equations are as follows:

$$m \left( \frac{\partial \vec{v}_b}{\partial t} + \vec{v}_b \cdot \nabla \vec{v}_b \right) = -e\vec{E} - e\vec{v}_b \times \vec{B}$$

$$\vec{v}_b = v_{0z} \hat{z} + \vec{v}_{1b} \quad e^{-i(\omega t - kx)}$$

$$\frac{\partial \vec{v}_{1b}}{\partial t} + \vec{v}_{0z} \cdot \nabla \vec{v}_{1b} = -\frac{e\vec{E}}{m} - \frac{e}{m} \vec{v}_{0z} \times \vec{B}$$

z comp.

$$-i\omega v_{1bz} + (v_{0z} \cdot i k \hat{x}) v_{1bz} = -\frac{eE_z}{m}$$

$$v_{1bz} = \frac{eE_z}{m i \omega}$$

NPTEL logo is visible in the bottom left corner of the slide.

So, I am considering a non realistic equation of motion which is m mass of the charged particle into delta v by delta t plus v dot del v is equal to minus e E minus e v cross B well when there was no perturbation no electric field was there no velocity was there no magnetic field was there in that case the particle velocity was v 0 z cap this is beam response.



Well if I want to write this equation for the beam electrons, then I will put a subscript  $b$  here then this is called  $b$  here,  $b$  there  $b$  here and  $b$  there minus  $e$  is the charge of an electron.

So, the beam velocity was  $v_0$  when these fields were not there in the presence of these fields, we say that this becomes is modified by a quantity called  $v_1$  this is the modification. I will substitute this in this equation and linearize this equation linearize means ignore the product of perturbed quantity  $v_1$  with it is self or with the magnetic field because I will treat this  $b$  and  $e$  as perturbations.

I substitute this in this equations then and neglect the products of perturbed quantity then this equation becomes  $\Delta v_1$  by  $\Delta t$  plus  $v_0$  outside dot  $\Delta$  of  $v_1$  is equal to minus  $e E$  upon  $m$  minus  $e$  upon  $m$  I am dividing this equation by mass.

Now,  $v \times B$  is only  $v_0 \times B$  that comes in there  $v_1$  times  $b$  that product have has been ignored being the product of two perturbed quantities similarly, here  $v_1$  and  $v_1$  product has been ignored. And there is no  $v_0$  inside the  $\Delta$  operator because  $v_0$  is a constant and when you take  $\Delta$  operators will be 0.

So, this the linearized equation let me write down the  $z$  component of this equation. The  $z$  component of this equation will give you minus  $i \omega$  from here, because I am replacing  $\Delta \Delta t$  by minus  $i \omega$  the reason is that  $e$  and  $b$  this right hand side is the driver is the source which has a dependence on space time like exponential minus  $i \omega t$  minus  $k X$ .

It means that my response also in the quasi steady state should have similar dependence on time and space. So, whenever  $v_1$  varies with time and space like this I can replace  $\Delta \Delta t$  by minus  $i \omega$ . So, it becomes  $I$  this  $1$  and  $\Delta$  operator by  $i k$   $i k X$  rather.

But please remember  $i k X$  if I put here,  $i k X$  turns out to be well let me write down plus  $v_0$  for this dot  $i k X$  cap this is  $k \Delta$  operator is this much into  $v_1$  is equal to  $z$  component of right hand side would be minus  $e E_z$  upon  $m$ .

Now, this term  $v_0$  is in the  $z$  direction. So, this term does not contribute to  $z$  component this is all sorry I am writing  $z$  component. So, please I should remove this

vector sign here vector sign there write z component here this is what I want to do. So, this is the z component of the equation.

$V_0$  is in the z direction this is  $\hat{x}$  is in the x direction this also becomes 0. So, what you get is  $v_1$  z turns out to be is equal to  $e E_z$  upon  $m i \omega$ . I would also like to write down the x component of this equation the reason is  $v_0$  is in the z direction  $v$  is in the y direction. So, this has x component. So,  $v_1$  is expected to have x component.

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$$\frac{\partial v_{1bx}}{\partial t} + v_{0b} \cdot \nabla v_{1bx} = + \frac{e}{m} v_{0b} B_y$$

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = -\hat{y} \frac{k}{\omega} E_z$$

$$-i\omega v_{1bx} + i v_{0b} \cdot k \hat{x} v_{1bx} = -\frac{e k v_{0b}}{m\omega} E_z$$

$$v_{1bx} = \frac{e E_z}{m\omega} \frac{k v_{0b}}{i\omega} \quad \vec{k} \uparrow \quad \rightarrow \vec{v}_{0b}$$

$$\vec{J}_{1b} = (-e n_b \vec{v}_{1b})_{Part.} = -e n_{0b} \vec{v}_{1b} - e n_{1b} \vec{v}_{0b}$$

So, let me write down the x component of this equation. Delta delta t of  $v_1$  b x plus  $v_0$  b dot del well, when I put x component I should remove this vector sign I am sorry into  $v_1$  b x is equal to there is not electric force in the x direction because electric field is in the z direction. So, the  $v$  cross  $b$  term gives you minus  $e$  upon  $m$   $v$  cross  $b$  I want x component. So, that will be  $v_y$  b z which is 0 minus  $v_0$  z. So, which is  $v_0$  b in the z direction and b in the y direction, b in the y direction let me write.

And this b y I can really write in terms of electric field because we have already seen that  $B$  is equal to  $k$  cross  $E$  upon  $\omega$   $k$  is in the x direction  $E$  is in the z direction. So,  $x$  cross  $z$  is minus  $y$  cap  $k$  upon  $\omega$  into  $E_z$  just I substitute this in here and replace this delta delta t by minus  $i \omega$  and del operator by  $i k \hat{x}$ . So, I get minus  $i \omega$   $v_1$  b x this gives you plus  $i v_0$  b dot  $k \hat{x}$  into  $v_1$  b x.

So,  $\mathbf{k} \cdot \nabla$  is the value of del operator right hand side would be equal to minus  $e k_z$  upon  $\omega - v_0$ , this term is identically 0 because  $v_0$  is in the z direction. So, dot this with  $\mathbf{k}$  it will be 0 and you will get  $v_1$ . So, much is equal to  $e$  well I forgot to write this  $m$  should be written there.

So, you get  $e$  upon  $m \omega$  and this  $E_z$  is also there. So,  $e E_z$  upon  $m \omega$  sorry why did I write this  $\omega$  here this  $\omega$  is there one extra  $\omega$  is there. So, you get  $k_z v_0$  upon  $\omega$  this extra  $\omega$  and this I will also be there. So, this is what you get. Once I have obtained the velocity components and if you evaluate  $v_1$  it will be 0.

Now, current density due to the beam electrons perturbed current density due to the beam electrons is the product of charge into density into velocity I want the perturbed quantity.

So, let me write down this is called perturbed. Not the d.c. part the perturbed part that will give will be equal to minus  $e n_0 v_1$  minus  $e n_1 v_0$  this is the current density.

Now, the current density involves the density perturbation and hence I should find out whether there is any density perturbation or not please note your  $\mathbf{k}$  is in  $\mathbf{k}$  was in this direction this is your  $\mathbf{k}$  vector perturbation like the some sort of a wave is travelling in this direction beam is travelling in this direction.

So, what is happening here if the perturbed velocity is also in the same direction as  $\mathbf{k}$  vector then that will give rise to compression rarefaction of beam electrons. So, density perturbation arises because of this  $v_1$ . So, I would like evaluate the value and that can be done. So, before I evaluate this perturbed current density I need to solve the equation of continuity.

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$$\frac{\partial n_b}{\partial t} + \nabla \cdot (n_b \vec{v}_b) = 0$$

$$n_b = n_{0b} + n_{1b}$$

$$\vec{v}_b = v_{0b} \hat{z} + \vec{v}_{1b}$$

$$\frac{\partial n_{1b}}{\partial t} + \nabla \cdot (n_{0b} \vec{v}_{1b} + n_{1b} \vec{v}_{0b}) = 0$$

$$-i\omega n_{1b} + i\vec{k} \cdot (n_{0b} \vec{v}_{1b} + n_{1b} \vec{v}_{0b}) = 0$$

$$n_{1b} = n_{0b} \frac{\vec{k} \cdot \vec{v}_{1b}}{\omega} = n_{0b} \frac{k}{\omega} v_{1bx}$$

And the equation of continuity is for the beam electrons rate of change of density of the beam electrons plus divergence of beam density into beam perturbation velocity or beam velocity  $v_b$  is equal to 0. (( )) Write down beam density as the initial density or equilibrium density plus a perturbation and beam velocity already we had written as sorry  $v_0 b$  in the  $z$  direction plus a perturbed quantity this is what we have done.

Substitute these in this expression and ignore the products of  $n_1$  with  $v_1$  now, obviously,  $n_0 v_0$  product will have divergence 0 because they are both independent of position. So, when you take divergence of this product it will not appear here. So, what really you get here is on substituting the equation becomes  $\Delta n_{1b}$  by  $\Delta t$  plus divergence of  $n_0 v_1 b$  plus  $n_1 v_0 b$  equal to 0.

Now, replace this by minus  $\omega$  and then by  $i k$  what you get is minus  $i \omega n_{1b}$  plus  $i k$  vector which is in the  $x$  direction dot here  $n_0 v_1 b$  plus  $n_1 v_0 b$  equal to 0 this term does not contribute, because  $k$  is in the  $x$  direction  $v_0 b$  is in the  $z$  direction.

So, this product of  $k$  with  $v_0 b$  will vanish and hence these two terms survive and they give you density perturbation is equal to  $n_0 b k \cdot v_1 b$  upon  $\omega$  and  $v_1 b$  has  $x$  component and  $z$  component because  $x k$  is in the  $x$  direction. So, only  $x$  component of  $v_1 b$  will survive here  $n_0 b k$  upon  $\omega$  into  $v_1 b_x$

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$$\vec{J}_{1b} = -n_{0b} e \vec{v}_{1bx} - n_{1b} e \vec{v}_{0bz}$$

$$J_{1bx} = -n_{0b} e v_{1bx} = -n_{0b} e \frac{e E_z}{m \omega} \frac{k v_{0b}}{\omega}$$

$$J_{1bz} = -n_{0b} e v_{1bz} - n_{1b} e v_{0bz}$$

$$= -\frac{n_{0b} e^2}{m \omega} E_z - n_{0b} e \frac{k}{\omega} v_{1bx} v_{0b}$$

$$= -\frac{n_{0b} e^2}{m \omega} E_z \left( 1 + \frac{k^2 v_{0b}^2}{\omega^2} \right)$$

$$J_{1px} = +n_0 e \frac{e E_z}{m \omega} \frac{k v_0}{\omega}$$

So, now let me visualize the equation for current density the current density equation is  $J_{1b}$  perturbed current density  $J_{1b}$  of the beam electrons is equal to minus  $n_{0b} e v_{1b}$  vector minus  $n_{1b} e v_{0b}$  vector let me write down its components x component and because  $v_{1b}$  has x component and z component  $v_{0b}$  has only z component. So, both I have to retain. So,  $J_{1bx}$  turns out to be current perturbed current density due to the beam electrons is minus  $n_{0b} e v_{1bx}$  this has no x component. So, I do not write it.

And  $J_{1bz}$  turns out to be minus  $n_{0b} e v_{1bz}$  component minus  $n_{1b} e v_{0bz}$  z component and if I substitute the value of  $n_{1b} v_{0bz}$  what you get here is actually this z component I need not write because  $v_{0b}$  is in the z direction anyways.

So, I do not have to write this z component here this equation I can simplify little bit minus well if I put the value of  $v_{1b}$  this becomes minus  $n_{0b} e$  square upon  $m$  let me just  $\frac{1}{\omega} \frac{1}{\omega} E_z$  this is the value of the first term second term is  $n_{1b}$  if I substitute, then I get  $n_{0b} e$  is here and there is another  $e$  in the expression there.

So, let me just write in two steps  $n_{1b} n_{0b}$  into  $e k$  upon  $\omega$  into  $v_{1bx}$  into  $v_{0bz}$  this  $v_{0bz}$  I have written here and this is the value  $e$  is already there  $n_{0b} k$  upon  $\omega v_{1b}$  x I have written.

And let me put the value of  $v_{1x}$  here as well as here  $v_{1b}$  that I had obtained was. So, this becomes  $-n_0 b e$  and this is  $e E_z$  upon  $m \omega$  into  $k v_0 b$  upon  $I \omega$  the same expression I will substitute here.

So, this becomes  $-n_0 b e^2$  upon  $m I \omega$  common and you will get  $E_z$  also common and it turns out to be  $1 + k^2 v_0^2 b^2$  from  $v_{1b}$  where I put this expression here. So,  $k^2 v_0^2 b^2$  and  $1$  over  $\omega^2$  comes. So,  $v^2$  upon  $\omega^2$  what you get is this that is something interesting.

Please note that the beam electron perturbed current density due to the electric field in the  $z$  direction is modified by the beam velocity by this term, if the beams electrons were not moving then this term will be 0 and this is the conventional r f conductivity into  $e$ .

And interesting thing is that the beam current acquires a  $x$  component also the electric field of the wave is in the  $z$  direction, but, the beam acquires a current density in the  $x$  direction, but, this depends on the particle velocity first power.

So, if I evaluate similar quantity for plasma electrons plasma electrons have same density that we have assumed same charge same mass, but velocity is in the opposite direction. So, if I calculate the perturbed current density for the plasma electrons, I will call this  $J_{1p}$  its  $x$  component turns out to be exactly opposite of this because velocity of the electron plasma is negative  $d c$  velocity.

So, that gives me  $-n_0 e e E_z$  upon  $m \omega$  multiplied by  $k v_0$  upon  $\omega$  into  $I$  this  $v_0 b$  is equal to  $-v_0$  to be replaced. So, this is a magnitude of beam velocity this is a magnitude of they are equal I am taking them to be equal I will take  $v_0 b$  equal to  $v_0$  magnitude wise.

So, because of the reversal in sign of velocity propagation this term has become actually positive and when you add the two they will cancel out the net current perturbed current is the sum of beam current density plus plasma current density and because they are of opposite sign they will cancel each other.

So, the net current and what about  $z$  component of current density because it depends on the square of velocity they add up rather than cancelling each other.

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$$\vec{J}_1 = \vec{J}_{1b} + \vec{J}_{1p}$$

$$J_{1x} = 0$$

$$J_{1z} = - \frac{2 n_0 e^2}{m i \omega} E_z \left( 1 + \frac{k^2 v_0^2}{\omega^2} \right)$$

$$n_{0b} = n_0$$

$$v_{0b} = v_0$$

$$\nabla \times [\nabla \times \vec{E}] = -i \omega \mu_0 \vec{H}$$

$$\nabla \times \vec{H} = \vec{J}_1 + \frac{\partial \vec{D}}{\partial t} = \vec{J}_1 - i \omega \epsilon_0 \vec{E}$$

So, I would like to write down the net current density in the system  $J_1$  which is equal to  $J_1$  of the beam plus  $J_1$  of the plasma electrons and this term has  $J_1$  x total is equal to 0, because of the cancellation of beam current density to plasma electron current density and  $J_1$  z turns out to be additive it becomes minus twice  $n_0 e$  square upon  $m i \omega$   $E_z$  plus  $k$  square  $v_0$  square by  $\omega$  square where I have taken  $n_{0b}$  is equal to  $n_0$   $v_{0b}$  is equal to  $v_0$  this is interesting situation.

This is the additional term that has come up because of the beam electrons having some finite velocity and equal velocity is being having being held by the plasma electrons though in the opposite direction. Now let us examine the consequence of this current on the in the wave equation by wave equation general electromagnetic wave equation is can be reduced from Maxwell's equations curl of e equation I have already written which is equal to minus delta b by delta t and which was equal to  $i \omega \mu_0 \vec{H}$  and the last Maxwell equation is curl of H is equal to J.

But actually I want  $J_1$  basically plus  $\frac{\partial \vec{D}}{\partial t}$  for plasma displacement current density is the same as  $\epsilon_0 \nabla \cdot \vec{E}$  and if I replace  $\frac{\partial \vec{D}}{\partial t}$  by  $-i \omega \epsilon_0 \vec{E}$  I will get this is equal to  $J_1 - i \omega \epsilon_0 \vec{E}$  I should work I should substitute for  $J_1$  from this expression and they are coupled equations for e and H. I think before I substitute for  $J_1$  let me take curl of this equation and use curl of H from here in

this equation. So, when I take curl of this equation multiply this by curl from both sides on both sides and then use this equation here.

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The image shows a handwritten derivation on a grid background. The equations are as follows:

$$\nabla \times \nabla \times \vec{E} = i\omega\mu_0 \nabla \times \vec{H}$$

$$= \frac{i\omega\mu_0}{\epsilon_0} (\vec{J}_1 - i\omega\epsilon_0 \vec{E}) \epsilon_0$$

$$-\nabla^2 \vec{E} + \nabla(\nabla \cdot \vec{E}) = \frac{i\omega}{c^2\epsilon_0} \vec{J}_1 + \frac{\omega^2}{c^2} \vec{E}$$

$$\vec{k}^2 \vec{E} + i\vec{k}(\vec{k} \cdot \vec{E}) = \frac{i\omega}{c^2\epsilon_0} \vec{J}_1 + \frac{\omega^2}{c^2} \vec{E}$$

z comp.

$$k^2 E_z = \frac{i\omega}{c^2\epsilon_0} J_{1z} + \frac{\omega^2}{c^2} E_z$$

NPTEL logo is visible in the bottom left corner of the slide.

So, you will get curl curl of E curl curl of E is equal to  $i\omega\mu_0$  curl of H use the second equation the last equation rather and this becomes is equal to  $i\omega\mu_0$  multiplied by  $J_1$  minus  $i\omega\epsilon_0 E$  it is useful to multiply  $\epsilon_0$  up and down. So, if I multiply this divide by  $\epsilon_0$  and multiply by  $\epsilon_0$ .

Then  $\mu_0\epsilon_0$  is called  $1$  upon  $c$  square. So, I can write down this equation as  $i\omega$  upon  $c$  square  $\epsilon_0$  here  $J_1$  vector and then you take this in the interior this becomes  $\epsilon_0$  will cancel with this is one upon  $c$  square. So, you get plus  $\omega$  square by  $c$  square  $E$  this is perturbed current density this break it this is minus  $\nabla^2$  of  $E$  plus gradient divergence of  $E$  replace  $\nabla$  operator by  $i\vec{k}$  this gives you  $k^2 E$  vector plus  $i\vec{k}$  here for  $\nabla$  and  $i\vec{k}$  for this  $\nabla$  in the interior is equal to  $i\omega$  by  $c$  square  $\epsilon_0 J$  plus  $\omega$  square by  $c$  square  $E$  vector take the  $z$  component of this equation.

This  $k$  does not have any  $z$  components this term is identically  $0$  and this you gives you  $k^2 E_z$  is equal to  $i\omega$  upon  $c$  square  $\epsilon_0 J_{1z}$  plus  $\omega$  square by  $c$  square  $E_z$  and  $J_{1z}$  already we have obtained in terms of  $E_z$ .



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$$\begin{aligned} \left(k^2 - \frac{\omega^2}{c^2}\right) E_z &= \frac{i\omega}{c^2 \epsilon_0} J_{1z} \\ &= -\frac{\omega_p^2}{c^2} \left(1 + \frac{k^2 v_0^2}{\omega^2}\right) E_z \\ \omega_p^2 &= \frac{2n_0 e^2}{m \epsilon_0} \\ k^2 - \frac{\omega^2}{c^2} &= -\frac{\omega_p^2}{c^2} \left(1 + \frac{k^2 v_0^2}{\omega^2}\right) \\ \omega^2 - \omega_p^2 - k^2 c^2 - \frac{\omega_p^2}{\omega^2} k^2 v_0^2 &= 0 \end{aligned}$$

So, let me substitute and when you substitute you get  $k^2$  square minus  $\omega^2$  square by  $c^2$  square into  $E_z$  is equal to  $i\omega$  by  $c^2$  square  $\epsilon_0$  into  $J_{1z}$ . Then I use the value of  $J_{1z}$  that I had deduced it turns out to be equal to minus  $\omega_p^2$  square upon  $c^2$  square into  $1 + \frac{k^2 v_0^2}{\omega^2}$  multiplied by  $E_z$  where I have introduced a quantity  $\omega_p^2$  which is equal to twice  $n_0 e^2$  upon  $m \epsilon_0$ , two factor comes because there are 2. The total density of electrons due to the beam plus the plasma electrons is  $2n_0$  each spaces 2 by  $n_0$  comes in here this is the equation.

Now, this equation if there was no beam it gives you  $k^2$  the dispersion relation you can obtain by  $E_z$  is there  $E_z$  is there cancel the  $E_z$  and you get the dispersion relation. So, the dispersion relation that you get is  $k^2$  square minus  $\omega^2$  square by  $c^2$  square well this term I can put here is equal to minus  $\omega_p^2$  square by  $c^2$  square  $1 + \frac{k^2 v_0^2}{\omega^2}$  square by  $\omega^2$  square what I can do multiply this equation by  $c^2$  square and this equation take the following form.

I can write down this  $\omega^2$  square minus  $\omega_p^2$  square minus  $k^2 c^2$  square. So, if I take multiply by  $c^2$  square take everything on side side you will get  $\omega^2$  square minus  $\omega_p^2$  square minus  $k^2 c^2$  square minus  $\omega_p^2 k^2 v_0^2$  square upon  $\omega^2$  square equal to 0.

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$$\omega^4 - \omega^2(\omega_p^2 + k^2 c^2) - \omega_p^2 k^2 v_0^2 = 0$$

$$\omega^2 = \frac{1}{2} \left[ \omega_p^2 + k^2 c^2 - \sqrt{(\omega_p^2 + k^2 c^2)^2 + 4\omega_p^2 k^2 v_0^2} \right]$$

$$\approx \frac{1}{2} \left[ \omega_p^2 + k^2 c^2 - (\omega_p^2 + k^2 c^2) \left( 1 + \frac{2\omega_p^2 k^2 v_0^2}{(\omega_p^2 + k^2 c^2)^2} \right) \right]$$

$$= - \frac{\omega_p^2 k^2 v_0^2}{\omega_p^2 + k^2 c^2}$$

$$\omega = i\gamma, \quad \gamma = \frac{\omega_p v_0}{c} \frac{1}{(1 + \omega_p^2/k^2 c^2)^{1/2}}$$

Now, multiply by omega square also then this becomes a quadratic equation omega square the result is it becomes omega four minus omega square multiplied by omega p square plus k square c square minus omega p square k square v 0 square is equal to 0.

Such a rich physics of current filamentation is contained in the very simple algebraic equation and we can easily solve this to obtain omega a square the roots is equal to one upon two this is like a x square plus b x plus c equal to 0 the roots would be 1 upon two a into minus b which is negative of this quantity. So, becomes omega p square plus k square c square minus plus or plus minus you write here we will be (( )) negative sign because we want omega square to be negative. So, that there is instability.

So, I will consider only negative sign omega p square plus k square c square whole square plus four omega p square k square v 0 square this is called the dispersion relation. If my beam has v 0 non relativistic it means v 0 square is much less than C Square.

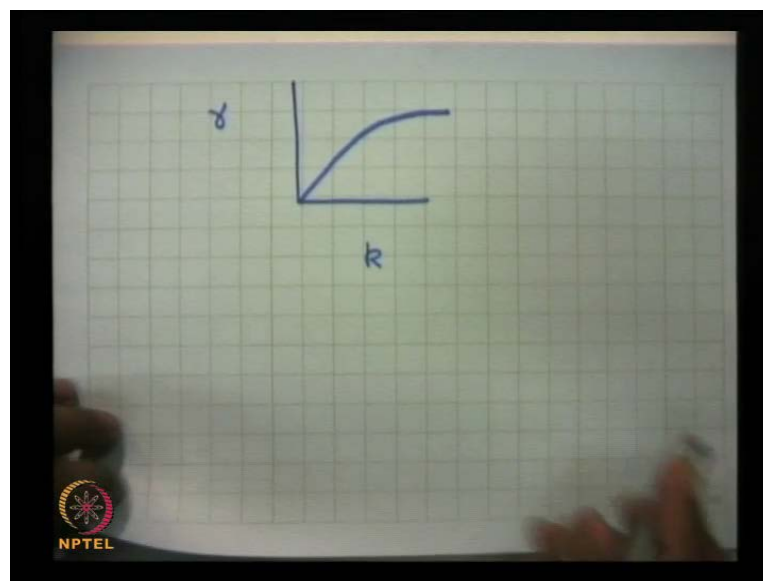
So, this is a much larger term than this. So, in that case I can carry out binomial expansion of this and this turns out to be approximately equal to half omega p square plus k square c square minus when I take this quantity outside it gives me or taking the under root same factor omega p square plus k square c square and then you will get within this 1 plus when you take the under root and use binomial expansion it becomes two omega p square k square v 0 square upon this factor omega p square plus k square c square whole square.

You may note that this first term one here will cancel this entire term and this is the only that survives two will cancel with this two and this becomes is equal to negative times its negative  $\omega p^2 k^2 v_0^2$  divided by  $\omega p^2 + k^2 c^2$ .

That is a neat expression and if negative sign is there then  $\omega$  can be written as  $i$  times  $\gamma$  imaginary term and  $\gamma$  is then called the growth rate and the value of  $\gamma$  is equal to just you can rewrite this as  $\omega p v_0 b c$  into  $1$  upon  $\left(\left(\right)\right)$  which will go as  $1 + \omega p^2$  upon  $k^2 c^2$  under root.

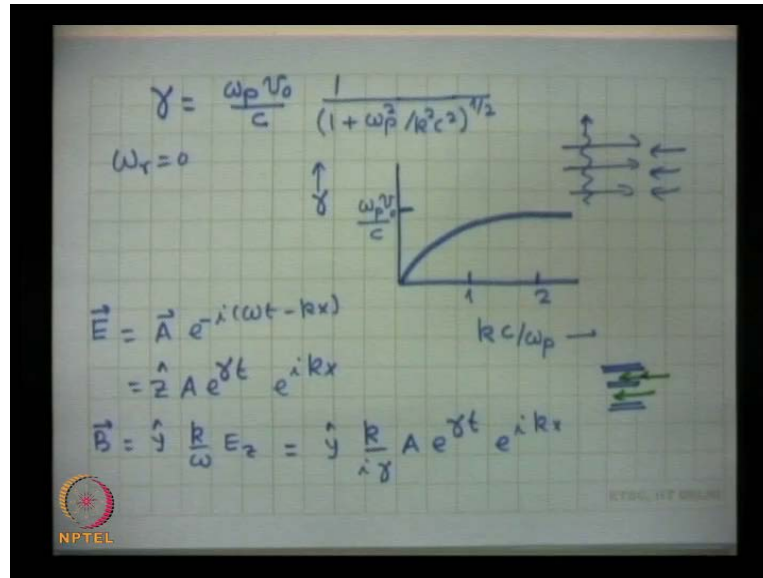
So, growth rate is the under root of this quantity  $\omega p$  is like this I have taken  $k^2 c^2$  common  $k^2$  will cancel out  $v_0$  by  $c$  you will get and this is the expression, very neat expression the growth rate goes as proportional to  $\omega p$  proportional to beam velocity and it depends on  $k$  also in a this fashion when  $k$  is  $0$  in that case what will happen when  $k$  is  $0$  this term is very large and the growth rate is  $0$  and when  $k$  becomes infinitely large very large in that when  $k$  tends to infinity in that case this term is  $0$  and this is the maximum value. So, the growth rate increases with  $k$ . Here also you can see because if  $k$  is  $0$  then this term you can ignore and this is proportional to  $k$ .

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So, if I plot a graph here gamma versus k here, then this will increase and you will go like this I made a mistake I made a mistake should not do this is not right this is opposite. I think let me remove this page.

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I am having growth rate is equal to  $\omega_p v_0 / c$  I think I was not wrong into 1 upon 1 plus  $\omega_p^2 / (k^2 c^2)$  sorry I think I was right. If I want to plot this I am plotting k here I think interesting quantity will be  $k c / \omega_p$   $k c$  upon  $\omega_p$  and growth rate here. When k is 0 this term is very large 1 can be ignored and very large in the denominator means gamma is 0 actually this really goes like this and at very large values of k this term is negligible. So, growth rate tends to a value here which is  $\omega_p v_0 / c$   $\omega_p v_0 / c$  this is the value growth rate goes like this and this is typically one is here somewhere here this is like two.

So, what does happen we are considering a situation the beams and plasma electrons were travelling in the opposite direction and you are considering the growth of a wave whose magnetic field varies in this direction.

But the 0 frequency real part is 0 imaginary part of frequency is gamma. So, if I look at this expression for the electric field or magnetic field, e is equal to a exponential minus i omega t minus k X if I put the value of omega as i gamma then this becomes this was in z direction A e to the power gamma t exponential of i k X and b field if you calculate b field was in the y direction  $k / \omega$  into e this is the value of b.

You may note here because  $\omega$  is  $\Gamma$ . So, this is equal to  $y \cap k$  upon  $\Gamma$  and this is a  $e$  to the power  $\Gamma t$  exponential of  $i k X$  now take the real part  $b$  and  $e$  are out of phase by  $\pi/2$  because of this  $\Gamma$  being here that is a very interesting thing. Secondly, if you evaluate the quantity  $k$  upon  $\Gamma$  because  $\Gamma$  is like  $\omega_p / v_0$  by  $c$  this is  $n k c$  by  $\omega$  if you look at this quantity this is much bigger than it is quite large actually this turns out to be  $c/v_0$ .

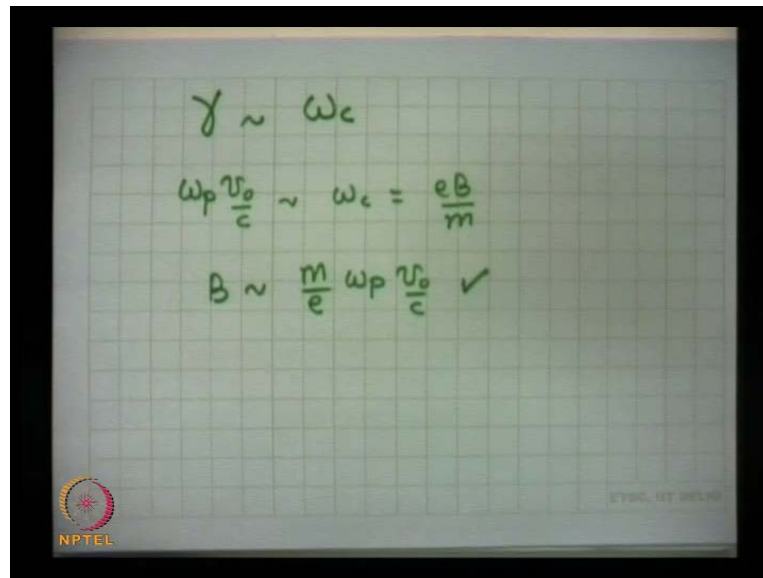
So, this  $b$  field has much stronger influence on this mode than the electric field this is a very strong magnetic. So, primarily this is a magnetic perturbation though this has a finite  $E_z$  also, but, this is primarily magnetic perturbation this causes filamentation of the beam current beam  $(( ))$  travels through a plasma filaments current filaments are formed you get regions of high current beam current then plasma current like  $I$  if I draw by a green pen then in between these there will be plasma current here then there will be a beam current here and then there will be a plasma current in between this kind of thing situation is created.

So, you have created filamentation of plasma current as well as beam current and this instability has a typical wave length of the order of  $c/\omega_p$  and which is quite a small wavelength.

So, this produces fluctuations magnetic fluctuations in the plasma well if you include some other effects like pressure etcetera there is a possibility that you can introduce some finite real part and frequency also, but that is not important issue important issue is that this magnetic field has a special structure it produces magnetic fluctuation in space and they could be very large finally, I would like to mention how this instable will saturate.

Because as these this magnetic field grows the beam turns around and the  $(( ))$  of the beam will be stopped and this happens instability saturates when growth rate becomes comparable to let me write down.

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The image shows a grid background with handwritten mathematical equations in green ink. The equations are:

$$\gamma \sim \omega_c$$
$$\omega_p \frac{v_0}{c} \sim \omega_c = \frac{eB}{m}$$
$$B \sim \frac{m}{e} \omega_p \frac{v_0}{c} \checkmark$$

In the bottom left corner, there is a logo for NPTEL (National Programme on Technology Enhanced Learning) featuring a stylized sun or starburst design. In the bottom right corner, there is a small, faint text that reads "EPSC, IIT DELHI".

When this becomes comparable to the growth rate becomes comparable to cyclotron frequency cyclotron frequency characterize the electron motion. So, whenever the growth becomes comparable to cyclotron frequency instability saturates.

And gamma as I mentioned to you is of the order of  $\omega_p v_0 / c$  and. So,  $\omega_c$  that you produce which is equal to  $eB / m$  the magnetic field that is produced will correspond to a cyclotron frequency of the order of this quantity.

So,  $B$  is of the order of  $m / e \omega_p v_0 / c$  and this could be quite large quite high whenever in plasmas you are encountered with the generation of beams you are encountered with very strong magnetic fields.

Well certainly there are other schemes of (( )) generation and I think I do not have time now, but I will discuss it sometime later. Thank you.