Group Theory Methods in Physics
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Lecture - 07
Permutation Group

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Ok. So, now, I am going to get on to Permutation Groups because the symmetric group has lot of resemblance to permutation groups, ok. So, basically what is permutation? If you have n identical objects I give a tag to it as 1 2 3 4, I can take object 1 and object 2 and exchange or I can take object 1 to take the object 2 position, object 2 to take the object 3 position and object 3 taking the object 1 position, all possibilities are there. How many possibilities are there? How many permutations you can do with n objects, n objects?

Student: n factorial (Refer Time: 01:07).

n factorial, why are you all hesitating. n factorial exchanges or permutations. So, it is a finite order group, ok. So, it is not that if n is very large it has infinite, but countably many right. So, it is a very well named group to handle. So, we will now look at the permutation groups and understand the permutation groups. I said, ok.

So, what do we want to understand the group structure and then we want to also see conjugacy classes a number of elements in each conjugacy class, just like here there are 3 conjugacy class. There is one conjugacy class with 3 elements, one conjugacy class with 2 elements.

So, we want to see whether we could get some nice handle on combinatorics of permutations in understanding how many conjugacy classes should be there, how many elements in conjugacy classes are there, ok. So, these are questions which one could ask. So, just for specific clarity let us look at permutations of 3 objects, ok, and then extending it to 4, 5 can be done as a you know you can try and play around and get a clarity it, ok.

So, there will be 3 factorial which is 6 elements, ok. So, this symmetric group why I call with 6 elements will be clear that there will be an isomorphism between this permutation group and this abstract group, ok. So, we will see that this abstract group with two generators a and b will have some isomorphism to the permutation of 3 objects. We will see that, that is why this being termed as symmetric, ok.

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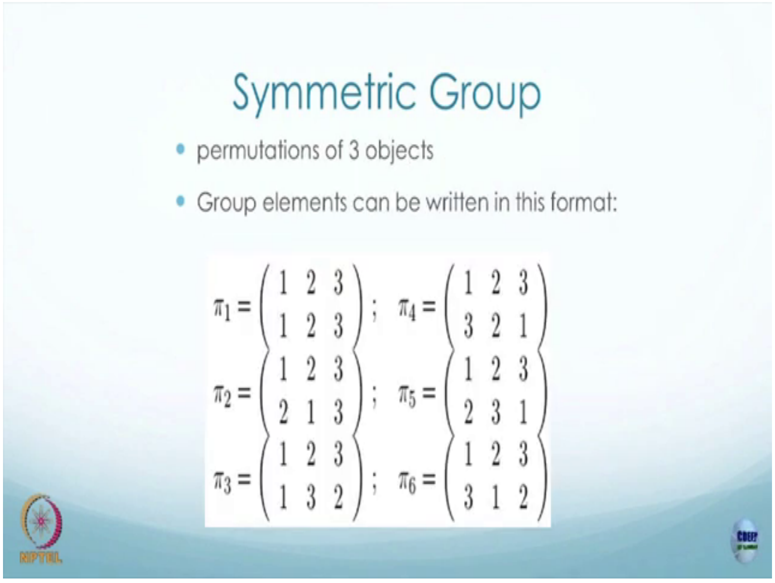

So, first of all how to write these group elements in the permutation of 3 objects, ok.

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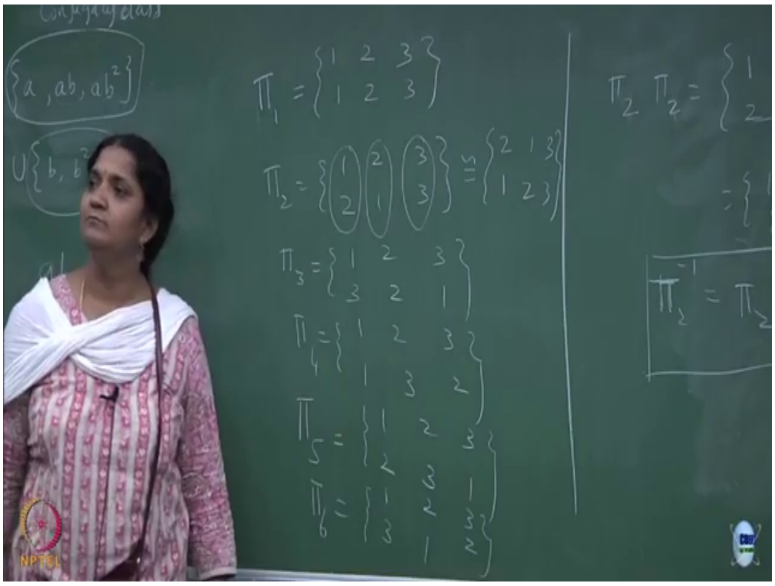

So, you start with saying that object 1, object 2, object 3 does not permute at all, ok. So, it remains as 1, 2 and 3. This is the initial configuration, this is the final configuration. This I will call it as pi 1, actually it is like a identity element in this set, ok, no change, no permutation. The second one could be 1 goes, 1 changes to the second position, the second one goes to the first position and the 3 is not touched, ok. This is one possibility and so on. We can start listing everything.

What are the one? You can have pi 3 with 1 2 3 let us say 1 goes to 3, 2 is not touched. This is one possibility. You can have pi 4, 2 can go to 3, 3 can go to 2, and 1 is not touched. The beauty of these elements are if you do pi 2 again you will get back identity right, 1 goes to 2, 2 will go to 1, so 1 will go to 1. How do we do that? Ok. So, let me also explain in this notation how to write inverse.

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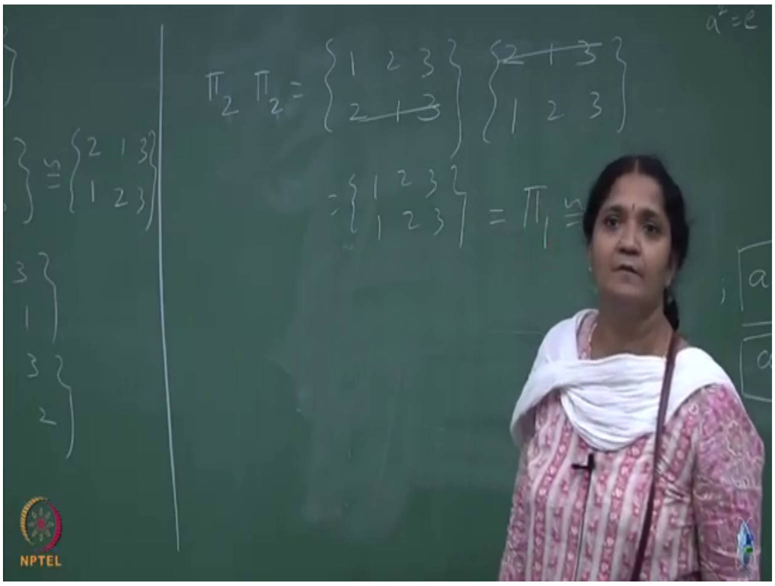

So, if you want to do pi 1 into pi 1 I will do it as 1 2 3 sorry, pi 2 and pi 1 is trivial, so there is no point in doing this pi 2 into pi 2. The one more thing which I want to notice for you is that the ordering is unimportant. I could have called this, if suppose I can interchange these columns, does not really matter. So, this is equivalent to both are same, ok. Idea is that 2 goes to 1, 1 goes to 2 is important. I can interchange the columns, I can just take the second column put it as first column, first column as second column, the element is the same. What is that operation is 1 goes to 2, 2 goes to 1, so you see that 1 goes to 2, 2 goes to 1, ok. So, no change.

So, now, we will exploit this property to do multiplications. So, I want to do pi 2 and then I want to do pi 2, ok. So, here usually when I want to do it when I multiply fractions I make sure that I can cancel denominator with a numerator. So, let us write it in this form and then figure out what is the equivalent. So, 2 1 3 will go to 1 2 3, ok. So, this is the multiplication procedure. When you do that you get, so you write the initial and the final and this is what they call it as pi 1 which is the identity, ok.

So, the multiplication in this case is little you know different, elements are written differently. I say the initial 3 objects, this is the permutate 3 objects, no permutation I call that as an identity element, one permutation between object 1 and object 2 is what I have written here. This element whether I call the first column, second column, third column in whichever order does not matter, it depends on these elements as an entity, you can swap whichever way you want, ok. Is that clear? That is what I have done here.

And this is useful then I want to do the multiplication. I will take this final state to be the initial state and then see how it changes. Finally, the initial state goes into this final state which is nothing but the identity.

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So, what does this tell us? Pi 2 inverse, pi 2 squared is identity. So, ordered two group, pi 2 inverse is pi 2. Is that clear? So, this is what you have seen. Can you also tell me how to write the inverse? Let us look at some more elements. There should be actually 3 factorial elements right, 6 elements. So, you will have a pi 5 and a pi 6. So, we can write that, ok.

So, pi 2, pi 3, so I probably have done on the slide differently, but let me correct it later. But what is this going to be? 1 2 3, 1 can go to 2, 2 can go to 3, 3 can go to this is one possibility, cyclic permutation. The other one is 1 goes to 3, 3 goes to 2, 2 goes to 1, am I right, acyclic permutation. 1 will go to 3, then 3 will go to 2, 2 will go to 1. So, these are ways of playing around all the 6 elements and I have shown this here on the slide, ok. So, I think I have followed a different notation here, but you understand what I am saying.

Pi 1 is an identity element. Pi 2 exchanges 1 and 2 that is also consistent with what I did. Pi 3 is 2 and 3 and pi 4 is 1 and 3, so the pi 3 and pi 4 has to be exchanged. Pi 5 is 1 becomes 2, 2 becomes 3, 3 becomes 1. Is that ok, is that clear. And pi 6 is the acyclic one, 1 becomes 3, 3 becomes 2, 2 becomes 1, ok. I do not think any other permutations are allowed. You know this very well, so I thought let me take the 3 objects in experiment, ok.

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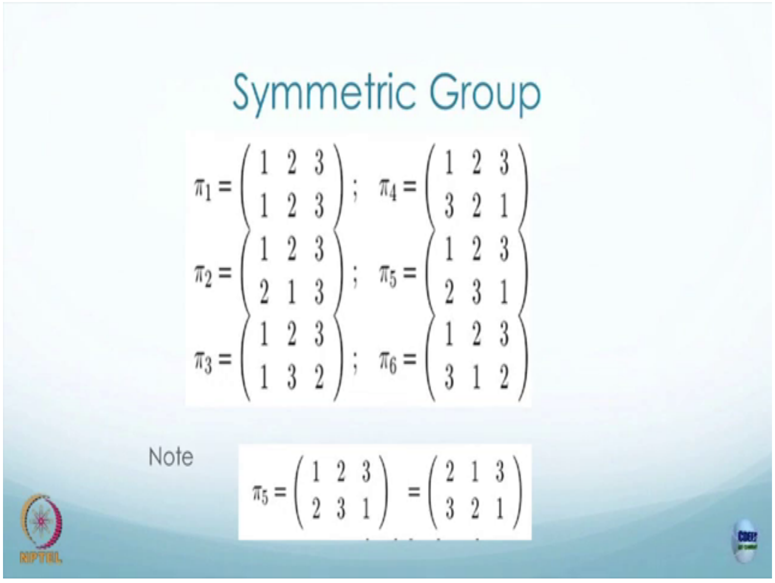

I also try to stress the fact that exchanging the columns should not change the elements, so if you exchange the columns both are actually the same element, ok. So, this I have tried to stress here that you could exchange the second column and the first column, ok, even exchange all the columns in any combination, but the element will be this, ok. So, only 2 4 5 pi 5 any element actually. I just tried to show that.

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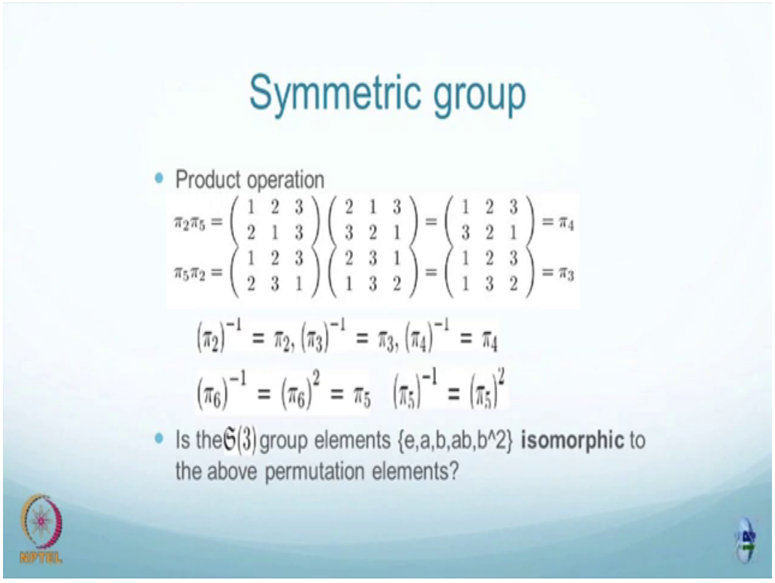

So, product operation also I have explained now, I put it here explicitly. Pi 2, you can multiply pi 2 with the pi 5. As I said before you do that make sure that the resulting state of pi 2 becomes the initial state and you shuffle the column and write the final state there, ok, and then see what that element is and you will find that; please check it out pi 2 pi 5 will be pi 4 and pi 5 pi 2 will be pi 3, ok. So, please check this out. I think my pi 4 which I wrote on the board is pi 3 there and vice versa. But you can check it out, ok.

This also I have stressed already, pi 2 is order 2 element and similarly you can show that any two particle exchange to object exchange alone, these two are involving all the 3 objects 1 goes to 2, 2 goes to 3, 3 goes to 1, right. Whereas, here if you see it is only either 2 and 3 are getting exchanged, 1 is untouched.

So, any two particle exchange, so pi 2, pi 3, pi 4 are only involving only two objects, they will always be ordered two element right. If you do it again you will get back to the same position, right. So, there ordered two elements and then you come to pi 6 and pi 5 relation. I leave it you to check pi 6 squared, can somebody do it, let us do it maybe. So, pi, pi 6 is what? This one, right.

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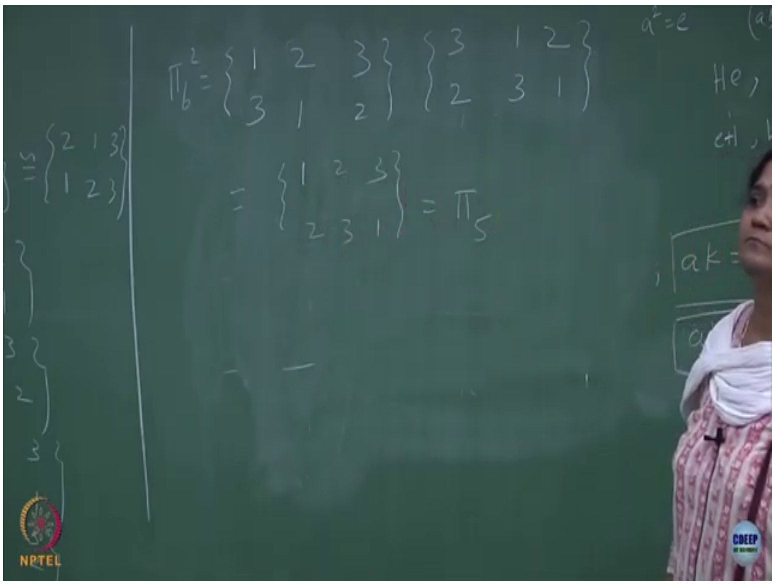

So, I want to write pi 6 squared. Is that right? Are you all with me?

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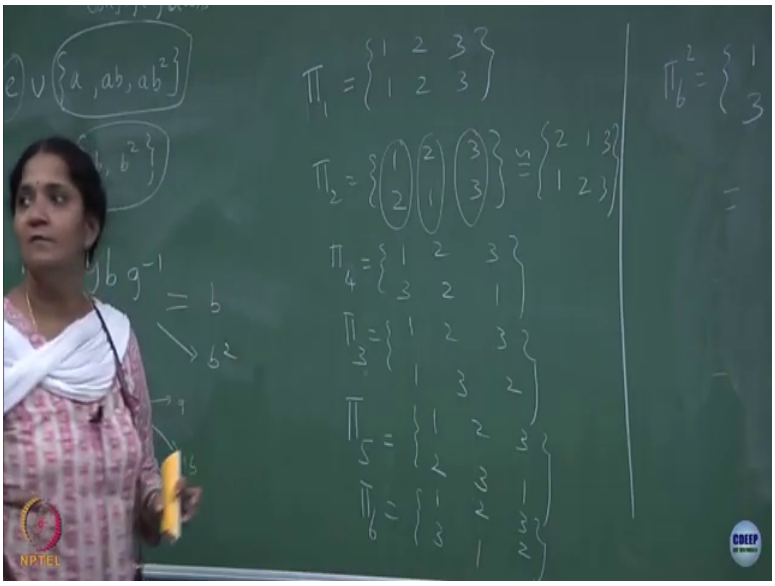

It is just that here I called pi 3 and pi 4 maybe you can change this to pi 4 and pi 3 just to be in synching with the power point. So, this is what we get, ok. Pi 6 squared turns out to be pi 5, ok. And similarly, you can show, you can check the other one also. What is pi phi square? Ok.

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So, you can check this out pi 5 squared will be pi 5 inverse, this also. How do you find an inverse? What is an inverse? Inverse is if you undo the process you have to get identity element, right. So, whenever I want to look at inverse, suppose I want to look at the inverse what I will do is I will look at here and write the inverse, so 1 goes to 3, 1 goes to 3, the reverse 3 goes to 2; 3 goes to 2 and 2 goes to 1.

So, this will be my inverse. And what is that element? This, I am have checked, but please check it, but I am writing what will be the inverse. Whenever you want to look at the inverse you do the reverse process. When I write the forward element I say that 1 goes to 2, 2 goes to 3, 3 goes to 1. When I want to find the inverse 1 will go to 3, 2 will go to 1, 3 will go to 2 because when you multiply these two you will get it to be an identity. Is that clear? Ok. So, this is one way of writing the inverse and checking this out.

So, a couple of things will be now really clear to you that you find pi 6 squared to be pi 5 and also you have pi 5 cube is identity, similarly you can also show pi 6 cube is identity. Now, we can see an isomorphism. What is the isomorphism? Isomorphism between the abstract symmetric group with generator a and b, and this one.

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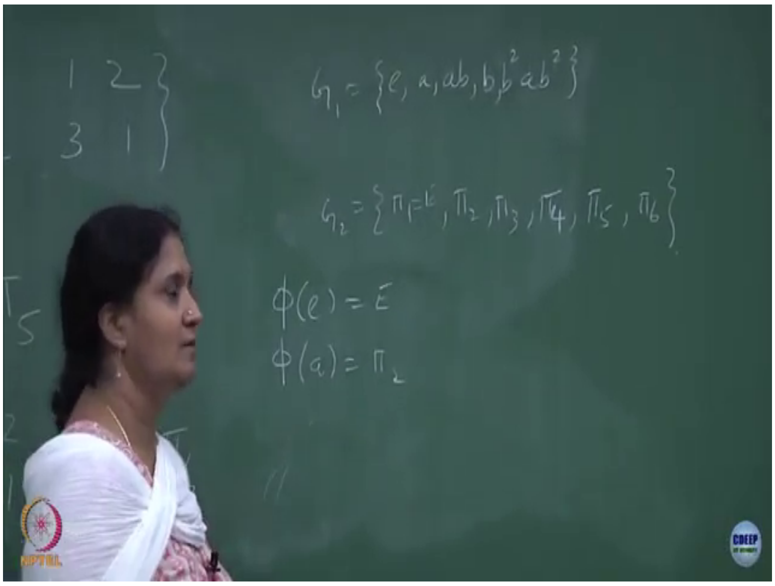

So, you have a G 1, you have a G 2, pi 1 is identity, ok. Is it an; it should be an isomorphism, order of the two groups are same, right, homomorphism will come only when the orders are different.

Student: (Refer Time: 18:27) b, b is basic?

B is basic. Yeah, b should also, thank you. Totally 6 elements, ok. So, I leave it you to find which one is a, which one is b, which one is a b, which one is a b squared, please check it out, then you have an isomorphism between two different looking definitions that is clarifying how to see isomorphism, ok. So, the phi map will be such that phi of e as E, phi of probably a as pi 2 and so on and so on. So, figure this out, ok. So, that is isomorphism's.

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Now, you can generalize, permutation groups, permutation of n objects, there will be n factorial elements and then you say that such a permutation of n objects is called symmetric group of degree n. n is for the number of objects nothing to do with the order of the group, order of the group is n factorial and subgroups will all have some properties, ok.

So, these subgroups of what they call it as a permutation group. The symmetric group is also the trivial subgroup. It is also called as a permutation group. Identity element is a trivial subgroup, total group is a sub it is also a trivial subgroup, non-trivial subgroups will also be some kind of what they call it as a permutation group.

The reason why we are stressing on permutation group is whenever in the solid state, crystallography or something you see a finite group symmetry you can always show that there is an isomorphism to a subgroup of the symmetric group, ok. So, that is why we are trying to stress on understanding the universal group which is a symmetric group.

If you understand that then we know how to do subgroups there, how to understand conjugacy class and this may be useful, suppose I give you a finite group of order 6, for example, in this case when I gave G 1 to be this then you can see that there is some kind of an isomorphism to the symmetric group of degree 3, right. So, the same thing can be done for other finite groups by using this invoking Cayley's theorem. Yeah.

Student: (Refer Time: 21:38).

Just the way the mathematicians have defined and we just want to keep it as a some symmetry transformations [FL]. So, permutations is what we call it, ok. Some books do say them to be permutation group also, but permutation groups, the subgroups of them are technically what they call it as permutation groups, but you can call this symmetric group also as a permutation group in that sense of an (Refer Time: 22:10). Yeah.

Student: (Refer Time: 22:12).

Yes, tell me what is the one? Good.

Student: (Refer Time: 22:20) single generator (Refer Time: 22:24).

It has single generator, here also, because pi 6 squared is pi 5 you can still generate with e pi 5 and pi 5 squared the cyclic group of order 3. I am giving you an example it looks like different element, but pi 5 can be generated from pi 6 or vice versa. So, pi 6 and pi 5 are not independent elements they are the elements of the set.

Student: (Refer Time: 23:00) pi 6 are not (Refer Time: 23:01).

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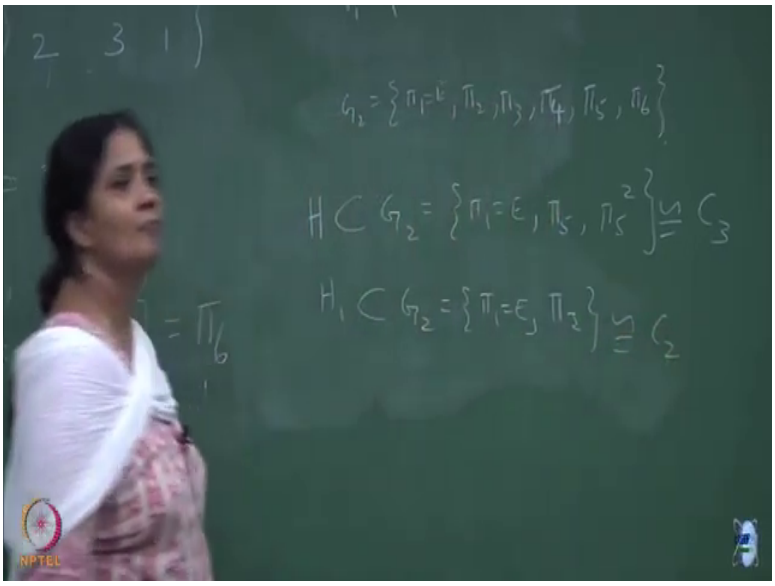

A subgroup; subgroup of the symmetric group of degree 3. So, let me write it; will be isomorphic to C 3. So, that is what Cayley's theorem says any finite group can be seen to be isomorphic to subgroups of symmetric groups. I am just giving you an example. Is that ok. You can find other examples also, ok. So, so cyclic group is something which we have been studying, so it is it is trivially you can find, ok.

So, now, I just want to straighten out one more notation which is many books follow it and its much more you know clearer, this becomes more elaborate if I have to write n objects in such a complicated fashion. Instead they use something called as a permutation cycles, ok. So, let me explain what that permutation cycles in these examples and then we will get to.

The aim is given a symmetric group of degree n it has n factorial elements, that n factorial elements I should be able to write it as a disjoint union of conjugacy classes and I should also be able to determine in how many elements are then there in each conjugacy class, ok. So, this example you can keep it at your back of your mind that it has 3 conjugacy class, one conjugacy classes identity, other conjugacy class is 3 elements, third conjugacy class had two elements. How do we understand that in a combinatorial way? Ok. So, this is what is the.

You can do it by brute force, suppose I say do it for 5 objects I am sure tomorrow 10 people can form 3 groups and come back tomorrow and tell me how many classes are there, how many. But it will be interesting that without doing anything can you come up that when you have permutation of 3 objects which is isomorphic to that group which shares the same multiplication table, right.

I want to say that by closing my eyes that there are 3 conjugacy class and identity of course, is conjugacy class in every group, so we do not need to worry. The other two non-trivial conjugacy class why one has 3, one has 2, all these kinds; why cannot I have 4 conjugacy class, why cannot it you know you can ask various questions and you should be able to give a nice way of understanding that in an algorithm way. So, that is why the cycle structure will play a very important, ok.

So, just bear with me for some more notations. But it is, I will give you a simple example so that it is clear in your mind, so that any arbitrary n you can do it, even though it becomes abstract for arbitrary n you understand always go back to this simple example. That is why I have try to keep this non-trivial example in the back of my mind, ok.