Group Theory Methods in Physics
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Lecture – 09
Cycle Structures Continued

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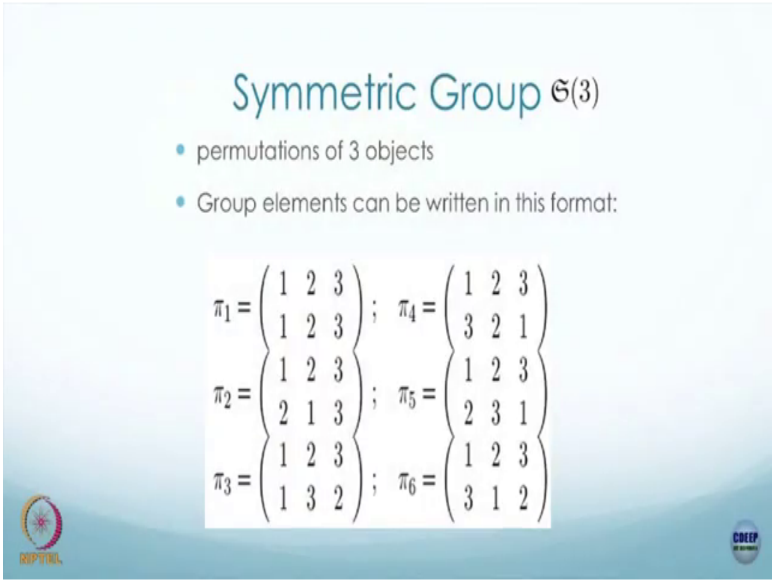

So, just let us a warm up we have got into permutation of objects which I called it as a Symmetric Group. Since, we had a week break, just say the warm up let me just go through the slide which you already seen, but just for continuity, ok. So, permutation of 3 objects is the symmetric group, so that, ok. So, is the symmetry group, group elements can be written in this format which we seen it, right.

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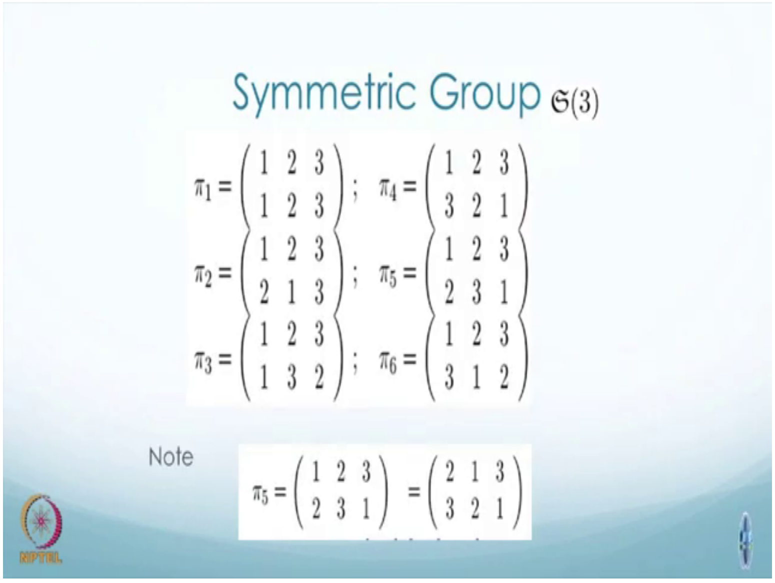

One format is of this type and then we also could see that if you interchange columns the element is the same, you do not get a new element. Everybody is with me?

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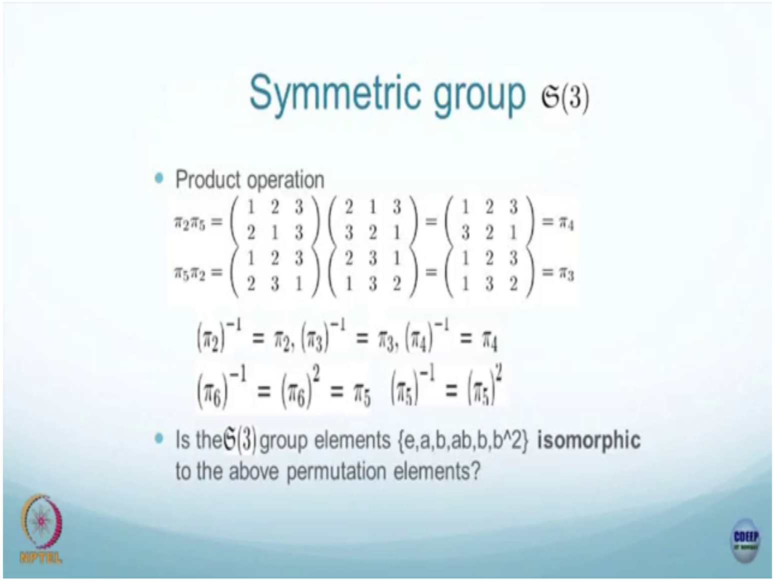

And then we also understood how to do because you can exchange the columns and it is the same element. You can do the multiplication so that the final result after pi 2 becomes the initial one and then you write it, ok. So, then you get the final result which is showing you the group multiplication property, right. We did these things elaborately in the last lecture and I also said how to see the inverse, you can go in the reverse direction, looking at how the if a goes to b, then you see that b goes to a will be the reverse operation on every column then you will get the reverse is that, right.

So, then a couple of things; pi 6 was having this property that pi 6 cube will be identity, similarly pi 5 cube will also be identity. So, essentially your symmetric group of degree 3 has elements. We have seen this earlier. As writing it in letters where a and b were generators of this abstract group, has some meaning by comparing it with the permutation of 3 objects, ok. So, such comparison of one to one onto correspondence is what we call it is an isomorphism between one group and another group, ok. So, this is isomorphic to the permutation elements. Is that clear?

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And order of a permutation group of degree n will be a symmetric group involving permutation of n objects, ok. So, this is a symmetric group. What is the order? Order is going to be factorial n. And we say that it is a symmetric group of degree n. And subgroups, you can start forming subgroups out of this n factorial elements you can take some subset which satisfies the 4 axioms of the group which is the subgroup those subgroups are called permutation groups and what Cayley's theorem says is that any finite group, like the way we wrote an abstract group with two generators it gives you a finite number of element finite order group.

You can show them that they are isomorphic to some subgroups of the symmetric group of degree n, ok. For example, we saw that the permutation group sorry the abstract group where you had a and b as generators with ab equal to ba squared you had 6 elements you can show it to be exactly same as your symmetric group of order degree 3, right. This is what we have seen. Is that clear.

So, you had a group identity a, b, ab, ab squared and what else b square and this is isomorphic two identity pi 1, pi 2, sorry this I think is pi 2, this is pi 1 I think. And this is nothing, but your symmetric group of order 3, ok. So, this was generated with a and b and this one is the permutation of 3 objects and these two are isomorphic to each other, ok.

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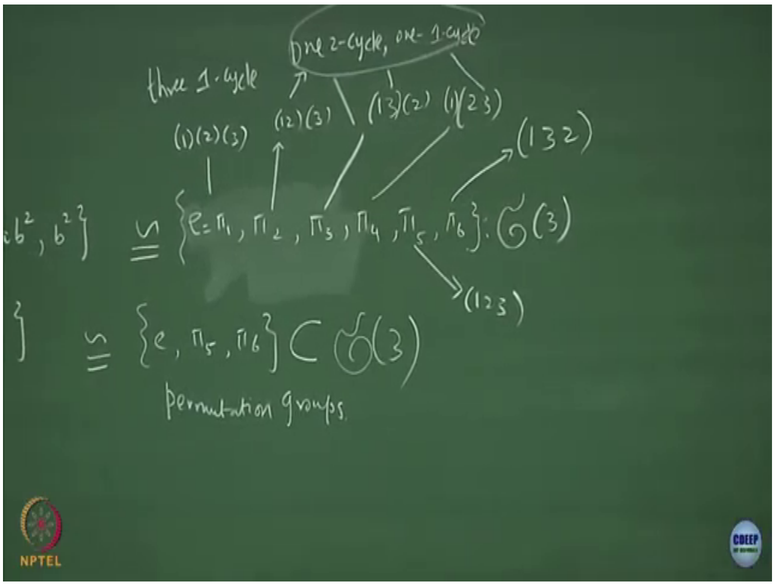

So, I am just saying that a finite group of some degree of some order, you can always find it to be isomorphic either to the symmetric group of degree n or subgroups of symmetric group of degree n, that is all I am trying to say.

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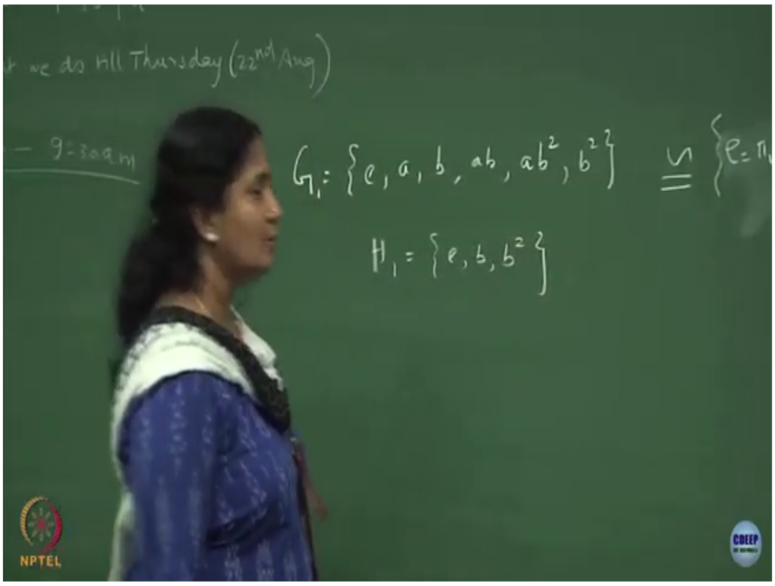

Like for example if you just take a group H 1 which is e, b, b squared, you can show this to be isomorphic to e pi 5 and pi 6 which is a subgroup of the symmetric group, ok. You all agree. So, any finite group will always be isomorphic to subgroups of the symmetric group of degree n and that the subgroups are sometimes called as a permutation groups, ok. This is trivial permutation group, the total set. The sub subsets which satisfy group axioms are called the permutation groups, ok. So, that is the Cayley's theorem.

And any permutation element is also we saw that you can equivalently be represented as a product of disjoint permutations cycles. So, if you remember these elements this one can be written as 1 2 3, which is disjoint cycles which you do not these are one cycles, right and this one was 1 2 exchange 3 was not touched and so on, right. This one is I do not know which one was 1 3 and which one was 2 3. Is this 2 3? Anyway, one of them is 2 3 and the other one is; this one is 1 2 3, this is a 3 cycle and this one is 1 3 2. Am I right?

So, this many times people do not write this one cycles they just write only the non-trivial cycles, but it you can see that any element will be a product of; so, no two elements here will have an overlap with the next one. So, if it is 1 2, 1 2 is a non-trivial cycle 3 has no overlap with it. So, that is what I mean by saying that any permutation element can be written as a product of disjoint permutation cycles. So, this is three 1-cycles, ok. This is one 2-cycle and one 1-cycle, ok. So, in this class all these 3 all of them go into it, ok. All the 3 belong to the cycle structure where one of them is a two cycle and another one is a one cycle. This is an identity element which is trivial which is three 1-cycles and then these two are one 3-cycle, ok.

So, you can break every permutation element each element you can call it as a permutation element into a cycle structure and typically the cycle structure will have this cycle structure is respected by 3 elements and this cycle structure is respected by two elements there are two distinct elements, which share the same cycle structure. Clear. So, that is the any permutation element can be equivalently represented as a product of disjoint permutation cycles, ok.

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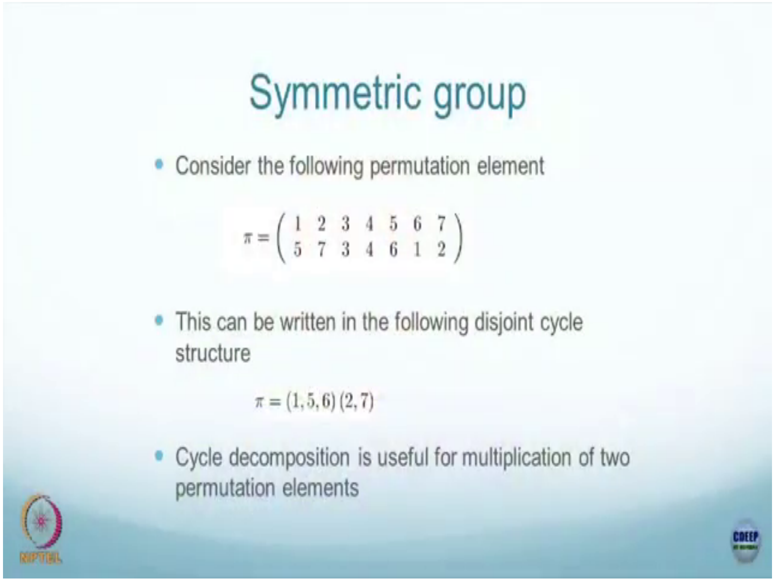

And this also I have already discussed that if you had a non-trivial permutation group of 7 objects you can write this disjoint cycle by looking at it looking at that element 1 goes to 5, 5 goes to 6, 6 goes to 1. So, 1 5 6 is one permutation element sorry the cycle and then 2 and 7 forms another cycle. The rest of the one cycles typically people do not mention, but if you want you can mention it, ok, fine.

So, cycle decomposition is generally helpful if you want to do multiplication because you know you can just concentrate on objects which are only subsets not on all the n objects when you do the multiplication, that way it is useful, yeah.

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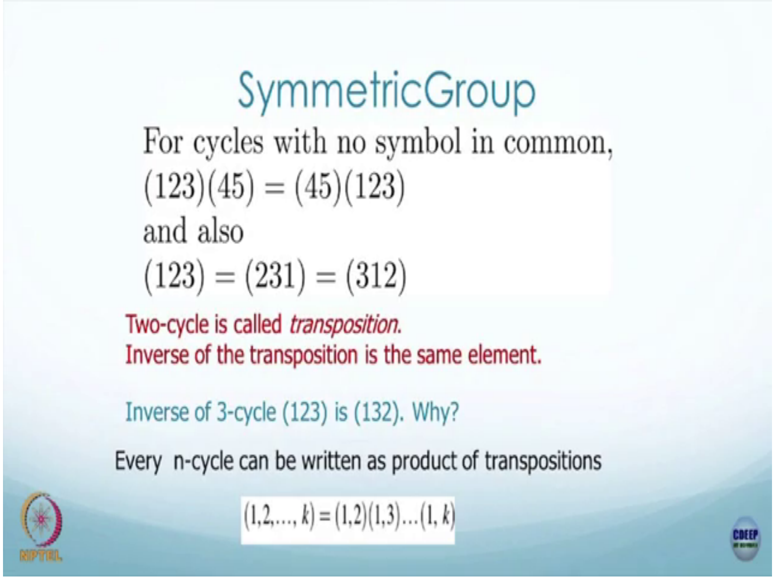

So, if you have no symbol in common, you can write either the 3 cycle followed by a 2 cycle or a 2 cycle followed by 3 cycle, basically the order does not matter it because you can do the exchange between 4 and 5 or permutation between 4 and 5th object, 1 2 3 you can do a cycle permutation, but those two are independent, so you can do it in whichever order you want. So, these are advantages if you write it in the cycle structure that you can use these properties, ok.

And this is that column exchange which I said is equivalent to just doing the cyclic way of writing it both are equivalent in the cycle structure the way of writing this is same as exchanging the columns. And whenever you have a two cycle, two cycle is nothing, but exchange of object one and object two or any two objects. So, like that here it is exchange of 1 to 2, 2 to 1 it is exchange of 1 to 3, 3 to 1. So, it is always an exchange of two objects it is called as a transposition.

Inverse of the transposition is the same element. So, inverse of a 3 cycle this also I have told you how to do it you have to instead of going in a cyclic manner you go in the anticlockwise manner. So, you say that in the 1 goes to 3, 3 goes to 2, 2 goes to 1. So, you write the inverse of the 3 cycle by reversing the direction of exchange, ok. So, 1 2 3, if you do 1 goes to 2, 2 goes to 3, 3 goes to 1 that is different, but instead you can do the other way round 1 goes to 3, 3 goes to 2 in the reverse direction then this will be the inverse of inverse of this element, ok. So, that is why these two are inverses of each other. Yeah, any questions, ok.

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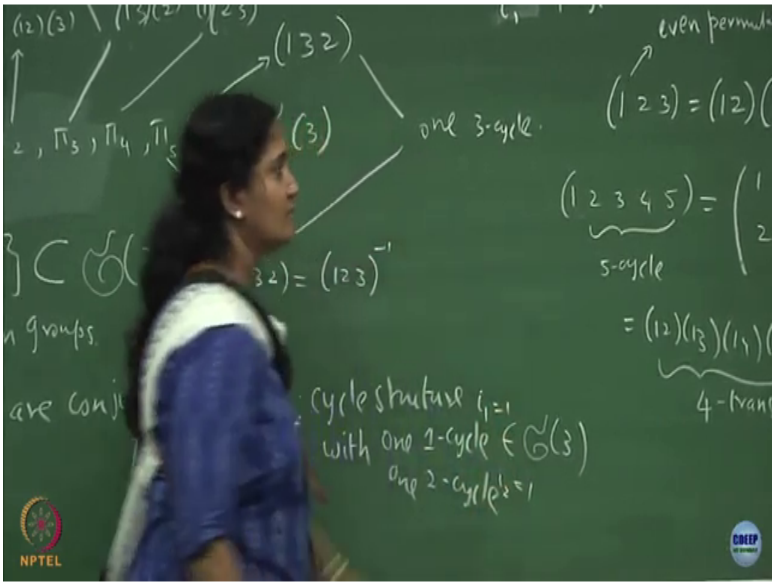

So, this also you saw any end cycle whenever you have a 3 cycle I could try to write it as a 1 2 3 can be written as a product of transposition, ok. So, in any n cycle you could try and write it as a product of transposition or any case cycle you can write it as a product of transposition. So, what is the meaning of this?

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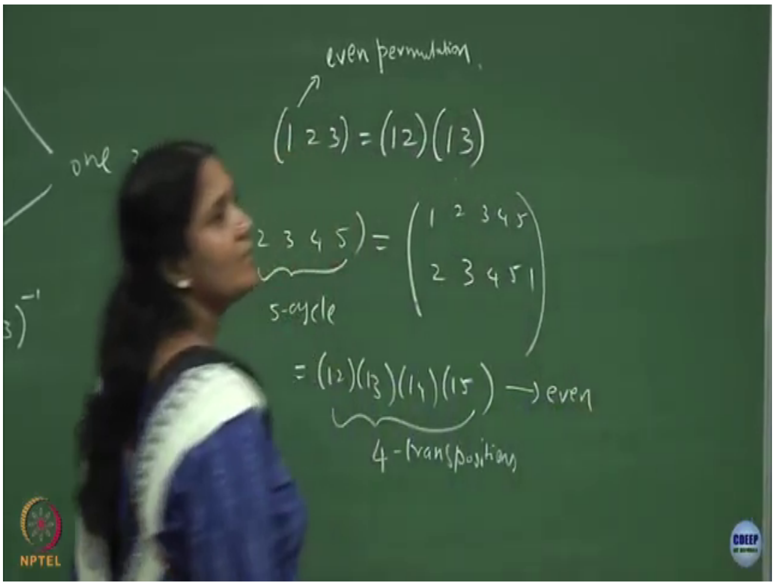

So, 1 goes to 2, 2 goes to 3, 3 goes to 4, 4 goes to 5, 5 goes to 1, if you write this in the element fashion 1, 2, 3, 4, 5, ok, so this is the meaning of the cycle structure. And you can rewrite this. So, this is a 5 cycle, you can rewrite this as 1 2, 1 3, 1 4, 1 5. We miss anything, ok. So, it in involves 4 transpositions, ok. Similarly, this one involves two transpositions and so on.

Student: (Refer Time: 15:16) how many (Refer Time: 15:20) try to (Refer Time: 15:21) how many (Refer Time: 15:22) are.

Ma’am, giving in terms of statement to you we will verify this last time, I want you to verify for some cases. Please verify it, but this will be generalizable. Just like mathematical induction you first verify for a couple of things and you can generalize and write a expression. So, that is all I have done that you have to check it out, ok, ok.

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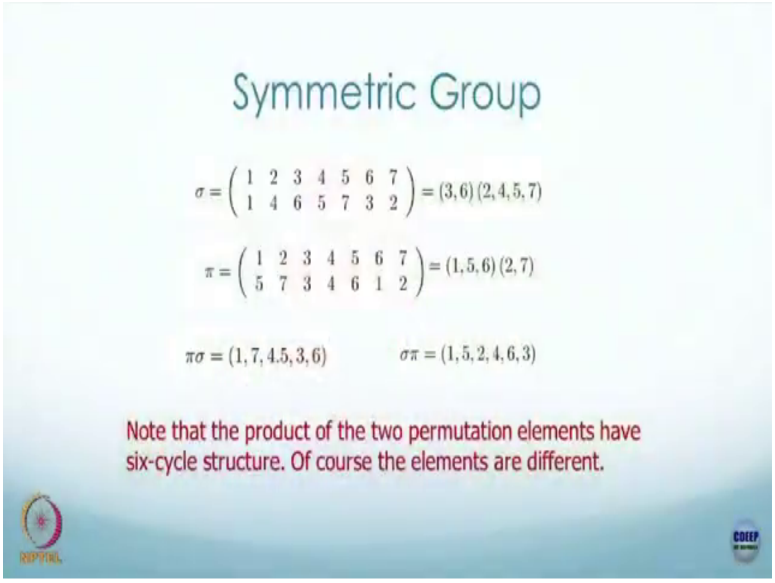

So, I have tried to recap some more examples sigma and pi and also we did pi sigma which gave you a 6 cycle structure and then sigma pi is another 6 cycles structure, and interestingly the order of multiplying two different elements gave you the same cycle structure irrespective of whatever. But the elements are different, ok. As an element this is very different from this element, but both have the same cycle structure. So, it is a 6 cycle structure. So, this is also a 6 cycle structure that is important to note here, ok.

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So, this I have already said any k cycle can be broken up into a product of transpositions. Transpositions are nothing, but two cycles. If you have odd number of transposition or even number of transposition the corresponding element is called as a even permutation or an odd permutation. So, this one is called even permutation and these are called odd permutation, ok.

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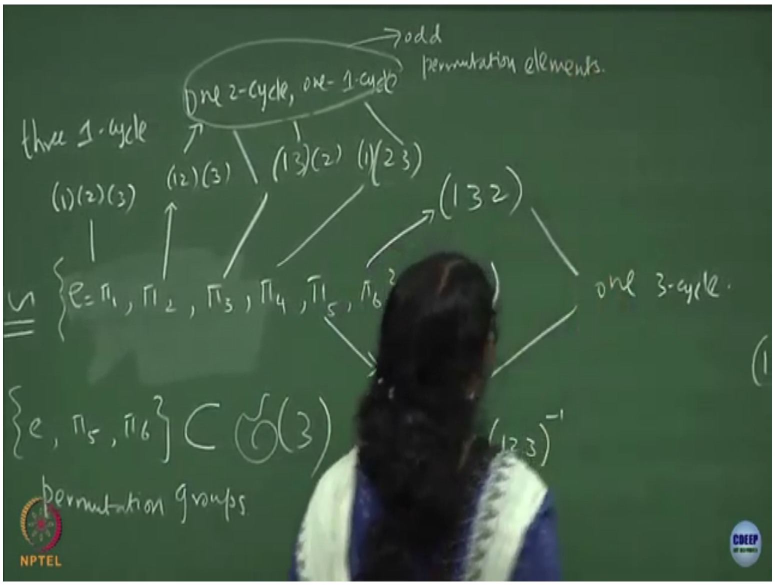

What about this? This is also even permutation, ok, fine. If you take an odd permutation cycle and multiply with an even permutation cycle what do you expect?

Student: (Refer Time: 17:47).

Louder.

Student: (Refer Time: 17:49).

It would be an odd permutation. You take two even permutation if you multiply, it will always be even. So, does that tell you something? Can a subset of only odd permutations satisfy group properties or can a subset of even permutation satisfy group property?

Student: (Refer Time: 18:16).

Loud.

Student: Even permutation.

Even permutation. So, even permutation elements that is what happened here. What is this? This is nothing, but this element was 1 3 2 and 1 2 3, right. It is a 3 cycle involves even permutation, right, and you find that it satisfies the group properties. So, this is a subgroup which is generated by the 3 cycle and it is going to be only even permutation elements, ok. Is that clear, ok.

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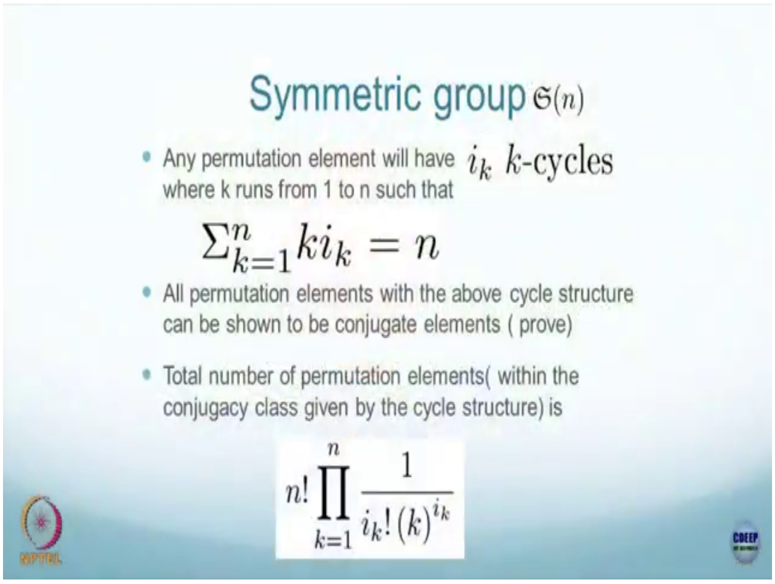

So, now coming to abstract definition with all these warm up which we have done. So, whenever I talk about permutation elements I look at it as if there is a set of elements with you know i 1 repetitions, ok.

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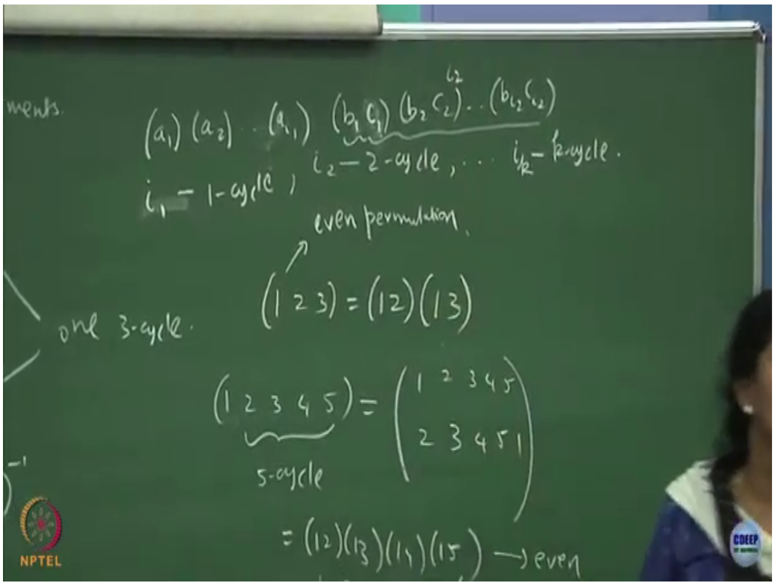

So, basically let me write it. So, one cycle I will write it this way i 1 1-cycle, i 2 2-cycle and so on. You get what I am saying. So, you will have objects which are let me call it as a 1, a 2, a i 1, and then I could have b 1, b 2, may be what is a better way of writing this, yeah. So, I am basically saying this one will be or let me called it c c 1, then b 2, c 2, ok. So, there will be an i 2 2-cycles, and so on; so, all these things which I have given as examples false into this picture.

This identity element means i 2 is 0, i k and all is 0, only i 1 will be 3, clear. Similarly, if I look at this it corresponds to i 2 equal to 1 and i 1 equal to 1; one 2-cycle and one 1-cycle. So, this will be the most general cycle structure for any symmetric group of order degree n, where I can write it in a cycle structure, but there is a constraint. What is the constraint? Here if you see if the group is of degree 3 the total number should add up to be 3. Is that right?

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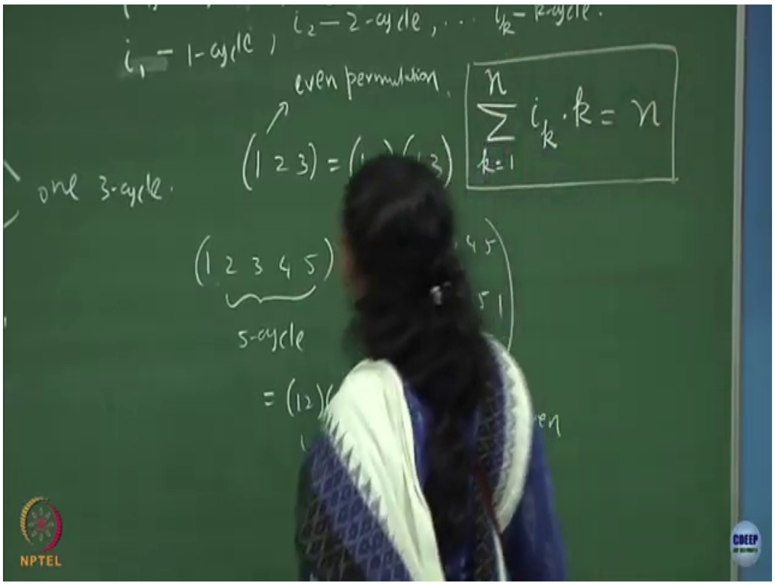

So, if you see here you see a constraint that summation over k which runs from 1 to n, with an i k multiplied with k has to be n. Is that right? How many n cycles or possible in a symmetric group of degree n? 1, right, in the cycle structure. So, this is what is important.

So, if you had an n cycle, so this is n then i k has to be 1, ok, so then only then this is satisfied. Is this satisfied for these cases also you can check, right. This one is two, 1 times 2 plus 1 which is 3. This is very important you have to make sure that it is always a sum adds up to give you number of two cycles multiplied by two this subscript is multiplied here, ok. So, keep this in mind.

Student: (Refer Time: 24:02) this elements can be set that there are one (Refer Time: 24:06) one cycle and one 1-cycle.

Yeah.

Student: So, total this (Refer Time: 24:11).

No, this is only for a element, any element in the permutation or in the symmetric group of degree n can be broken up into a cycle structure with this constraint, any particular element if you write it should satisfy this constraint, the cycle structure should satisfy that condition. Good.

The next question I am coming to. What you are saying is nice that how do you know that there are 3 elements with the same cycle structure, that is what you are asking. That is not visible from this. This is for a element or any element there will be a cycle structure, only constraint is that the cycle structure what you write should satisfy this condition, good. So, that is what I have shown here. Any permutation element will have i k k-cycle, where k runs from 1 to n such that this constraint is satisfied.

All permutation elements with the above cycle structure, can be shown to be conjugate elements. So, if you see here this element, this element, this element all of them have i 1 equal to 1 and i 2 equal to 1, right. So, I am sure you would have check that some point that pi 2, pi 3, and pi 4 are conjugate to each other. This is same as showing that a, ab, ab squared, were conjugate to each other. Have you checked this?

We have check this many times, right. So, this is interestingly pi 2, pi 3, pi 4, have a cycle structure with one 1-cycle and one 2-cycle. So, I am just trying to stress the point that conjugate elements of the symmetric group if an element a is conjugate to another element you can be sure that these two will be having the same cycle structure. The elements will be different. You see that the element is very different from this element, but as a cycle structure, by cycle structure I mean I just give you the set of integers i 1, i 2, i k, ok. As a cycle structure they have the same cycle structure, i 1 and i 2 are same, ok.

So, then the next question is that how many how can you say that there will be only 3 elements with this cycle structure, ok. This is all elements of my permutation symmetric group of degree 3, right. How can I say that there will be only 3 elements not more not less with this particular cycle structure? Ok. So, this is i 1 and this is i 2, this is 1 and this is 1 in this notation. So, everyone with me; so, I am not going to prove this for you, but I leave it you to as a curiosity to sit back and look at how to prove this.

Total number of permutation elements like to get this 3 you can use this formula; so, also can be derived by combinatorics.

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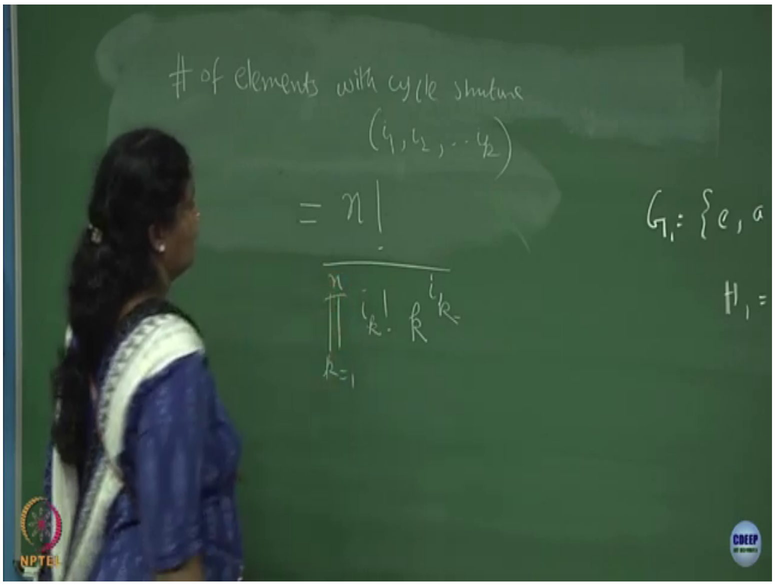

Number of elements with cycle structure; so, now, I am going to write cycle structure as i 1, i 2, i k you understand what it means, i 1 will be the number of 1-cycles, i 2 will be the number of 2-cycles and so on. So, in this case i 1 is 1 and i 2 is 1 and you found the answer to be 3. So, in general all these elements which you find with the particular cycle structure will be conjugate elements and I want to find that number. So, this number is going to be for a permutation group of object n or symmetric group of degree n, it is going to be this times, it is going to be i k factorial, k to the power of i k, ok. So, let us try this out here.

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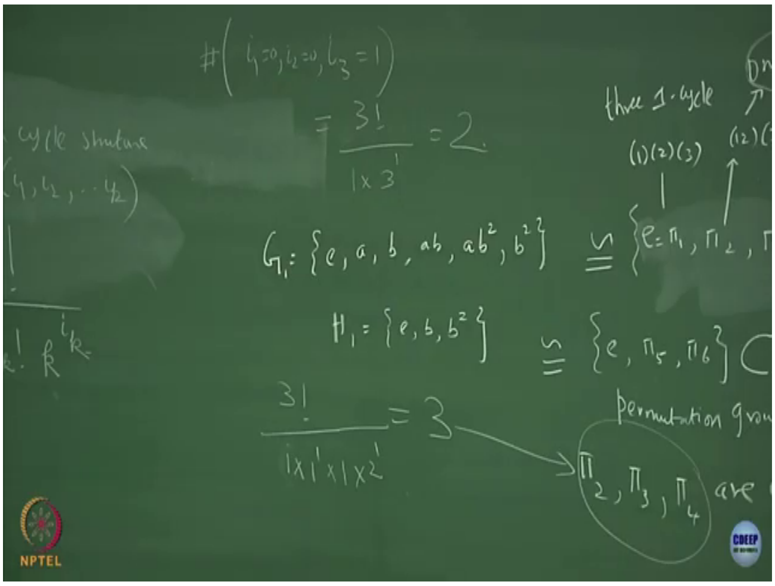

You will have in that case 3 factorial because this is a degree is 3 and then you have i 1, i 1 is 1, right it is 1 time 1 to the power of 1 and then i 2 is also 1 times 2 to the power of 1. You agree. Second one is a 2-cycle, second one is a 2-cycle. So, you have this. I have explicitly written the product in this case and 3 cycle is of course, 0. So, i 3 you can put it 0, 0 factorial is 1 and when you can use 3 to the power of 0 which is also 1. So, it does not contribute. So, this turns out to be 3 which is nothing, but these 3 elements, clear.

You can also check the second type which is one 3-cycle, right. How many elements are there? Can we check? So, i 3 is 1, i 1 is 0, i 2 is 0 number of elements with this cycle structure will be, you know the answer, but you I just want you to use the formula and check it out, ok.

So, you essentially can determine the conjugacy classes, how many elements are there in the conjugacy classes for which is given by a cycle structure. By conjugacy class I mean a cycle structure and you can find how many elements have that cycle structure and they are conjugate to each other, ok. So, idea is to somehow get to handle these conjugacy classes in a better way.

For 3 object it is simple, but if suppose I give you ten objects it become much simpler and you need to work only with one candidate of the conjugacy class. Physics just not determine you know it does not bother about all the elements of the conjugacy class, whether I work with this element the same physics I could get from here and here, so I can actually work with only one nontrivial element of a conjugacy class. I do not need to work with all the elements which is n factorial.

So, to break this, this methodology will help you to at least determine how many elements are in the conjugacy class which will be like a multiplicative factor and then you worked with the candidate of the conjugacy class, a candidate of the conjugacy, ok.