**Group Theory Methods in Physics**
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**Lecture - 08**
**Cycle Structures**

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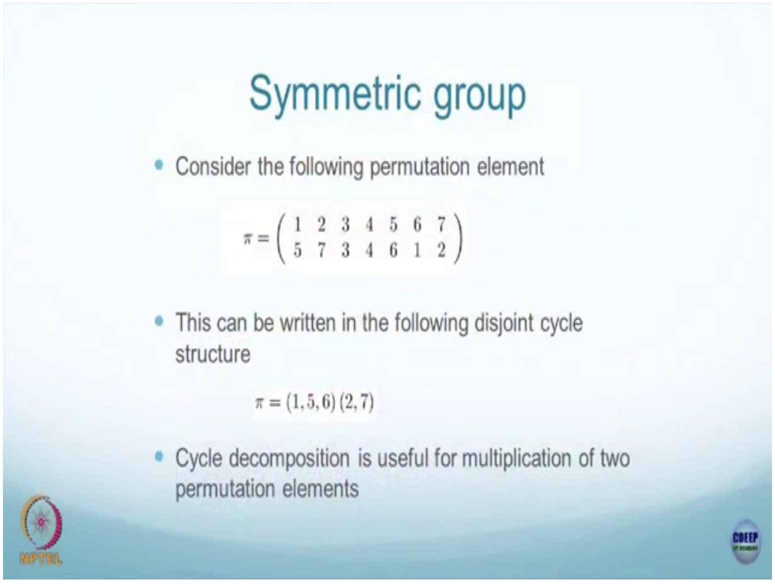


So permutation cycles. So, let us consider one particular element. So, now, let us look at this particular element which I have put it on the screen, ok. So, I have written an element; first of all looking at that element you should be able to tell me. So, this will be an element of a symmetric group of degree 7 ok.

This is one element in the symmetric group of degree 7 and you will see that 1 goes to 5 and 5 goes to 6, 6 goes to 1 ok, this is what you see. It can be compactly written as 1 goes to 5, 5 goes to 6, 6 goes to 1. It is an equivalent description rather than writing in the longhand way. Once I put it in parenthesis 1, 5, 6; it means 1 goes to 5, 5 goes to 6, 6 goes to 1 ok.

And then let us see the other one; 2 goes to 7, 7 goes to 2 right sorry, this is 1 and this is 2. So, 2 goes to 7, 7 goes to 2 this is the better notation which takes care of that element. And what else 3 remains 3, 4 remains 4 that is it; something else 1, 2, 3, 4, 5, 6, 7 yeah seven elements. This is what I call it as a, the same element written in cycle structure.

So I am just writing the same element in a cycle structure ok. That these objects did not change, there is no need to mention it at all; whatever will change is what we mentioned in the cycle structure, the rest which I do not mention are going to remain as unchanged, that is what is the meaning.

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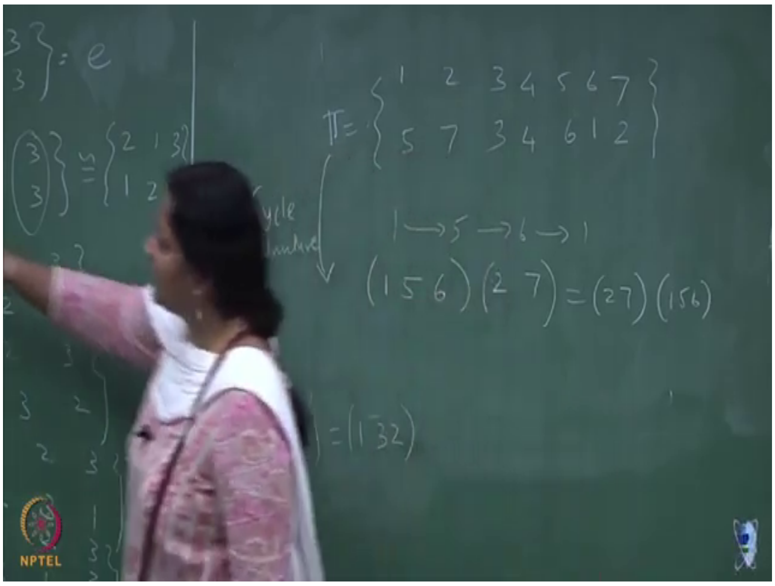

So, this is what is a cycle structure. So, let us write the cycle structure for these elements.

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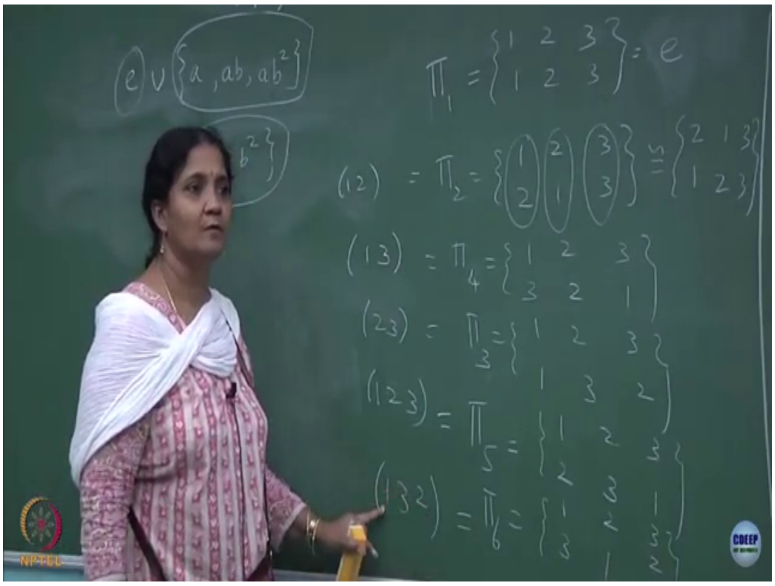

This you can write it as 1, 2 and 3; anyway it is equivalent to not mentioning right, you can just leave it as if it is some kind of a, call it as an identity element. What about this? 1 2; this one will be 1 3 this one; this one is 2 3 cycle structure. What about this? 1 2 3; this one, am I right; 1 goes to 3, 3 goes to 2, 2 goes to 1. Are you all with me? If you understand the simple example others you will follow, ok.

So, this is the cycle structure for an arbitrary element I have just taken an order degree 7 symmetric group just to say that; looking at this you could write the cycle structure. The advantage is that, the elements in the cycle structure are also in disjoint subsets. You cannot have one element common in the; if it is common, then everything will get into one higher cycle structure, right. They are all disjoint and you can play around when you do the multiplication little better; that is why the cycle structure becomes much more handleable, if you go to higher degree groups ok.

So, that is why I am trying to push you to this ok. So, I will explain that, you can write it as a cycle structure. So, you can write this one and cycle decomposition is useful for multiplications of two permutation elements.

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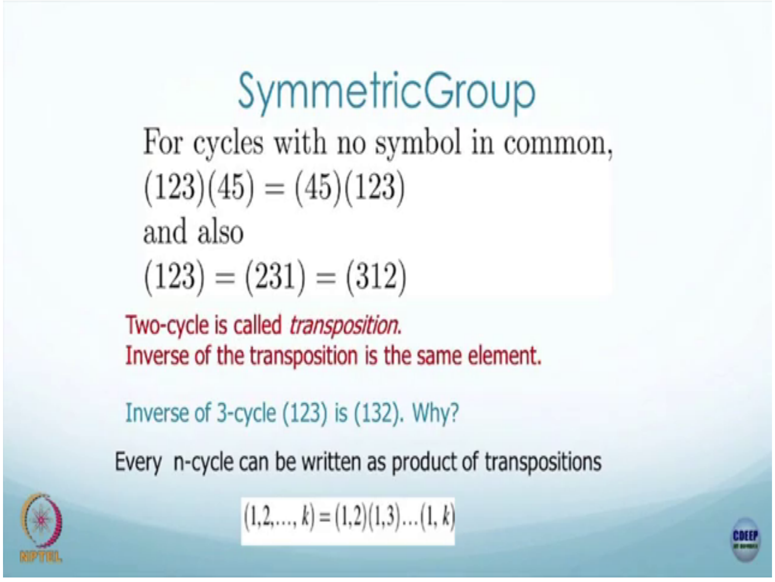


Because if you see here, if I write it as a 1 2 cycle structure; if I do it twice 1 becomes 2, 2 becomes 1; again you know you can see that 1 2 square is identity, similarly 1 3 square is identity. And you can see that 1 2 3, you can cyclically permute it and when you do it third time you will come back to the identity; you know there are a lot of advantages of seeing in the cycle structures.

So, order really does not matter, whether I call 1 5 6 permutation followed by 2 7 or 2 7. So, this will be same as you could have done 2 7 and 1 5 6. So, these kinds of you know commuting elements are easy, ok. So, this is what I show here. And also when you do 1 2 3 when you write it; this order also does not matter, the initial starting point could have been 2, 2 3 1 that is same as 1 2 3 ok.

So, these also I have said in the slide here; that 1 2 3 is same as 2 3 1 which is same as 3 1 2 clearly does not matter. There are these technical definition; any two cycle we call it as a transposition. And then whenever I want to write an inverse of a transposition, it is the opposite one 2 1 go; you know it goes to this and then by this equivalence you can show that pi 2 is same as pi 2 inverse, right.

So, transposition or order two elements, transposition is on a two cycle; cycle with two elements and transposition will always be on order two elements. This one if you do it again you will get back identity. An inverse of the transposition is the same element ok, is this clear ok. Inverse of 1 2 3 is 1 3 2, can you check that also you could probably do it here; if you take the inverse here, do the inverse here.

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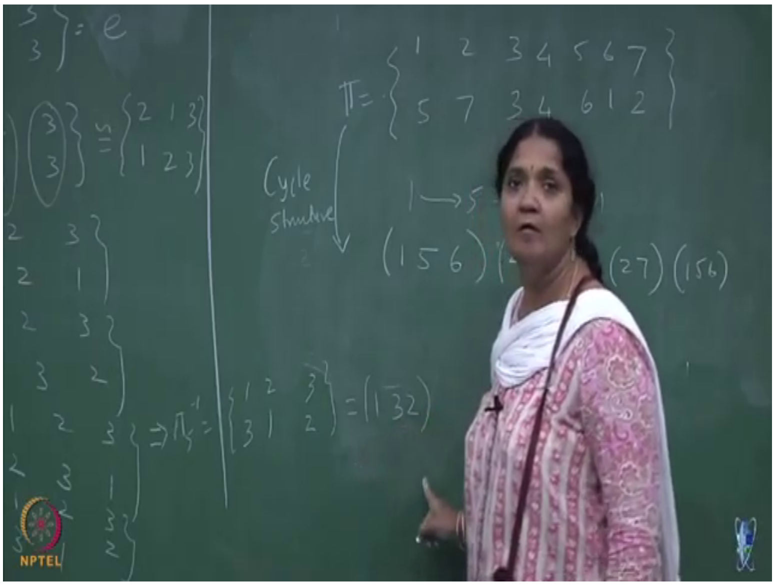

So, what does this mean in the inverse? I have already taught to you; that you have to see 2 goes to 1; 2 goes to 1, 3 goes to 2 and then 1 goes to 3. What is this? I did the reverse and you see that this is nothing but, which is nothing but the pi 6 element ok. So, that is why I said very, you just do the inverse of the 3 cycle and you will see that pi 5 and pi 6 are inverses of each other ok, is this clear.

So, I have tried to take you on a warm up, how to do multiplication, group multiplication in the notation of symmetric group elements of this type. I have also said now, you can also equivalently write it in the cycle structure fashion; any two cycle is what is called transposition, it is an order two element this element is it is own inverse.

And the other elements you can start finding; if you want to find the inverse of any arbitrary element you do the reverse and write down the element, I will give you the inverse. And you can check whether the inverse lies in that set 1 2 3 and 1 3 2 lies in that set, so 1 2 3 inverses 1 3 2, ok. So, this much I have given you a clarity.

The next one is, suppose I have this was a simple example; I could have had a cycle which had four or five elements, this is a three cycle, this is the two cycles. So, this particular element is a product of a three cycle and a two cycle. You could have four cycle and a three cycle and then the question is; can what do we mean by a three cycle? It is actually a composition of transpositions Any three cycle will be a composition of two transposition, any n cycle there will be a composition of n transpositions ok.

You can play around with transposition; transposition is a fundamental object and you can play around with these transposition to generate all possible n cycles ok. So, this is what I am trying to say here; every n cycle can be written as a product of transpositions, ok. So, I have given you a k cycle, 1 to k. So, the first check you have to do is, do it for 1 2 3 whether it is satisfied, can you check that out for this case. So, what have I written 1 to k is 1, 2, 1, 3, 1 k.

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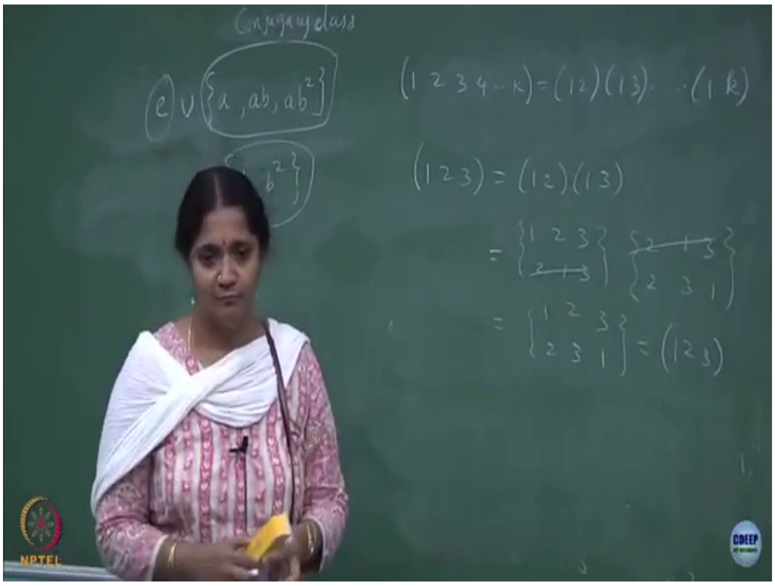


So, let us write it. So, let us do it for 1 2 3 ok. So, let us write it in the longhand ok. So, I am just verifying the statement which I am writing, that k cycle can be broken up into k transpositions and then just verifying that 1 2 3 is given by 1 2 multiplying 1 3, ok.

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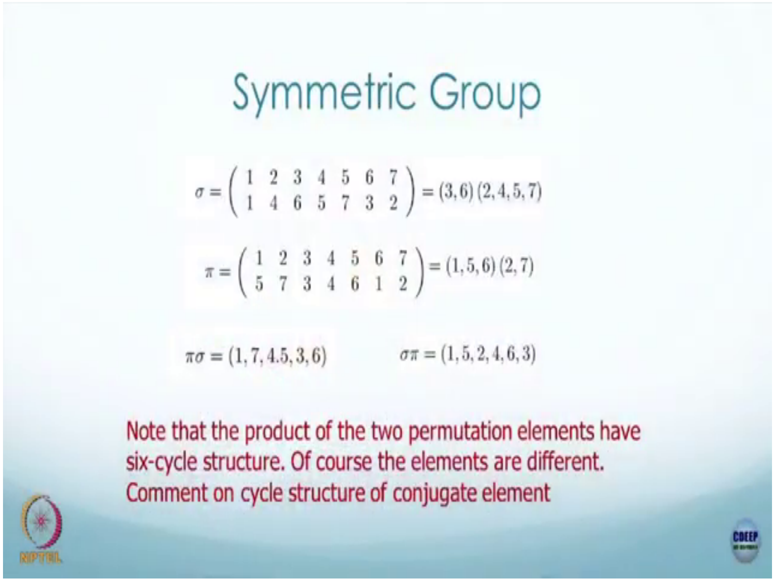


So let us take another element of this s 7, the symmetric group of degree seven. Now the cycle structure of this element sigma which I have shown has the two cycle under four cycle ok; others are all 1 is not changed, you can see that; 3 goes to 6, 6 goes to 3. So, that is why it is a two cycle.

And then you have a 4 going to 5, 5 going to 7, 7 goes to 2, 2 goes to 4 ok. So, this is 2 4 5 7, is that clear notation. Now, the next thing is do multiplication; pi I have anyway gave you. Let us multiply pi with sigma; what will be the cycle structure of pi with sigma, ok. That is one question you can ask; explicitly do it, please do this and verify. You will see that you get a cycle structure with six elements ok.

And if you do sigma pi, it is some other element it is not the same element. Why it is not a same element? Permutations are not abelian group; if you do pi with sigma, the answer need not match with sigma with pi, you agree right. But what is seeing here is something you know; little bit I want you to think. So, what is happening? You do pi sigma you find a cycle structure with six elements or a six cycle; you do sigma pi you again find only a six cycle, you are not finding a cycle which is different.

What does that tell you, ok? You can have pi with a different cycle structure sigma with a different cycle structure and you know pi sigma is not same as sigma pi. Interestingly if I find a six cycle structure for pi sigma product; I find the same six cycle structure, element is different. This is pi 5 and pi 6 are three cycles, but they are different elements; elements could be different but the cycle structure remains the same, that is very something interesting and that leads to making one more comment.

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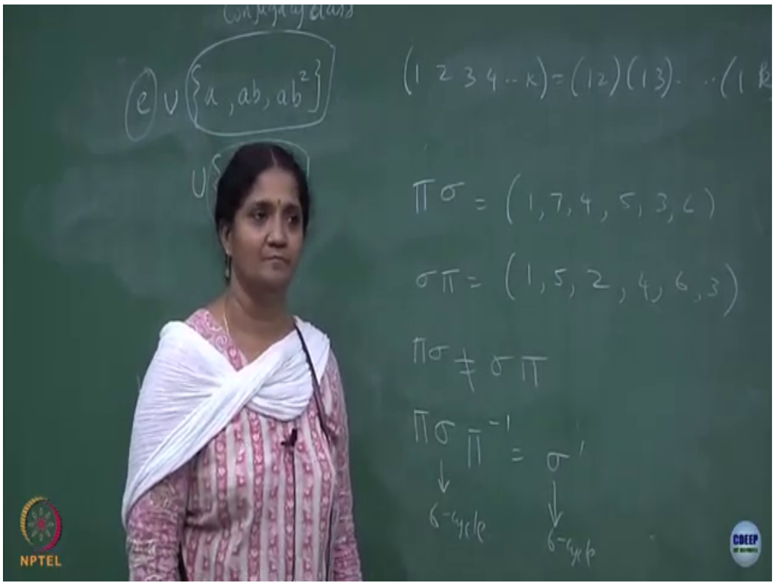

So, what we find is pi sigma has a six cycle structure. So, let me 1, 7, 4 5, 3, 6; sigma pi is 1, 5, 2, 4, 6, 3; these two are not same ok, pi sigma is not equal to sigma pi. But you can see that pi sigma pi inverse, is that right. Can give you a new element, so pi sigma pi inverse can give you a new element in general right and this element are called what, we did this in the beginning of the lecture; conjugate element ok, they are conjugate elements.

What you can show is that, if this has a six cycle the conjugation will give you the same six cycle; it will be a different element, but the cycle structure remains the same this is what we can show. So, now, I have given you a hint on the conjugacy classes; what does the hint?

In the objects, three object cases you had a cycle structure; there was a two cycle structure, you had a three cycle structures, right. So, a three cycle cannot be conjugate to a two cycle, clear. If you have a two cycle structure by conjugation it will go only to a two cycle structure, cannot go to a three cycle structure.

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So, that tells you that, you will have these three which are 1 2 3 and 1 3 belong to a conjugacy class; because all the three will have a two cycle structure, the elements are different but they have a cycle structure which is two cycle structure, these two are 1 2 3 and 1 3 2.

There are only two non trivial 3 cycles and the two will belong to one conjugacy class; there are three transposition, independent transposition they belong to one conjugacy class; identity element is like one cycles and that is always a class by itself, ok. So, this is one way of motivating in terms of the cycle structures; first of all given a symmetry group of degree four suppose, you can break it up right. You can write situations where, let us do that, maybe I will stop with that.

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So, you can have degree 4 identity element. So, I am calling that as degree 4 group symmetric group, degree 4. So, identity element and then let us say what all you can have and 4; you have to help me otherwise I will, that is the two cycles. You can have 1 2 3, then 1 3 2 1 4 3 you know you can list out all the possibilities ok. I am not writing the; how many will be there totally? Four factorial, 24 elements ok.

These cycle structures are all two cycle structures, we will all belong to one conjugacy class and similarly three cycle structures will belong to one conjugacy class and so on, ok. This is also still not at a simpler level, we will do a little more simpler level in the next class; but at least you get the feel of that how many classes will be there, is how many partitioning you can do, of an integer 4, right.

So, here it is this means what; there are four 1 cycles ok. This means what? There is 1 2 cycle and two 1 cycle, we should add up to 4, you should add up to 4. What does this mean? This is 1 into 3 plus one 1 cycle right; the one cycle I never mention. What else can I have? 1 into 4, yeah.

Student: (Refer Time: 21:40).

Yeah. So, I am saying identity element is nothing but 1 2 3 4; four 1 cycles. So, 1 cycle plus 1 cycle plus 1 cycle plus 1 cycle which you can call it as four 1 cycles if you want; 4 into 1 maybe that is a better notation; 4 into 1 that is 4. This one is 1 2 is a two cycle, then you have a 3, then you have a 4. So, one 2 cycle and two 1 cycles ok. So, maybe this also I can write it as 2 into 1.

How about this? It is 1 3 cycle and 1 1 cycle. What is this? It is 1 4 cycle ok. So, this is the way I am trying to interpret the diagrams and then you can break it up into partitioning off an integer 4 into various ways. Do you know what is the combinatorial formula for that; maybe you will know it, but maybe next class we could discuss this ok, is this clear yeah.

Student: What about the cycle once we use?

You will have.

Student: It is a different cycle or it is same as a old cycle.

No, it is different right; good point, I am not listed everything.

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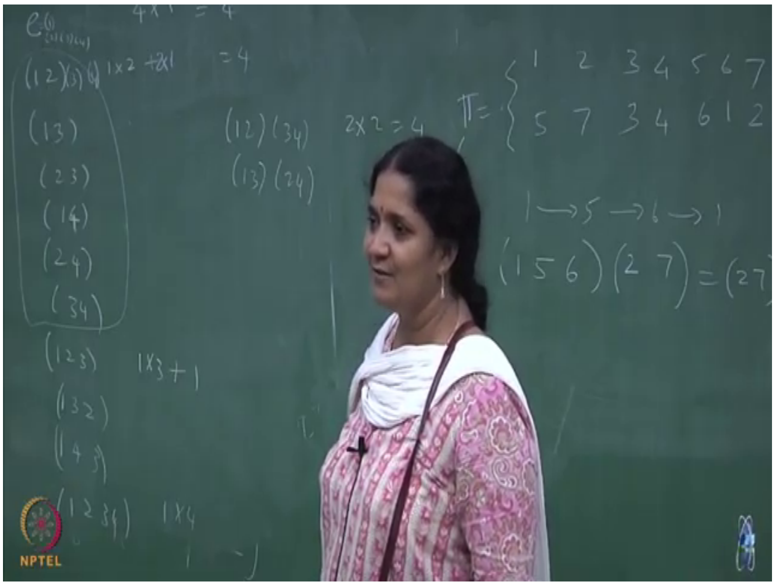


So, you can have a 1 2 and a 3 4; can have a 1 3 and a 2 4. So, this one will be 2 into 2, two cycles two of them and no other one cycle; this will also be 4, this is also there.

Student: (Refer Slide Time: 23:37).

It should permanent. In fact, you can figure it out at least how many classes are there. How many elements in the class maybe there is a combinatorial; but you can figure out that how many classes are there. So, this is the breaking of an integer 4 into various partitioning. So, let me write that.

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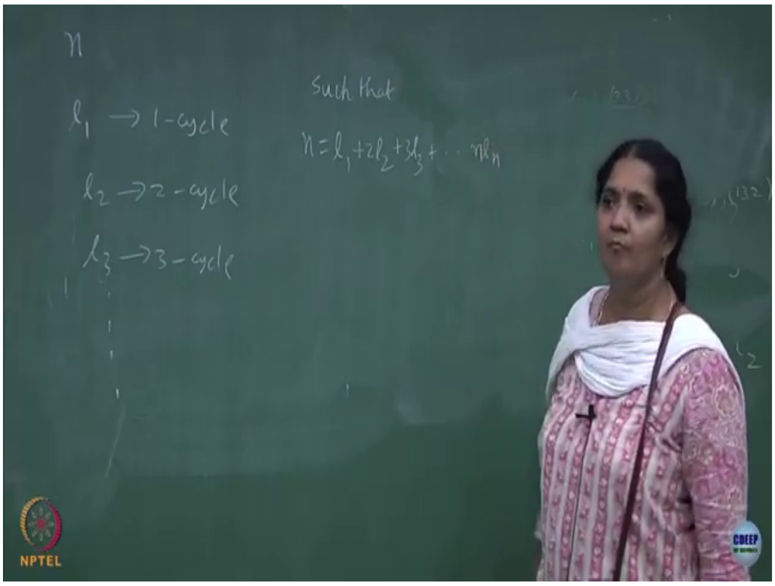


So, integer n into l 1 1 cycle, l 2 2 cycle, l 3 3 cycle and so on ok. With the restriction such that, some one l 1, 2 l 2, 3 l 3 is that right. They should add up to, how much it can go to; not beyond l n, cannot have a n plus 1 cycle on a degree and object. This is a good constraint.

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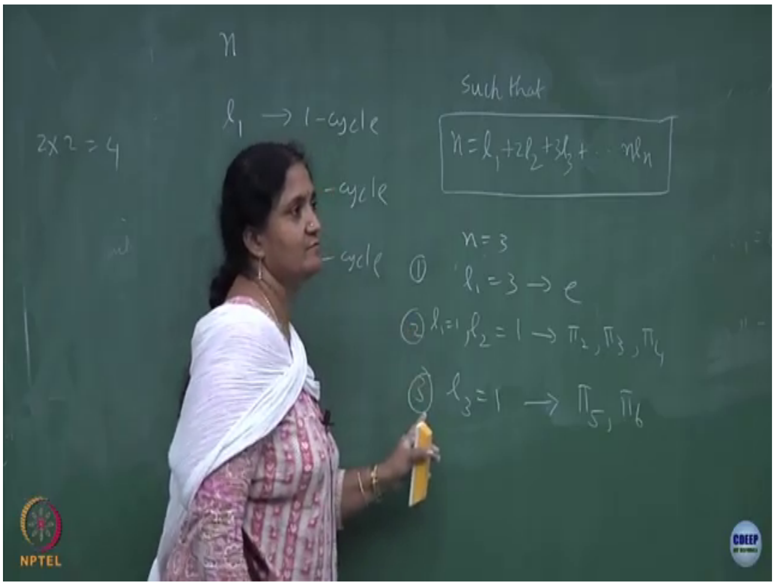


For the three objects it is simple; what are the options? You can have 3; if you want n equal to 3, l 1 is 3. What is that? That is the identity element, 3 1 cycle; l 2 is 1, that will be your pi 2, pi 3, pi 4; l 3 is 1, satisfies this condition.

So, I have broken them into basically the division of integers satisfying this condition, then once how many such divisions are possible will tell you how many classes are there. So, you have class 1, class 2 and class 3; no other possibility here. So, you do the same thing for n equal to 4 just to get some practice.

After you do that, you have to also figure out, this numbering is three; why is number 2 having three elements, number 3 having two elements, can you find the combinatorial factors for that also.