Plasma Physics

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Lecture No. # 11
Energy Flow With an Electrostatic Wave

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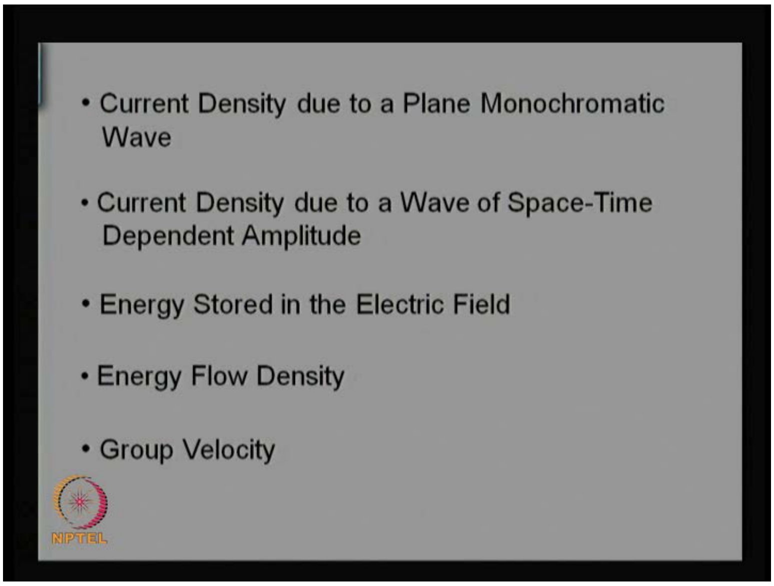

Well friends. Today I would like to discuss the energy flow with an electrostatic wave. We shall discuss current density due to a plane monochromatic wave, then current density due to a wave whose amplitude is a function of space and time, and then deduce an expression for energy stored in the electric field of a wave. And deduce the expression for energy flow per unit area per unit time with the electro static wave, and we shall also discuss the group velocity.

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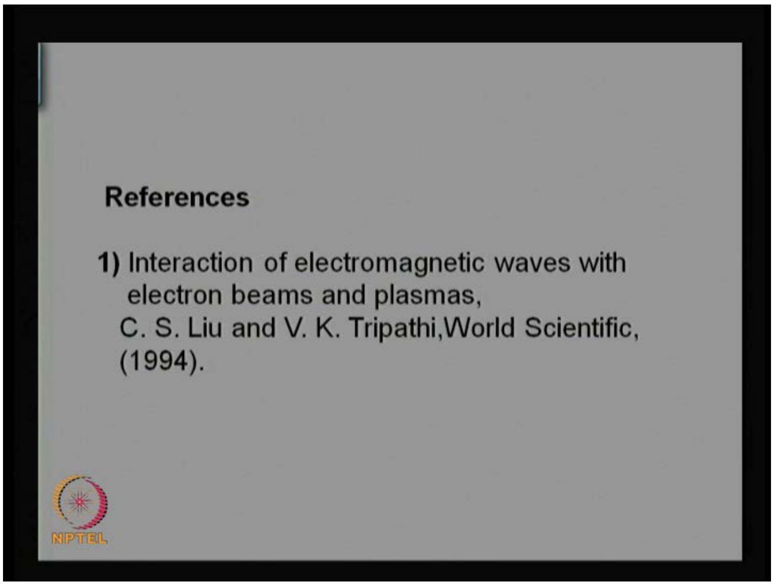

Well, this is based on this book interaction of electromagnetic waves with electron beams and plasmas by Professor C.S. Liu and myself. However, you can find discussion of today's theme in any book in plasma physics.

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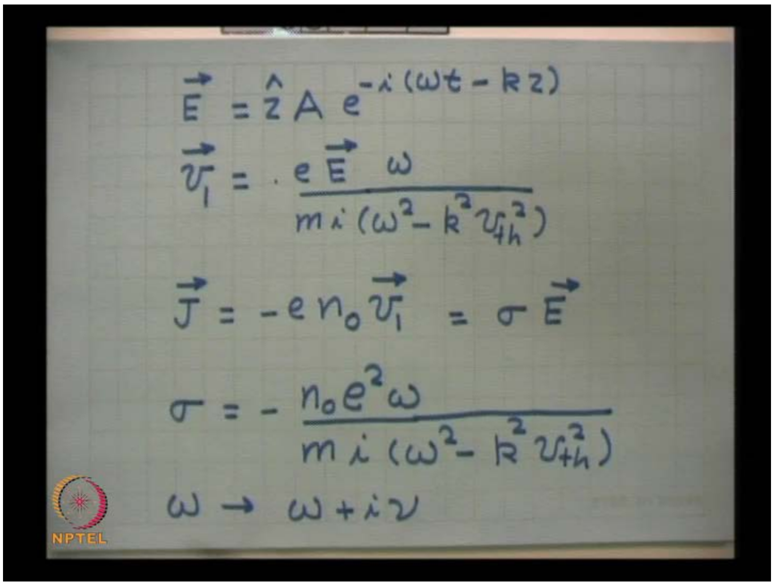

Well, let me begin with the response of a plasma to a monochromatic plane electrostatic wave whose electric field is E is equal to A to be simple or a specific lets the wave is longitudinally polarized in the z direction, and amplitude is constant in the time variation is i omega t, and space variation is like k z. This is the electric field of an electrostatic wave, we have already solved the equation of motion for electrons including the pressure term, and we deduced an expression for electron velocity from there we wrote that the perturbed velocity of electrons due to this waves is equal to e E upon m i omega square minus k square v thermal square into omega.

If thermal velocity is not there this is the thermal velocity of electrons then, this is the same expression for drift velocity as due to an electromagnetic wave, but the electrostatic wave is difference because of the thermal effects the denominator is modified and from there if you write down the perturbed current density due to the wave which is a product of electron charge equilibrium electron density, and perturbed velocity or drift velocity of electrons.

If you substitute this there then, this is expressible as sigma into E, and sigma turns out to be minus n 0 e square and omega in the denominator upon m i omega square minus k square v thermal square in this derivation I have ignored the effect of collisions. If collisions are finite then wherever omega occurs you have to replace omega by omega plus i nu then this equation is valid for coalitional plasma also. Usually when we are talking of Linear waves omega is around omega p, and collision frequency is orders of magnitude is smaller than omega p, hence this effect is small though it may be significant, but it is small. This is the current density due to an electrostatic wave whose amplitude is constant in time as well as in space. In order to deduce an expression for energy flow with the wave, we like to examine the current density of plasma response to a wave whose amplitude has a slow dependence, but significant dependence on time and position. So, let me go over to that stage.

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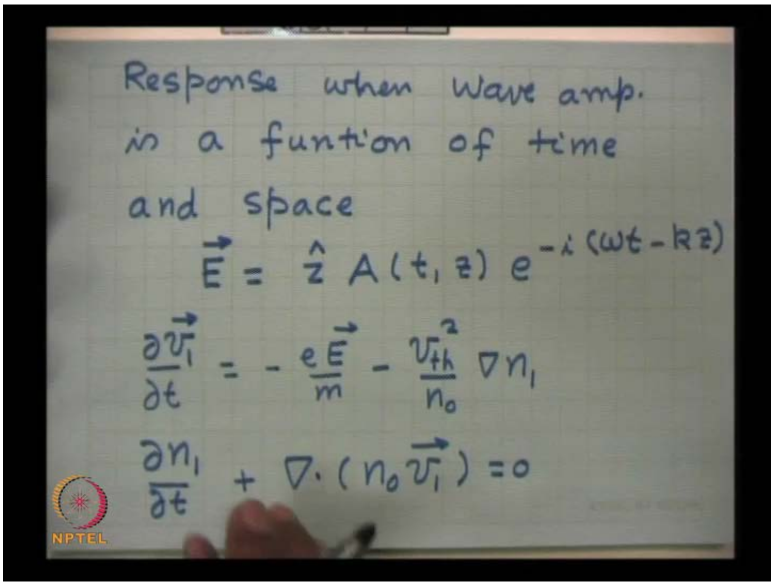

Now, I would like to discuss the electron response when wave amplitude is a function of time and space. I think we shall learn something very significant from this deduction. I am considering that my system has an electric field which is directly along z because it is a longitudinal electrostatic wave propagating on z axis, but amplitude I will permit to have a dependence of time and z Exponential minus i omega t minus k z. So, besides having fast phase dependence like this the way has a slow dependence on time and amplitude in terms of it is amplitude. Let us see what the consequence of this. If you write down the equation of motion is if I divide the equation of motion by mass it reads as delta v by delta t is equal to minus e E upon m minus pressure term which reads as v thermal square which is t e upon m upon n 0 and gradient of n.

Here I have used the linearized equation of motion. Where v dot Del v term has been dropped means the products of perturbed quantities have be dropped. This is what we call as a linear zed equation of motion that we had written last time. And actually this is for the perturbed velocity and perturbed density. So, I have written like this. And the equation of continuity for the perturbed density is delta n 1 by delta t plus divergence of n0 v1 is equal to 0. I wish to solve these equations when is not a constant, but a slowly varying function of t and z. I will slowly following a procedure of iteration solve these equations and derive some a result and then I will point out a general procedure which is very simple, but in order to appreciate that simple procedure. I need to do some deduction from these equations. I will be little slow and elaborate in deriving or rather in obtaining the solution to these equations.

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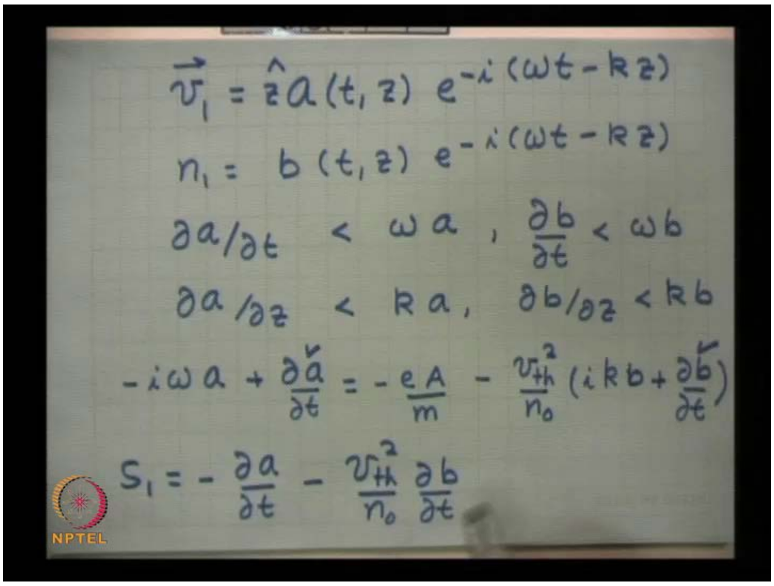

What I do I express my perturbed electron velocity n terms of some quantity a and I am taking all quantities vector quantities along z because it is a longitudinal wave. Suppose this is slowly varying amplitude of velocity which is upon t and z exponential minus i omega t minus k z. I have written the past time and z dependence the same way as the electric field because there is a source there is a response, but I am permitting my amplitude to have some dependence on t and z. Then I can write down my perturbed density also as say b which is a function of t and z exponential minus i omega t minus k z. Now what I do, I substitute these two expressions in the equation of motion continuity and treat delta a by delta t to be small as compared to minus i omega and delta b by delta t also to be small.

So, I am presuming that delta a by delta t is less than omega into a and delta b by delta t less than omega v. This is called iteration approximation. What I am going to do, first of all I will ignore the dependence of a on time and b on time and similarly I am assuming there z derivative also to be small as compared to k, means delta a by delta z is less than k a and delta b by delta z less than k b. This is the approximation I am going to make. What I do first of all when I substitute this I ignore the derivatives of a and b in the equation of motion continuity then those equations read like this. Let me actually substitute this and see how those equations look when you substitute this expansion for v 1 in the equation of motion you get minus i omega a plus delta a by delta t. This is d delta a by delta t term exponential factor is common and will cancel out in both sides is equal to minus e into a minus the pressure term which is v thermal square upon n 0 delta z of n 1.

So, when you m also is here. When you evaluate that term there are two terms i k into b plus delta b upon delta t. This is the term and this is term. These two terms together are called perturbed terms and I can call them if I take this term on the right hand side and add these two terms together I will call them as S 1. Then this equation becomes, let me define S 1 is equal to minus delta a by delta t minus b thermally square upon n 0 delta v by delta t. These two terms must I ignore and obtain the value of in terms of A; obviously, there is b here.

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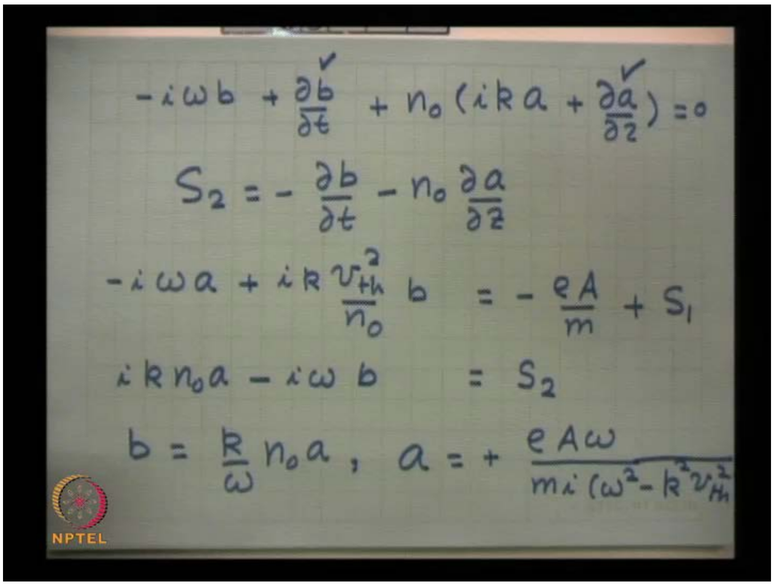

So, I should write down the equation of continuity as well and the equation of continuity is, if you substitute the expansion for density it gives you minus i omega b plus delta b by delta t plus n 0 divergence of delta z of v 1 which gives me i k a plus delta a by delta z is equal to 0. It looks little complicated, but simple substitution of v 1 and n 1 in the equation of motion and continuity. Here also there are two small terms delta v by delta t and delta a by delta z. I can take these two terms on the right hand side and define the sum of these 2 as S 2 which is equal to minus delta b by delta t, because I take on the right hand side and then this becomes and there is another term minus n 0 delta a by delta z.

This is the second term S 2 that I get what you should have is in terms of S 1 and S 2 the equation of motion then is minus i omega a plus i k v thermal square upon n 0 into b is equal to minus e A upon m plus S 1. The equation of motion takes this form and the equation of continuity this one, when I transfer these partial derivative terms from the right hand side takes the form i k. This term I will write first n 0 a then minus i omega b is equal to S 2. What I am suggesting is that first ignore S 1 and S 2, when you ignore S 1 and S 2, you can solve these equations to obtain b is equal to from this equation k upon omega into n 0 a and when you use this value of b in this equation and ignore this one you will get a is equal to plus e A omega upon m i omega square minus k square v thermally square.

This is the same velocity. If you multiply the exponential term as we had obtained in the local approximation when the amplitude was a constant and this is the density perturbation when you multiply the phase term. So, as before, we obtain the same thing when you ignore this S 1 and S2, but one use this value of b and a in S 2 and S 1. S 2 is put this value here and this value of a here and solve these two equations you can get in the new case when S1 and S2 are retained. This is when S 1 and S 2 are ignored. If S 1 is equal to 0 and S 2 is equal to 0 these are the values of b and a. Then I am suggesting that now evaluate S 1 and S 2 using these values of b and a and then solve these two equations returning these two terms and you will get the following. I just give the result, this algebra is simple and I am pretty sure you can easily simplify this. The net result is this.

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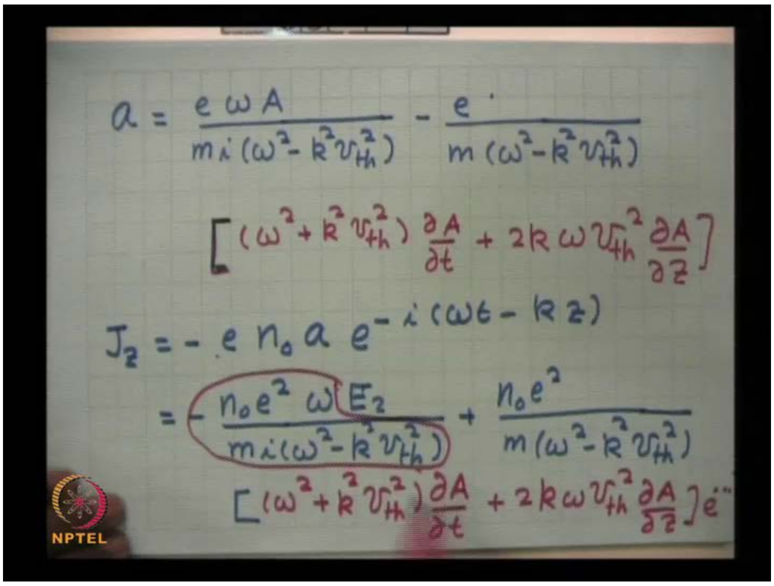

So, to the next approximation when you evaluate a and b. T hey turn out to be a is equal to e omega a upon m i omega square minus k square v thermal square where v thermal is the electron thermal velocity this term as before. Now the additional terms are these minus e upon m omega square minus k square v thermally square multiplied by I will write down this in different ink because these are the 2 important contributions. Because of S 1 and S 2 which is omega square plus k square v thermally square into delta a by delta t plus twice k omega v thermally square delta a by delta z. This is an important addition to the solution, this is if you ignore these two terms this last two terms then a is the same as due to a uniform amplitude wave. And these are the contributions that take into account that time variation of amplitude and space variation of amplitude. These two terms together. They are important terms and we shall learn in little while the role played them in energy flow.

Well, if you know the velocity the current density can be written and current density J z in this case will be minus e into n 0 into velocity. Velocity is a into exponential minus i omega t minus k z. This is the value of J z and when you substitute the value of a here you get an expression which is equal to minus n 0 e square just multiply n 0 e here. Because minus n 0 e square upon m i omega square minus k square v thermally square into omega this omega into a into this quantity is simply e E z. This is first term and second term is plus because plus n 0 e square upon m omega square minus k square v thermally square multiplied by these two terms in the bracket. Let me write them with red ink omega square plus k square v thermally square delta a by delta t plus twice k omega v thermally square delta a by delta z into exponential term this exponential. So, I do not have room here to write exponential. So, I will write e and then put dot there this exponential factor phase factor is to be there.

Now the beauty here is that if you call the coefficient of e z as sigma then, you can easily see that this quantity this term which is multiplying delta a by delta t is simply delta sigma by delta omega with an iota and similarly this quantity is exactly same as delta sigma by delta k you can just verify. Let me write this, I will define this coefficient from here through here this quantity is conductivity. If I called sigma then these other coefficients can be easily shown and you must show yourself lf must check yourself.

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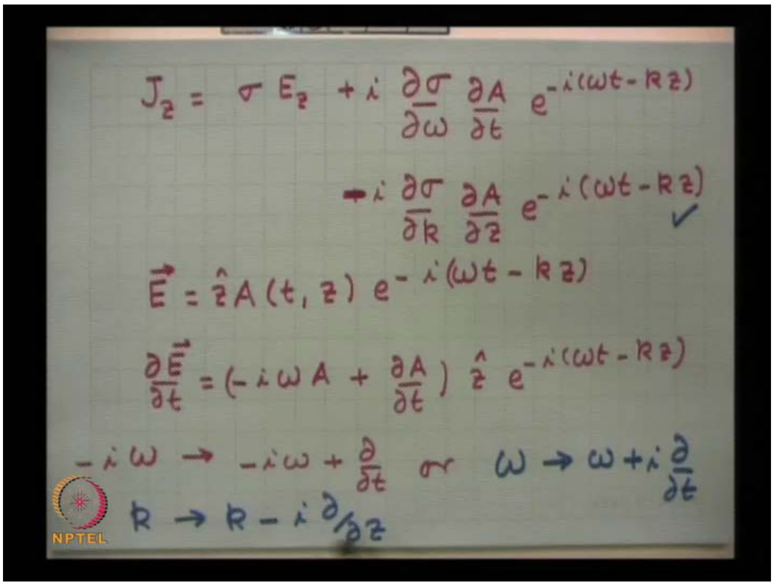

That J can be written as sigma E z plus i delta sigma upon delta omega into delta A by delta t exponential minus i omega t minus k z, and another term plus i delta sigma by delta k, actually this negative sign delta A by delta z exponential minus i omega t minus k z.

This is a very simple comprehensive way of writing an expression for J, because only in terms of a single quantity called conductivity and it is derivatives. I had been able to write J due to an electrostatic due to an electrostatic wave whose amplitude is a function of time and z. Of course, I have assumed that a depend slowly on time and slowly on z. This is the approximations I have made, but in that limit this is good. Well this was a very special case of a wave when there was no static magnetic field when I did not consider collisions and so, on. But in general I like to point out a general procedure. whenever you have to you are encountered with waves of time varying amplitude and position what to do if you know the response to a monochromatic wave whose amplitude is constant you can easily deduce the response due to time varying amplitude and time varying and the space varying amplitude the procedure is as follows.

You know if you have electric field is equal to say some function of t and z exponential minus I omega t minus k z, suppose this is my electric field. If I differentiate this partially with time I will get delta E by delta t how much that would be. That would be simply first of all I will differentiate this quantities will gives you minus i omega into this A will be there and then I will differentiate this. We will get delta A by delta t and of course, z unit vector is there and exponential minus i omega t minus k z is there. What I am saying, that if the wave has constant amplitude then this term is 0. So, delta t is simply equal to minus i omega. But when the wave has a time varying amplitude then delta t is not equal to this, but minus i omega plus delta t has to be there means omega has to be replaced by or minus i omega has to be replaced by minus i omega plus delta t.

If you compare this case with the case when there is no time variation of amplitude. Then this term is not there. The difference in the two cases is that minus i omega has to change by this or simply if I divide this equation by minus i, then this says that omega just to replace by omega plus i delta t. Because if I divide this equation by minus i this becomes omega goes to omega plus i delta t. And this delta t operator has to operate not on the whole field, but only on the amplitude. In general your sigma the conductivity is a function of omega and k. But you should do, replace omega by omega plus i delta t and similarly if you differentiate this with respect to z you can show that k has to go from k minus i delta z. If you can make these two changes in sigma and operate this sigma over the amplitude of the electric field you will get the right expression.

So, this expression that I have written here complicated expression can be deduced from J is equal to sigma e in a simple way if I substitute sigma omega by this and k by this let us see how do we do it.

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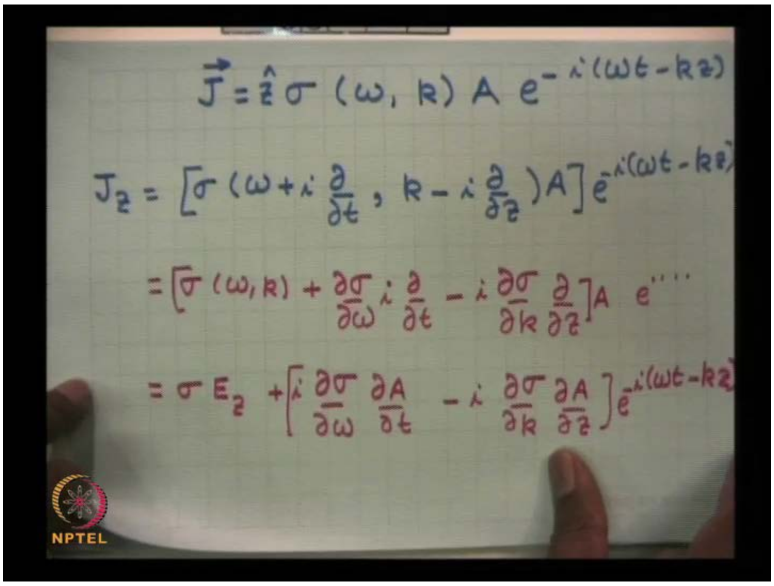

We begin with J is equal to sigma which is a function of frequency and k in general and this direction is z into a exponential minus i omega t minus k z, but what I am saying that I am going to replace this omega by let me write down J z as how much then sigma which is a omega plus i delta t and k is to be replaced by k minus i delta z operating over a exponential factor minus i omega t minus k z. Well, if this is a slow dependence as compared to omega as we have been assuming then you carry out the Taylor expansion. Any function, if there is a small increment in the variable can be expanded by using Taylor series. This expression becomes simply sigma at omega and k plus there is an increment here. So, it becomes delta sigma upon delta omega into I delta t and because of this will be a term minus i delta sigma upon delta k into delta z and these are operating over a into the exponential term. This same factor phase factor.

You may note that this is exactly same as before. Rather than going over the entire calculation and in this is valid in general. Even if there is a magnetic field using this is valid, this is a very general thing. This procedure is very similar to the procedure adopted by Schrödinger when he deduced the wave equation Schrödinger equation from de Broglie's hypothesis. Essentially, he said that omega should be replaced by an operator and you should also replace k by an operator. The same way I am suggesting here that omega can be replaced by because it is not operating over the entire electric field, it is operating this you know over the amplitude. I am replacing omega by omega plus I delta t because the time dependence of the phase term was already incorporated in deducing sigma. It is only the time dependence of amplitude that was ignored. I have to replace this by omega by omega plus i delta t and so on.

This is a very standard procedure in all areas of physics whenever you are dealing with waves or fields whose amplitudes vary with time and position slowly. This is a 1 important thing that I wanted to mention to you. Well, now this question of simple interpretation I think if time permits, then I would like to say something more significant from in a different way you can arrive at the same expression. Let me go a step further, let me write this expression sigma e bring this a in the interior, sigma into e is z rather this becomes i delta sigma upon delta omega into delta a by delta z delta t minus i delta sigma upon delta k and delta a by delta z and obviously, in these two terms I am multiplying an exponential term minus i omega t minus k z. I have to interpret this. Well, I am going to make an approximation here. As we have already seen the sigma that we have derived here is almost an imaginary quantity. Because the expression for sigma if you look is has a i there. It is imaginary. However, if we include collisions then it may have been small real part because collisions are a small quantity.

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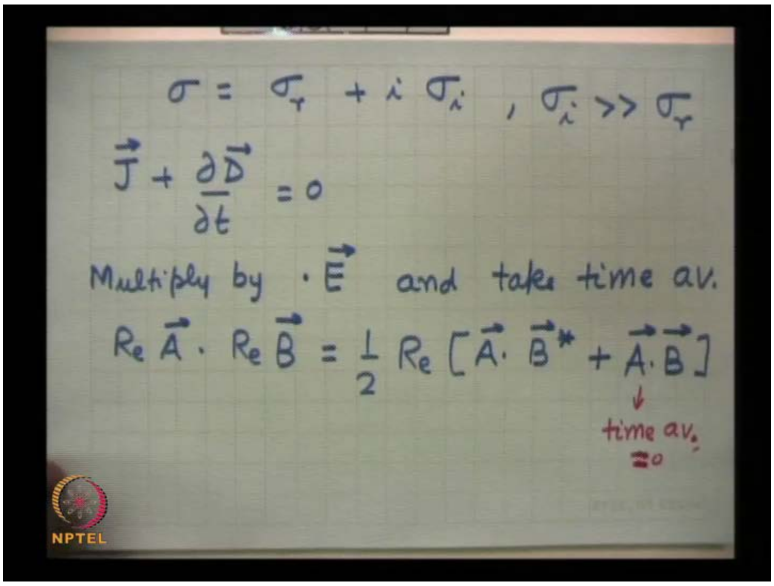

I can write down conductivity a sum a small real part and a very large imaginary part. where sigma i is much bigger than sigma r or magnitude bigger, but let me retain this small sigma r to have a physical interpretation. Now for electrostatic waves there is no magnetic field in the wave. Convention point in vector is 0 e cross h is 0 because h is 0. Now, let us understand the implication of J from the last Maxwell equation if curl of h is 0 then J plus delta D by delta t is 0. The last Maxwell equation becomes this if h is 0. I want to multiply this equation by dot E and take the time average. So, multiply this equation by dot E and let us see and take time average. Let us see what I get, please remember J is always a real quantity though I have written in a complex way, but whatever expression I have written real part of that expression is J.

So, J is a physical quantity and electric field also is a physical quantity though written in terms of a complex expression just takes the real part. So, I am going to use this identity that real part of A into real part of B is equal to half real part of A dot B take complex conjugate of any quantity either A or B plus a dot b. This is the identity that we had proved earlier and you can verify even now. Take A as small a 1 plus I a 2 and B as small b 1 plus I b 2 where a 1 b 1 a 2 b 2 are all real you can verify this easily. Now electric field changes with time exponential minus i omega t minus k z first time dependence. Similarly, J also has the similar time dependence. So, if I replace this a by J and b by e, but I get this term will have 2 omega frequency variation and time average will be 0 and this term will have the exponential phase terms will cancel because this is a complex conjugate. When I multiply this equation by dot E only these terms survives this goes this goes away and let us see what the consequence is.

So, time average of this is 0. Let me just mention this. This is time average is 0, you can forget this term is 0 rather time average is not exactly 0 and how about this term. This is the only thing I have to retain a dot b star by 2 and take the real part. I will write down term by term J has three terms and this may also have two terms. Let me simplify these, I think I want to be slow on this because this derivation is very fundamental to plasma physics. I want you to follow it with greater (audio not clear).

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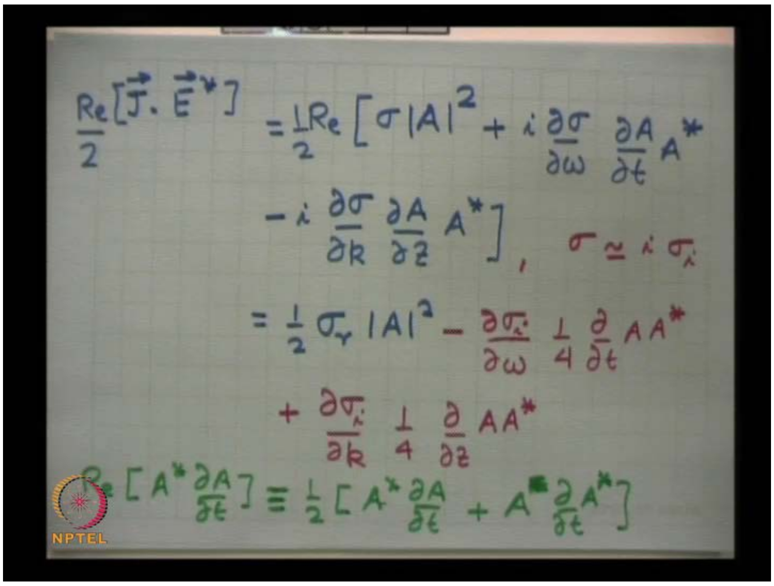

Let me write down J dot E a star and I take the real part of this quantity and divide by 2. This is what I am going to evaluate if I do this the first term J is equal to sigma e. It becomes real part half of sigma into e dot e star which is the same thing as modulus of A square which is a real quantity. This real part means real part of sigma will survive. Second term was plus i delta sigma by delta omega into delta A by delta t.

If you recall I am multiplying this by E star they are on the same direction, just multiply by a star. A could be complex in general and similarly, there is another term which is minus i delta sigma by delta k delta A by delta z into A star, this is the meaning of real part of J dot E star by 2 means the time average value of J dot E is this the quantity. As, I mentioned sigma here is largely sigma i. I replace by i times sigma i. This becomes means if I take the real part of this quantity it becomes half sigma r modulus of a square because this is real. So, real part will be sigma r only. Only real part will survive. And sigma i is because sigma in these expressions is nearly equal to i times sigma i. If I substitute this sigma in here and in there then, this becomes also real quantity i and i will become minus. This becomes a real quantity; similarly this also becomes a real quantity and then I have to take the real part of this quantity, which is you can easily show is equal to minus delta sigma by delta omega sigma i rather delta omega into 1 by 4 delta t of A A star.

And similarly this term becomes when I plus delta sigma i upon delta k 1 by 4 delta 2 of a star. You, may wonder why I am getting 4 not 2. The reason is that I am getting a term here. I think one should understand this with clarity that real part of A star delta A by delta t, I have to evaluate which is the same thing as half this number a star delta a by delta t plus. It is complex conjugate. Any complex numbers real part is half of that number plus it is complex conjugate. Complex conjugate of this quantity means replace this A star by A and replace the un-starred quantity by the starred quantity. This should go away and this should be there this should not be there and this can be easily shown is equal to half delta t of A star same thing. I have used this is identity you can identical. This is a very important expression. It has three terms the one which involves the real part of conductivity it may be tiny, but it may this term may be comparable to this term because a is A very slowly varying function of time.

Though sigma i may be large, but because the slow dependence of A star on time this term may be these terms may be comparable. Let us understand now the physical implication of the fourth Maxwell equation that I have written as J plus delta d by delta t equal to 0. J dot e star i have obtained I have yet to obtain the other term d delta d by delta t dot E star.

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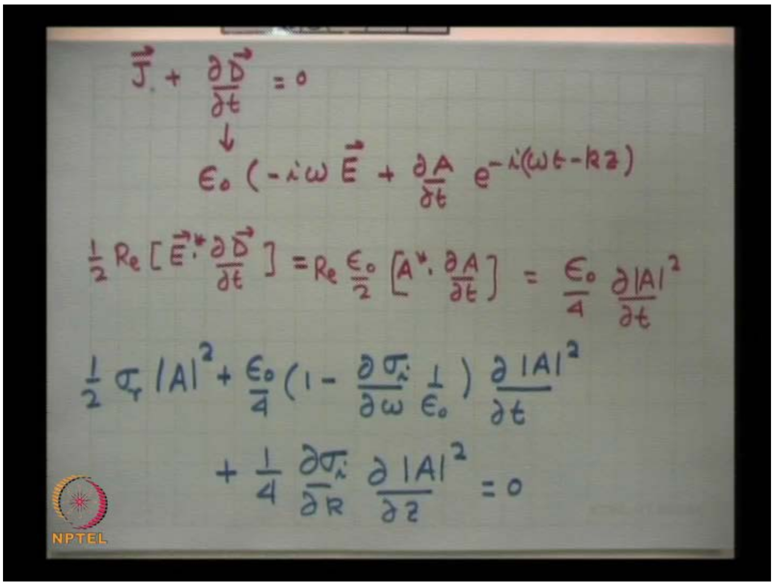

I am always keeping in focus J plus delta d by delta t is equal to 0. This term is how much D is epsilon 0 into delta e by delta t which is equal to minus i omega into E plus delta A by delta t exponential minus i omega t minus k z and when you take E dot of this quantity and time average this quantity. If I take E dot and this becomes real, but this is imaginary, real part of this is 0. If I take half real part of E dot delta D by delta t and take star here and then this term does not contribute. This is the only term that contributes and that gives you epsilon 0 is there; 2 is there and this becomes A star dot delta A by delta t and I take the real part of this quantity. Real part of this quantity has to be taken, which is, the same thing as epsilon 0 by 4 delta of modulus of a square by delta t. J dot to be obtained earlier E dot delta D by delta t we have obtained now add the two terms and put them equal to 0, what do you get essentially this expression.

If you just club these terms together what you get is this half sigma r A square plus epsilon 0 by 4 this term one comes from here minus delta of sigma i upon delta omega into 1 upon epsilon 0 into delta of a square upon delta t because of J dot E. The second term and E. This term is coming this one and then you get a last term plus 1 by 4 delta sigma i by delta k into delta of modulus of A square delta 2 is equal to 0. This has a simple interpretation because when there is conductivity real part means the current density has a component in phase with the electric field there is power dissipation. This is the power dissipation per unit volume. Then, this term should be something like rate of increase of energy density of the electric field. This factor multiplied by modulus of a square can be interpreted as energy density of the electric field and this term should be then expressible or can be interpreted as the pointing representing divergence of pointing flux.

This equation has lot of physical implication. Let us try to appreciate in plasma physics we do not talk that much in terms of conductivity. As, we talk in terms of effective plasma permittivity, but the two are unrelated as I defined earlier.

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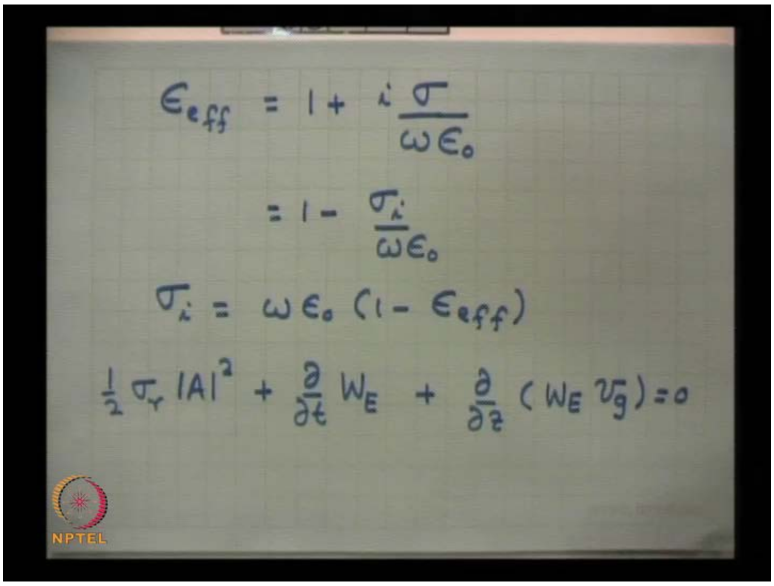

Effective plasma permittivity, we define as 1 plus i sigma upon omega epsilon 0 and sigma is almost imaginary at least in these terms that are appearing here the time derivative amplitude and space derivative amplitude in those terms. This is nearly equal to 1 minus sigma i upon omega epsilon 0, from here, I can write down sigma i is equal to omega epsilon 0 into 1 minus epsilon effective. This is a better way of expressing that equation rather than in terms of sigma. I would like to write that equation in terms of effective plasma permittivity and if you do this, the equation becomes rather simple. Half sigma r modulus of A square plus delta t of I will call a term w E plus delta z of w E into v g equal to 0. That equation can be put in this form; however, you may wonder there was delta k of epsilon effective. Where that is and what is W E. you just switch off this sigma i in that expression you will find it.

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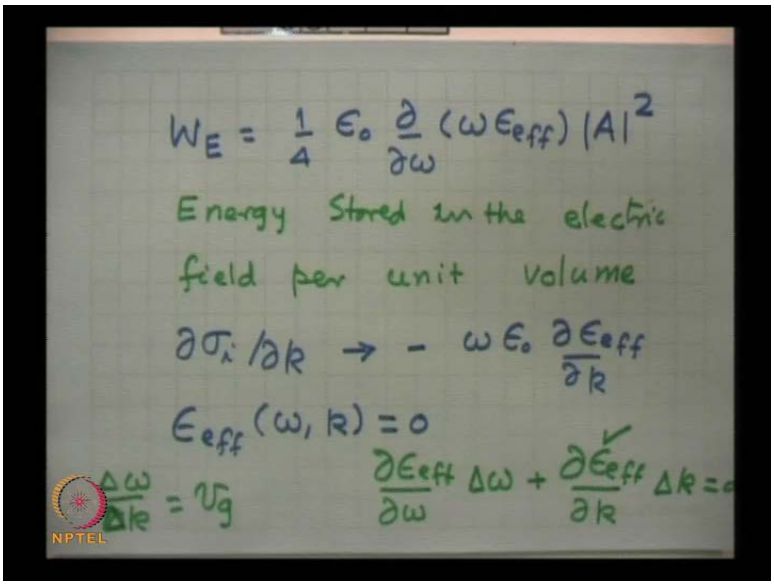

If W E turns out to be 1 by 4 epsilon 0 into delta omega of omega epsilon effective into modulus of A square or E star. Normally, a half vector comes in here, but because we have taken time average, half comes because of time average also. This is the energy stored in the electric field. This is called energy stored in the electric field.

Energy stored in the electric field of the wave per unit volume and what I have done regarding there was a term like delta sigma i by delta k what did I do with that. It was replaced by because sigma i. when, I wrote down in terms of epsilon effective. It became is equal to minus omega epsilon 0 because they were constant. I am differentiating this expression partially (audio not clear) to k. So, omega can be treated to be constant and then delta epsilon effective upon delta k. But one must remember one thing that epsilon effective is a function of omega and k and for electrostatic wave this is always equal to 0. If I differentiate this function any increment in this function due to increase in omega in change in k would of this form delta epsilon effective upon delta omega into increment in omega plus delta epsilon effective upon delta k into increment in k this should be 0 because this is 0 always at omega k, as well as, at omega plus delta omega and k plus delta k also this is 0.

So, from here I can obtain the value of this expression in terms of delta omega of this quantity you may just check and then you can write down the J dot E equation the way I had written and I have called delta omega by delta k or delta k as v g group velocity. This can be written as delta omega by delta k.

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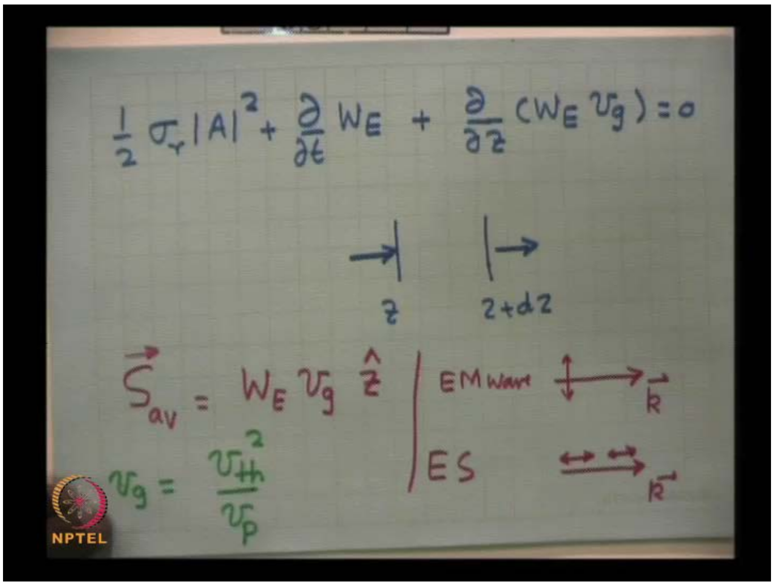

Let me write that J dot E equation again, which was half sigma r A square plus delta delta t of W E plus delta delta z of W E into v g is equal 0. Now simple interpretation would say that, this is the amount of heat dissipation, the power lost by the wave per unit volume. This is the increase in energy density per unit volume. If in certain region suppose consider a region between this is z and z plus dz, if in certain volume if I multiply by the volume of this region dz each term by dz, and area I am taking unit area of cross section of this plane as unity. Then the volume element will be dz, multiply this by dz, then the first term gives you the energy dissipation per unit volume. This gives you the increase in energy per unit time in that region.

Energy is being lost here more over energy is increasing here. It means energy should be coming in here from somewhere. So, more energy should pour in and less should get out only then increase in energy density and dissipation energy is possible. This quantity must represent the energy flow. Because divergence of energy flow is represents like in pointing theorem also. This is some sort of a pointing vector, less quantity more quantity of this comes in less gets out only then there is an increase in energy density of the electric field and increase and power loss there. The energy flow density or pointing vector for the electrostatic wave would be the pointing vector would be S. I am talking of average pointing vector is equal to W E into v g, the direction of energy flow will be in the z direction in this case because my wave was taken to be in the two direction.

This is equivalent of pointing vector for electromagnetic wave this is e cross h where as for electrostatic waves for which magnetic field is not there is no magnetic perturbation time average pointing vector is W E into v g, where W is the energy density of electric field per unit volume is energy stored in the electric field per unit volume. I have given you a derivation of two expressions one of W E in a dispersive medium and second in the derivation of time average pointing vector. One think, I would like to mention to you. For electrostatic waves the expression for conductivity was different then the expression for conductivity for electromagnetic waves. The thing was for an electromagnetic wave, if the direction of propagation is like this the electric field was in the transverse direction. So, there was no compression of charges this is E M wave.

Whereas for electrostatic wave your electric field is oscillating in the same direction as k this is electrostatic wave and there is a compression rare faction of charge and hence pressure term is significant. Here, there is no pressure term there. We had seen that in the case of electromagnetic waves conductivity does not involve any temperature, any thermal effects temperature comes through collision collisions only, but not through pressure, but here in case of electrostatic wave’s temperature comes or thermal velocity comes through pressure term. It is a separate dependence. The two conductivities are different and hence epsilon effectives are also different, and it will be useful for you to obtain the energy density for a plasma wave and energy density to a sound wave and also to calculate the group velocity. You can easily verify that for a plasma wave group velocity is equal to thermal velocity square upon v phase.