Group Theory Methods in Physics
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Lecture - 06
Conjugacy Classes

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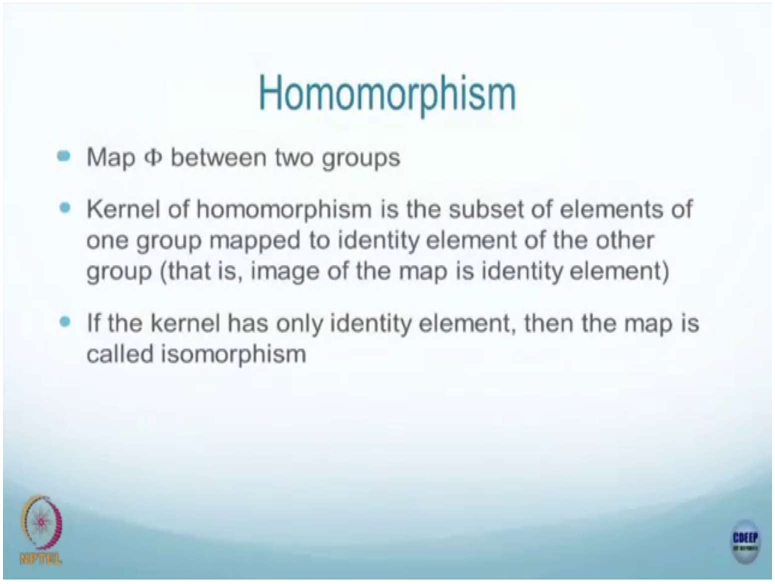

So, I just went on to homomorphism very quickly and there was some questions at the end so, I thought let me clarify homomorphism. So, the first thing is it is a map between two groups ok, we wrote this between group G 1 and G 2 we can write a map.

And, then we look for so, you have a map between G 1 to G 2 and then you want to find phi G which is going to be identity. So, identity I called it as capital E, this is an identity element belonging to G 2 right. And, this set so, set of elements in G 1 ok.

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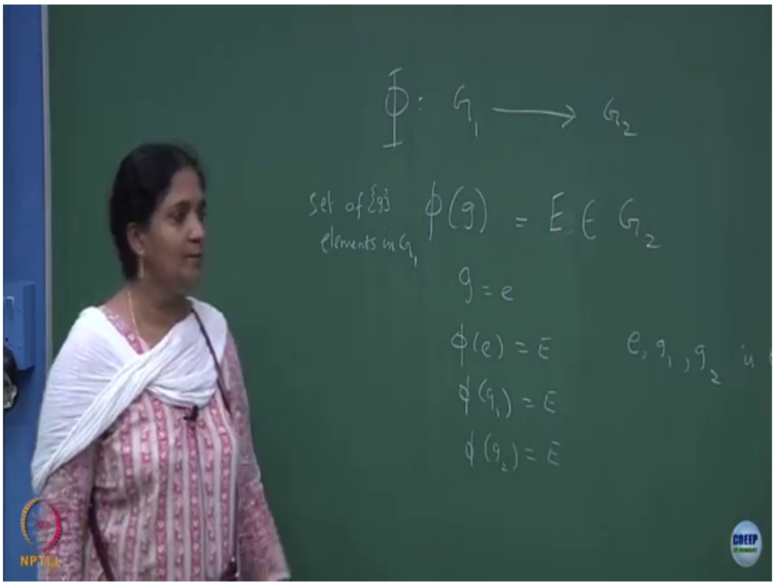

So, there is a set of elements let me call it as set g, which satisfies the condition that phi of g equal to E, then we say that this set whatever is this set, we call it include the set will also include the identity element. Because the map should be so, g 1 of the element will be g equal to be e will also satisfy phi of e equal to E, you can have phi of g 1 equal to E phi of g 2 equal to E suppose you have this, then you say that e g 1 g 2 is the kernel of the homomorphous.

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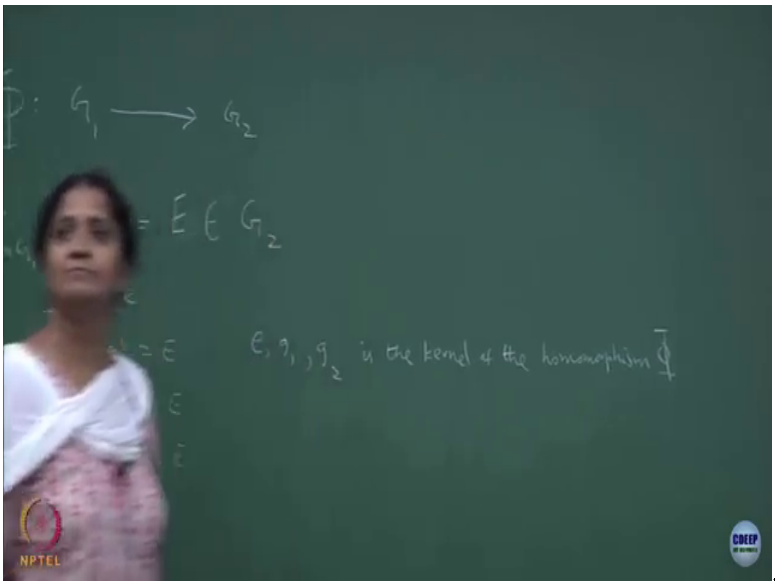

If, only this is the set which you find which is mapped to identity element, then you call it to be the kernel of the homomorphous.

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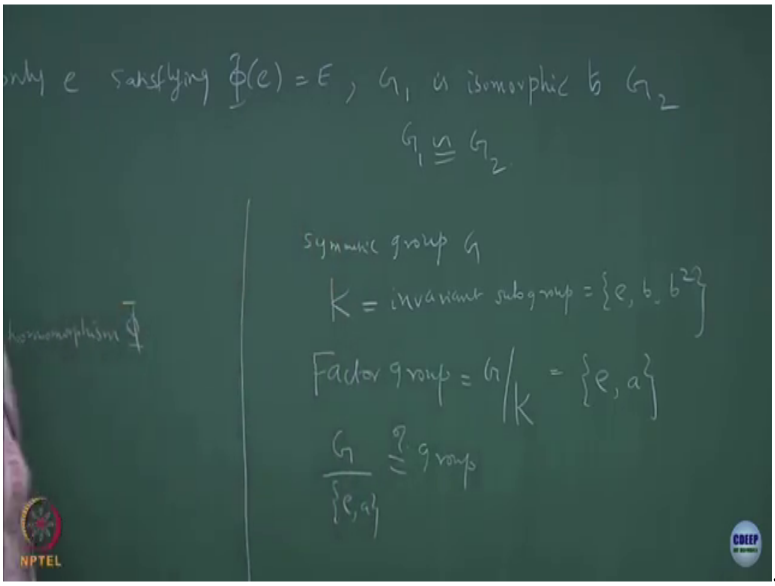

If, we find only e; only e satisfying phi of e equal to E and no other element ok; no other element. If, the identity of G 1 is mapped to identity of G 2 and no other element, then we say that G 1 is isomorphic G 2 ok. So, this we formally write it as fine.

Student: (Refer Time: 03:19) elements.

No other elements.

Student: (Refer Time: 03:22) already.

Once this is satisfied those things will also be not satisfied, because of the group properties, you can disprove it ok. So, this is a very stringent condition. Once, I put this condition all those 2 2 1 mapping will never be possible ok.

And, for this case we try to look at a simple example, which was symmetric group, where I took the K which is an invariant subgroup or a normal subgroup, that one had e, b and b squared right. This has an invariant subgroup, which is this. And, then we looked at a factor group, which is G mod K correct, and this G mod K was just e and a you are all with me.

And, then I said that the factor group should always be done with the invariant subgroup and then somebody point it out, that I could have factored it by this group and that is also fine. Why is it fine in this case can somebody tell me?

It turns out that the group G divided by a group e a the symmetric group let us take. Why this is also a group, why does it satisfy the group properties? Do you know why? This group has only one non-trivial element, there is no point in talking about left coset and right coset [nosie] right, invariant subgroup has this property that left coset is same as right coset right.

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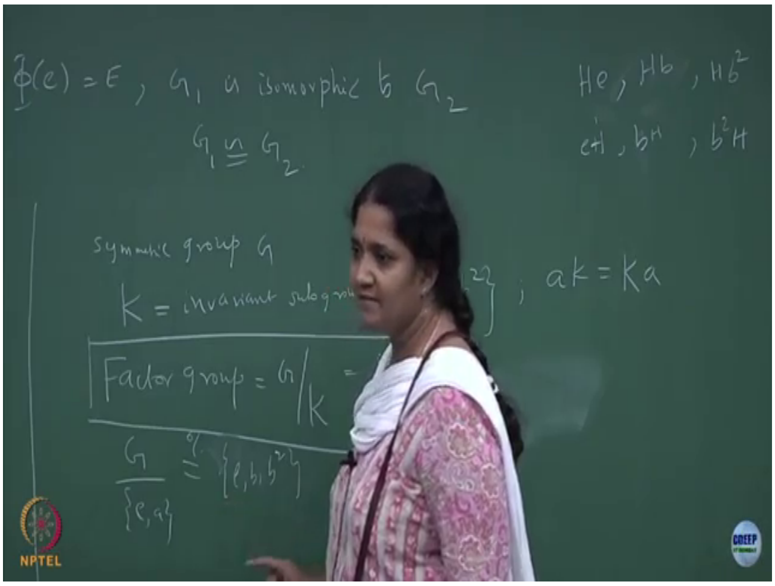

If, the group itself has only one element ok, which means that you have only two cosets to start with right, one coset with so, you have only two cosets to start with. So, there is no point in talking about left coset and right coset. So, that is why it accidently the G by e, a turned out to be your you know there will be only two cosets for this group.

And, you find that it turns out to be also a group, but in general for an abstract group G, if you find only an invariant subgroup the factor will be having the group properties ok. So, in this particular example it is satisfying us. So, that is one thing which was an accident here, because there are two element group; one is identity, other one is a non-trivial element, but this will not happen in general.

Student: Factor group will always be (Refer Time: 07:00).

Factor group will always be a group; factor group will always be a group provided the factoring on the group is done by an invariant subgroup. If, the group has only two elements subgroup has only two elements, then your cosets are only two cosets. So, that is trivially going to be the factor will also have a group properties.

Student: (Refer Time: 07:27) to the end of this group (Refer Time: 07:30) if a is also (Refer Time: 07:33).

It is also a factor group, but this factor group also has left coset equal to right coset trivially.

Student: (Refer Time: 07:41).

You see any group which will allow left coset being right coset right will always the factor will always be a group. In this case it is non-trivial in the two element groups order two groups if you have it becomes a very trivial statement. That is why it happened that that one is also a fact someone who was pointing it out at the end of the lectures so ok. So, this is a non-trivial statement for an arbitrary group, but this particular case it also turns out that it has the group properties, because it is just a two element [nosie]. So, whenever you have a two element subgroup, then the number of cosets will be only two right.

Right for a subgroup you can list out the cosets suppose the (Refer Time: 08:33) subgroup is e a the cosets are going to be H e and H a that is it. And, if you do the left or the right H e is same as e H right. So, which means H a should be same as a H also. So, this is why this property was satisfied left coset equal to right coset for the two order two group that is why this factoring factor group is also group ok. So, this is one thing which I want you to remember. So, I have put in here is a subset of elements of one group mapped to identity element of the other, this is what we call it as the image of the map is identity element.

Student: (Refer Time: 09:22).

Yeah I agree with you sorry I what I wrote here is yeah. So, what you are saying is that H itself is e a that is correct. So, there will be three. So, then argument is not right thank you, this is what you are saying and H b is it same as b H is the question.

Student: We can check it all the elements.

If, the left coset and right coset are same, then we can use this properties what I am thinking, but may be you will have a proof for a order to grow thank you, this is not what I wrote is not right H itself is e a. So, you have to multiply on the right or on the left. So, this is not clear now.

So, this is what you will get as left coset and right coset, most probably left coset and right coset are not same, but still we find this to be a group right. This one is nothing, but [nosie] e b b squared which is a group. Why is it happening is the question? Can you give a proof maybe you can think about it, I think it is the order two group, but it may be something more ok. Thank you fine ok.

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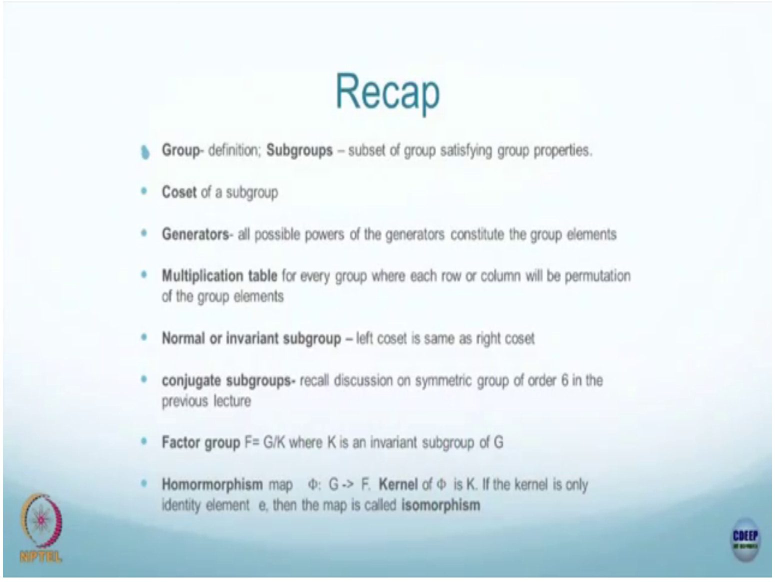

So, just to recap, so I have already done groups, subgroups, then coset of a subgroup, coset is always associated with the subgroup and generators or all possible powers of the generators ok, will constitute the group elements, we discussed this in the last lecture. Also told that for every group you can write a multiplication table along row and column, it will be some permutations of the group elements, but all the elements will be there along a row or along a column.

Normal or invariant subgroup in this case left coset is same as right coset, conjugate subgroups remember that we said that for the symmetric group you could write the conjugate subgroups, which were the conjugate subgroups? Conjugate subgroups are take e a as a subgroup and then do a conjugation right.

And, then you will end up getting H 1 will be some element b H b inverse right, H 2 will be a b H a b inverse I do not know I am just trying to write it like this, elements which does not belong to that set. So, if you do this then you know that, H 1 is conjugate to H 2 and H 1 is conjugate to H ok.

So, both have conjugate to H according to this equations and from these two you can also say that H 1 is conjugative ok. So, this statements you can say. So, I said that you start with the subgroup and generate other groups; if it is an invariant subgroup, can you find a conjugate to an invariant subgroup?

Invariant subgroup definition is that arbitrary element of a group a, multiplying that invariant subgroup is same as invariant subgroup times a. So, from here you can see that the ok. So, invariant subgroup is always self-conjugate ok. So, these are couple of points with the particular example in mind so, that you know you do not get confused.

So this also elaborated now factor group K is an invariant subgroup. So, this is a most general statement, but arbitrary groups whether it will the factor will be a group or not is something which one to try to prove even in the simple case ok. Homomorphism is a map that also I have said kernel of phi will be the invariant subgroup K. If the kernel is only identity element, then the map is an isomorphism is that ok.

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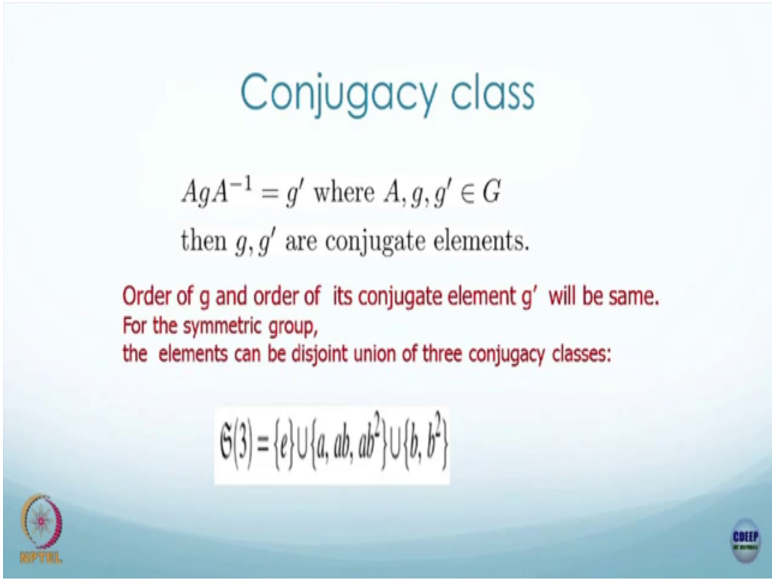

So, so far for the group structures subgroup structures, now let us get on to concrete, you know I want to do permutations of objects and for that as I have already said anything related by similarity transformation has no new physics. So, you write g and g prime by a similarity transmission, where a also belongs to the group g and g prime also belongs to the group, then this such a relation will tell you the element g is conjugate to g prime ok. Till now I was talking about groups generating conjugate subgroups, but now I am going to confine to conjugate of the group elements ok. And, then g and g prime are conjugate elements ok.

Suppose the element g had certain order like say that if the element g is like A, order of a was; a was 2 right. So, if a had 2 then the conjugate element which you find g prime, in case you find that then that order should also b 2. So, in this particular case H 1 had a set of elements to again be 2 elements right let me not write that what that element is.

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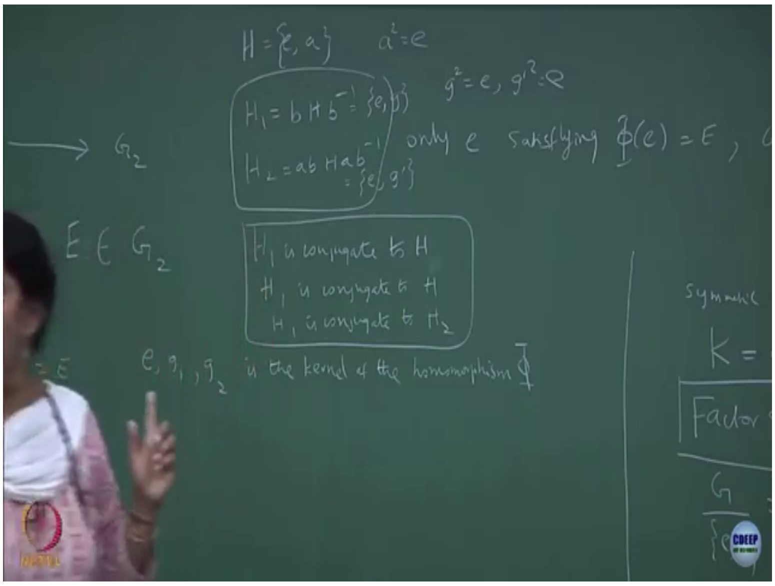

This one is e n another element ok, a had ordered a squared equal to identity ok. And, how can you find g? g will be conjugate to a g prime will also be conjugate to a. So, both g and g prime should also satisfy g squared equal to e g prime squared equal to e ok. So, the order of g also will dictate this also you can prove it use this conjugation and prove it, but as of now let me just give it as a statement, if g has certain order then it is conjugate elements will also have the same orders, that is what happened.

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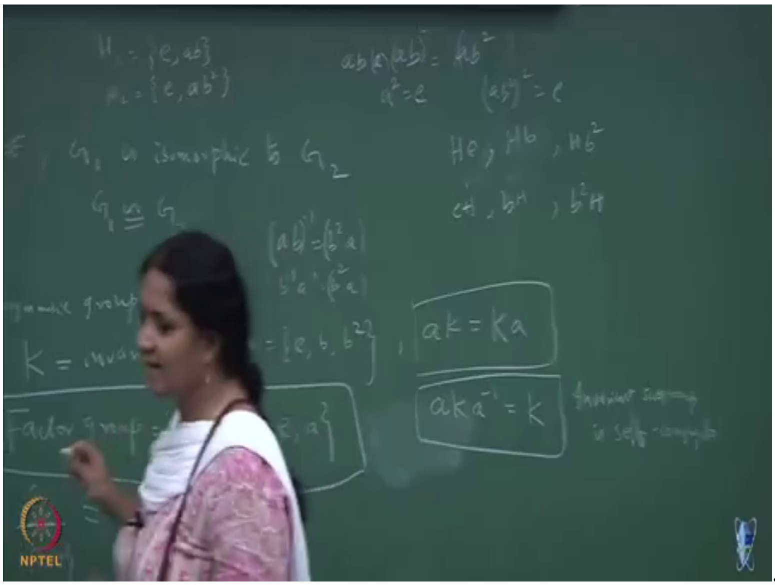

I do not know whether you recall we had H 1, one was e and a b other one was someone a squared b or a b squared, we can check a b a b and a b squared will be conjugate to a ok. Why is that we can show that a b a a b if you do it, you can get what b a was a b squared a squared cancel b squared a will become again a b. It will become b squared. I think I do something wrong.

Student: a b square.

A b square. So, this is what you will get. So, a is conjugate to a b squared, if a squared is identity you can do it twice here you can show that a b squared the whole squared is also identity, is that clear. Order of the group element, which is conjugate to another group element should share the same order, that is why H 1, H 2, H 3 each one had order 2 element. So, these two will have order two element only ok.

Student: Ma'am.

Yeah.

Student: In that a b a into a b inverse.

A b inverse did I make a mistake yeah should be inverse. Inverse I was just thinking it is an order two groups so, it is the same, but technically I have to put it yes.

Student: a b inverse (Refer Time: 19:27).

a b inverse is. So, it is same right.

Student: Yes.

A b already I have a definition that a b is; a b is b squared a. So, I am just saying that the inverse is also b squared a is that ok. Anyway this is a nice simple example to play around and to clarify all your notations. So, that is why I have given one non trivial, non abelian group ok.

For the symmetric group, now I want you to we did various things for symmetric group we wrote it as a disjoint union of cosets right. We also looked at what are the subgroups; we looked at what whether it has a invariant subgroup and so on. So, we did all this things for the symmetric group. I want you to find out the grouping of conjugate elements, can you do that now we have done it already. So, it is not very difficult. Can you tell me, what are the conjugate elements?

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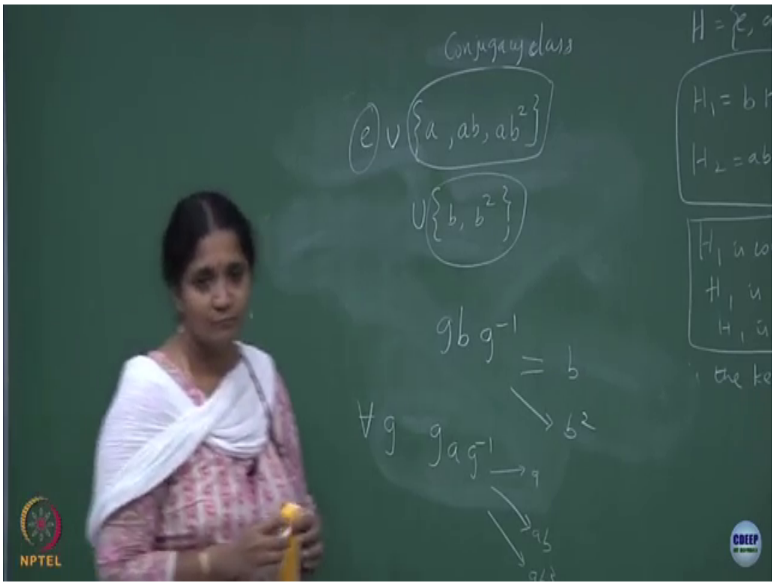

We have already shown there that, e; e has an any conjugate element? e is self-conjugate ok. Union one conjugacy class, what are the conjugate elements? Ok. Union what is left out? Right. So, you will see that this bracketed one is the set of elements conjugate to each other, by doing some similarity transformation you can show that b a b inverse will give you one of the elements in this set only ok. So, this is what we call it as a conjugacy class? Ok.

Similarly, this is another conjugacy class this is always alone element and which we call it as one conjugacy class ok. So, essentially your symmetric group can be having this conjugacy class decomposition with 3 conjugacy class ok; you will have 3 conjugacy class. So, this we have already checked a couple of elements and you will not be able to find any overlap between conjugacy class; conjugacy classes are distinct ok. So, you cannot make b and a to become conjugate and so on.

So, b and b squared are conjugate to each other they will be related by some similarity transmission ok. Any g b g inverse will only give you either b or it can give you, this way. Gives b is a trivial statement for all g you have to span it. If you do that you will find b and b squared are the only elements in that conjugacy class.

Similarly, take one element here g inverse do this for all g, it can either give you a, it may an either give you a b, or it can give you a b square I have given some assignment problems, which you have to submit and you have to see the conjugacy classes ok, this is clear. So, this is what I have put it in here, that the symmetric group which I was discussing, which is an order 6 group can be treated like a disjoint union of conjugacy classes.

The reason why I am stressing on conjugacy classes that we can work with one candidate, within the conjugacy class. So, you do not need to work with all the 6 elements instead we can take one candidate from every conjugacy class and the number of conjugacy class is very important. Different groups will have. If it is an abelian group how many conjugacy classes will be there? If it is an abelian group, what happens to the statement?

Student: All elements are self-conjugate.

All elements are self-conjugate. So, the number of conjugacy classes also the order of the group. So, the number of conjugacy class will only change, if you are looking at non abelian groups.

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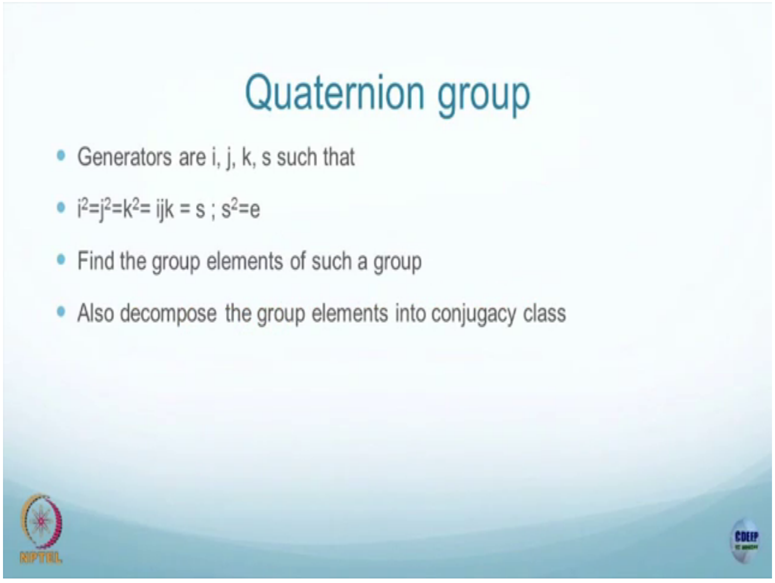

And, couple of things which is interesting about this group is that no complex numbers at times looks very unphysical, when you put an I factor and then you are used to I squared equal to minus 1 right.

So, here there is something called as a quaternion group, where you have i j k are like your imaginary numbers, s is negative 1 roughly as a mapping you can give, which satisfies properties like squares i squared equal to j squared equal to k square and equal to minus 1 and then a squared is identity ok.

So, these are some properties and then I have asked you to find the group elements of such a group those are generators and decompose the group elements into conjugacy class ok. So, take it as an exercise and try to do whether you can find the conjugacy classes here ok.

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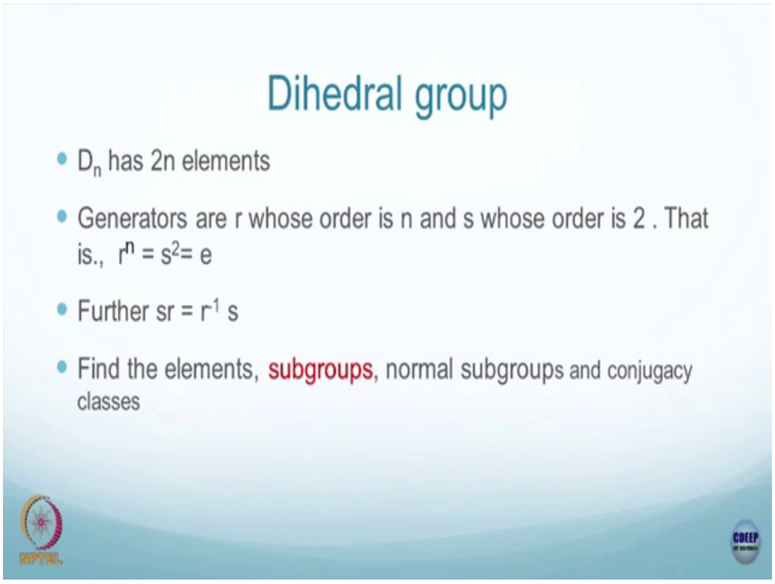

The other thing which we will find you know it will appear when I do the point groups of molecules, yes you can have additional symmetry ok. So, this the terminology of dihedral group will come back later D is just like I wrote cyclic groups see subscript n this subscript n has 2 n elements ok. It has 2 generators; one generator are order of that generator is order of the generator is r has what order n, order of that generator is n.

The second generator you have order of that generator s is 2 ok. Now, the question is given these 2 constraint ok. I also give you 1 more constraint s r; s r inverse s ok. Using these conditions just like we played around with a squared equal to b q equal to identity and then a b equal to b square a we made a multiplication table.

At least try and do it for some specific n n equal to 4 to get some clarity, an equal to 4 will have 8 elements and then you try and find out subgroups, normal subgroups, conjugacy classes, first non-trivial example where you can start playing around to get more clarity or what I was saying ok.

So, there are some crucial points here n being odd and n being even could play some different roles want you to figure out doing for D 3 and D 4 in general can you say how many conjugacy classes are there for a D n. Is that fine?