Group Theory Methods in Physics
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Lecture – 58
Dynamical symmetry in hydrogen atom: SO(4) algebra

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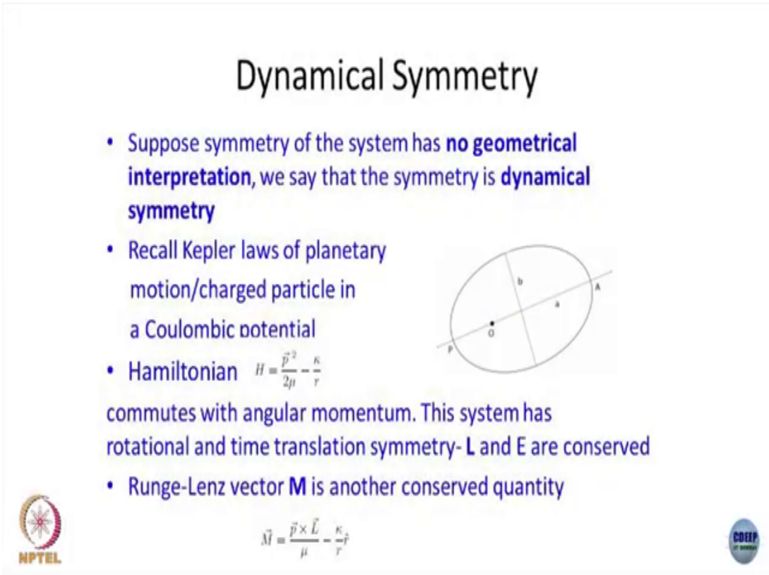

Now, I am getting on to one of the climax of group theory today; the Hydrogen Atom or a Kepler problem both are one over are potential, right. The planetary motion you are all familiar and also the hydrogen atom problem, only thing is the kappa coefficient is the Hamiltonian will be different, will be g m m. If you are looking at the center as the sun, if it is circular orbit; otherwise you have to put it at one of the what is that point called, the point O is called.

Student: (Refer Time: 01:02).

Focus of an elliptic orbit, no; and what is P and A called? Puerile and aphelion of an elliptical orbit. So, you know all these things; semi major, semi minor axis this is the way the planetary motion is, right. So, for such a situation, whether it is planetary motion or a charged particle in a polemic potential; Hamiltonian I can compactly write it as this is like a reduced mass of a system mu and Kappa will be different depending on whether it is columbic potential or it is a Kepler problem, ok.

What you all know already from classical physics is that, angular momentum is conserved for the system, right. Angular momentum is conserved, because it is a spherically symmetric potential v of r. So, for hydrogen atom you can show that the Hamiltonian for the hydrogen atom, it commutes with all the components of the orbital angular moment, right.

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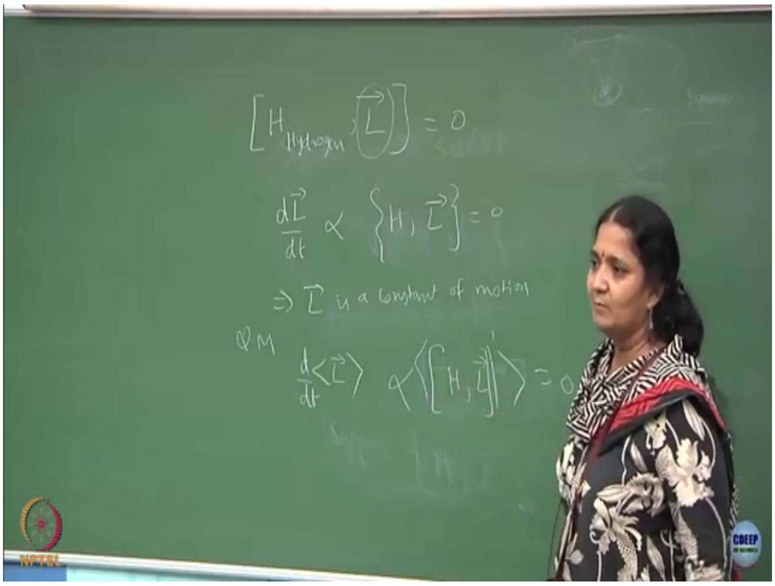

So, equivalently we say that the angular momentum is conserved in classical mechanics by showing that d L by d t which is proportional to the Hamiltonian methyl which is 0; implies L is a constant of motion, is that right. You were all not commutator, but a Poisson bracket in classical mechanics. But if you go to Quantum mechanics what is the difference?

You have d by d t of expectation value of L; sure you have done it in Quantum mechanics, what is this going to be proportional to commutator of L, right expectation. And this is going to be 0, that is why you say that the rate of change of angular momentum is 0 means; this is a conserved quantity. So, L is a expectation value of L is going to be constant.

So, associated with every symmetry, if you have a symmetry in the system the generators of that symmetry in the context of continuous groups and group elements automatically is taken care of, right. Once you put the generator, exponentiating the group generators are going to commute with the Hamiltonian, there is nothing new. But in the context of discrete groups, you put commutator with every group element; because then there is no concept of. You can have generators, but then they are not like exponentiating the generators to decide all the elements which is infinite; they are only finite elements, a finite number of generators and each one has some order, right.

If you had a cyclic group and if I give the order of the group is 11; you know that a to the 11 is identity, that kind of a constraint is not here. Here you can see the difference between what you have learnt in the first half of the semester and the second half of the semester, that here we concentrate on the generators and lie algebra. Even though we concentrate, the group elements should also continue these properties because you do exponentiation of these generators, ok. So, that is why I am going to concentrate only on the generators, which are orbital angular momentum which is familiar to you.

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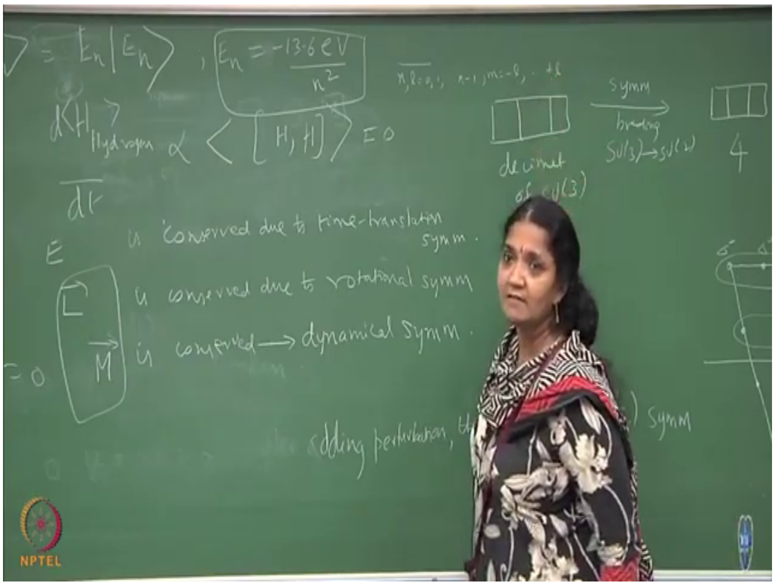

Other thing is the hydrogen atom prop hydrogen atom Hamiltonian which you are writing is does not have any time dependence right, it is independent of time. So, we always require that this one is going to be 0 always; there is no time dependence in the Hamiltonian. So, that is it, this being 0; because this should be proportional to Hamiltonian, right. What is the generator of the time translation? That is also Hamiltonian.

So, it is trivial statement. So, you have the fact that the Hamiltonian is also conserved quantity or the energy is conserved in these systems, ok; the total energy of the system for example, the hydrogen atom problem. And this being conserved is due to; so energy is conserved due to what symmetry? Time translations, ok; this is conserved due to time translation symmetry; just like L is conserved due to rotations, ok.

So, these are the conserved quantities which you have in your problem and you can associate some kind of a meaning, a geometric meaning in time coordinate there is a translation, some geometrical operation. Similarly you have a rotation in your physical x y z space, which also has a meaning, ok. So, this is why the hydrogen atom has these two symmetries; time translation symmetry as well as the rotational symmetry. On top of it there is some kind of a new conserved quantity, you see in this Kepler problem or the hydrogen atom problem; and that is what I am showing it on the screen here.

So, what you see here is that there is a Runge-Lenz vector, Lenz vector is what we call, is another conserved quantity, ok. So, it is an another conserved quantity, it does not have this kind of a geometrical way of looking at it. The way we look at here as time translation or rotation, we really do not have a geometrical way of looking at what is responsible, what symmetry it is responsible for giving this conserved quantity.

But Runge Lenz vector is conserved; d m by d t in classical mechanics is 0, you know you all know that. If you do not know please go back and look at Goldstein to check whatever I am saying, ok. The Runge Lenz vector is a constant of motion in classical physics, it is a conserved quantity even in the hydrogen atom problem; but we do not know what is the geometrical interpretation, ok

So, in that sense the symmetry is sometimes said that, you do not only have geometrical symmetry you can have something more, other than geometrical symmetry. So, we say hydrogen atom has a dynamical symmetry, ok. In fact, this vector will be the vector from in the directed along O to P; P is the perineal ok. And what happens is that, this vector acid undergoes in the planetary motion, the revolution that is also moved.

So, it is curious to see that that vector is a conserved quantity or a constant of motion, ok. So, this is all is the information we have for the hydrogen atom or planetary motion; we can associate generators for these L vector is having three components of generators. And Runge Lenz vector is conserved and we say that it is dynamical symmetry, ok.

Now, we are going to use that, there are three generators here; we want to construct the algebra of these generators, which is the symmetry for the hydrogen atom problem. This is a rotational symmetry, but this does not have any symmetry; but you can still say the algebra of L with L, M with M all the components, algebra mixing L and M that will give me the algebra of generators which corresponds to conserve quantities for the hydrogen atom problem, you all agree is that right.

Approach is very straightforward, this. So, once I have this, the next thing is I am going to exploit this symmetry algebra, Lie algebra generated using L and N; and I want to see whether I could get the energy spectrum. So, I want to find the energy spectrum for the hydrogen atom problem; if you know that the Hamiltonian on an energy state will give you E n; where E n is proportional to.

So, the exact value also you all know, n squared and you went through the derivation using Schrodinger equation in spherical polar coordinate, separating things and finding them to be special functions; and you worked out what are the energy eigen values given by those differential equations satisfied by those special functions.

What are the special functions? Lag of polynomial, right. So, you have these under spherical harmonics y L M, they are the solution and then you found those energy eigen value. It was not like; you cannot just say this is the answer, but you had to do some work.

Now, what we want to do is that, we want to achieve this using the algebra satisfied by L and M, this is all we are going to do, ok. There is something more here; energy levels are not, they are degenerate, right. Each energy level has an n, has an l which is running from 0 up to n minus 1 correct; or for each l you have m which is going from minus l to plus l, right. What is the degeneracy? If you include the spin it is 2 n squared; if since I am doing non-relativistic Quantum mechanics, I cannot detect the spin, I will only multiply by a two factor.

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Each level has a degeneracy and that degeneracy in the absence of spin is n square. All these information you derive by solving your spherical harmonic spreading it up and show that you have an n squared degeneracy for every level, right; s shell is non-degenerate, p shell is degenerate and so on, you know that right.

So, now what I want to do is, I want to see whether I can get this degeneracy and get this energy without going through Schrodinger equation, to show the power of my Lie algebra. Only thing I am going to see is; first given a system I find the conserved quantities and then I look at the algebra of those concerned quantity, is a motivation clear, ok. So, let me get to the slide. So, just for completeness a Runge Lenz vector involves p cross L; is it an axial vector or a polar vector?

Student: Polar vector.

L is an axial vector, polar vector; because the second term is a polar vector, so you can not add to it your section, ok.

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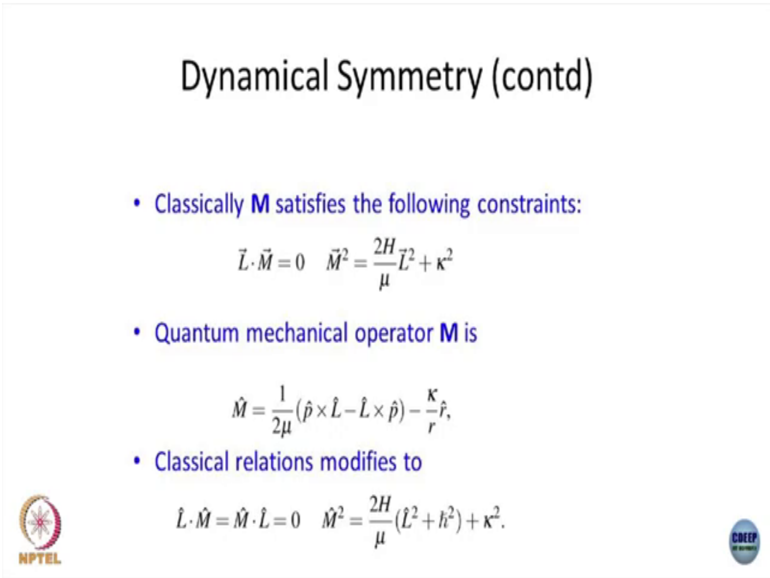

So, this has some properties; classically what we see is, because it is p cross L you can show L dot M is 0. Similarly you can show that L dot r hat is also 0, right; because L will be like r cross p, is that right. So, the two terms of the Runge Lenz vector, dot product is going to be 0; and the square of this Runge Lenz vector interestingly can be rewritten, just try and work this out.

You will see that it is proportional to the Hamiltonian and this term, ok. So, this I will leave it to you to check; I will put the slide today, but please check it, ok. This is classical; but if I go to Quantum mechanical, I need to make sure that the operators are Hermitian.

Whenever I write a dot product x dot p, I have to make sure that I add x dot p plus p dot x divided by 2; because when you do Hermitian, it has to be Hermitian conjugate it has to be self-conjugated, right. When you have a cross product, you have to do it with a relative sign negative sign. This is the Hermitian operator, Quantum mechanical operator for the Runge Lenz vector, ok.

Again you can show that, it is order matters in Quantum mechanics; so L dot M equal to M dot L equal to 0. And M squared when you try to write in the process of doing it you get an additional h cross squared here, ok. This also I will leave it you to check, this is the modification between classical and when you go to Quantum; because I want to solve the hydrogen atom spectrum, I am going into the Quantum mechanical operators, clear.

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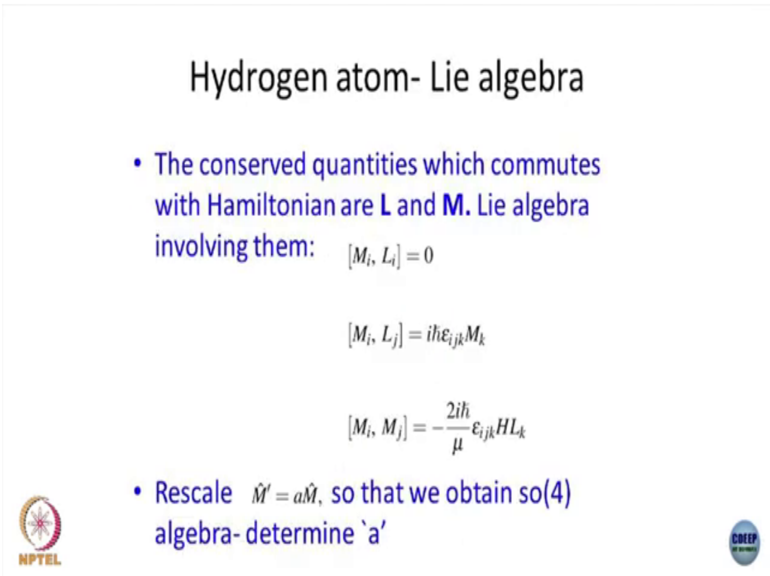

So, the conserved quantities will commute with Hamiltonian; the conserved quantities are angular momentum, orbital angular momentum and the Runge Lenz vector components. Lie algebra involving them; because they L dot M is 0, it is trivial for you to check that M i with L i is 0, ok. But you can show that M i with L j, there is a non-trivial transformation, ok.

The non trivial transformation involves the other components of the Runge Lenz vector. This is not enough, one more you have to see is amongst the Runge Lenz vector components what is the algebra. So, please use the Quantum mechanical form of the Runge Lenz operator to check that the algebra of the conserved quantities which are generators of the algebra, they are closed, ok. Is there a sub algebra L i L j I am not shown; L i L j is a sub algebra, on top of it you will have these algebras with an Runge Lenz vector. Make a scaling by an appropriate constant such that, this algebra will resemble like an S O 4 algebra, ok.

So, we did this. So, basically this constant which I have here, it is not exactly a constant; but it is dependent on the Hamiltonian, you apply on energy eigen states on both sides, it will pick up an energy of a particular state, ok. So, I want to make this scaling such that, I can make my algebra look like a S O 4 algebra.

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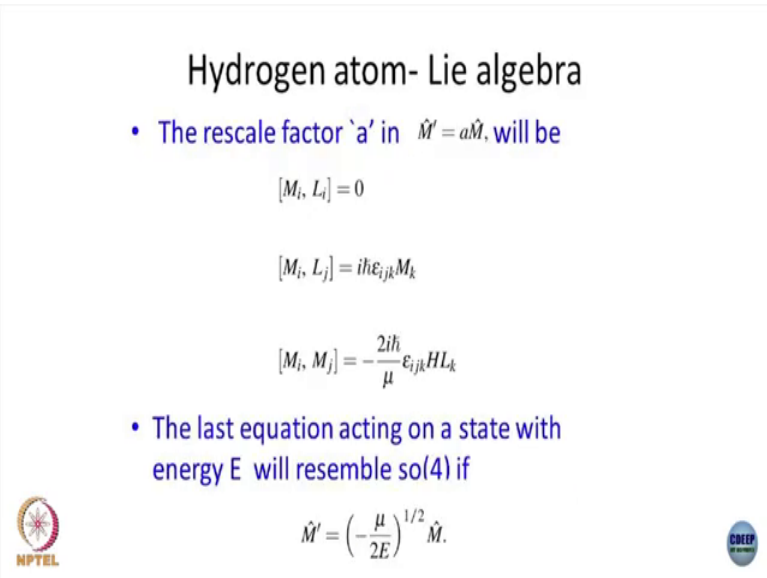

What is that scale? This also I think you did, but for completeness let me say. So, the last equation on the state with energy E will resemble. So, you operate it on the state with energy E and then you can scale your M prime, the scale factor will be mu by 2 E to the power of half.

So, you can see that here; because you have this factor it should become the inverse, so that this cancels and I want this factor to get canceled. We do this many times, whenever you have x p commutator this i h cross is familiar to you; when you have an A v operator with something, if you want it to resemble like x p, you rescale it, ok. So, that is what we are doing here, ok.

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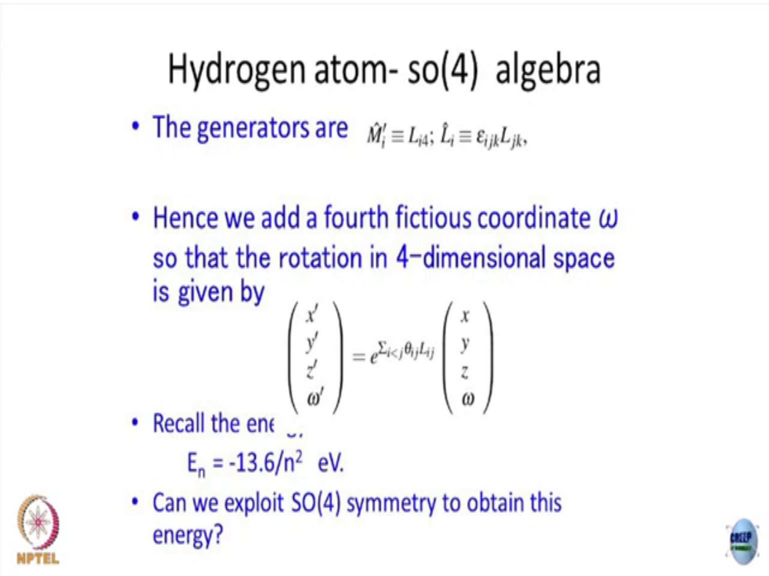

Now it should ring the bell after we did this; that the hydrogen atom M i prime if you identify it as L i 4 and if you identify the L hat i with the remaining spatial components i j k are all 1 2 and 3. So, there all the six generators I have compactly writen using the L i 4 and L j k. What does this tell us? I told you that S O 4; the 4 refers to this four spatial dimension, an abstract 4 dimensional space where we do rotations which gives you generators which are going to belong to a S O 4 algebra.

Now, I kind of put in a fictitious fourth coordinate to account for the Runge Lenz vector. So, Runge Lenz vector mimics as if the hydrogen atom problem can be treated like a 4 dimensional space; it is not exactly 4 dimensional space, hydrogen atom is only in a 3 dimensional space, ok. The potential is 1 over r; where r is x squared plus y squared plus z squared, ok. But you are introducing this fictitious coordinate just to mimic this algebra of S O 4, as I have shown in this slide, ok.

So, that is why it is called fictitious coordinate omega, so that the rotation in a 4 dimensional space, any arbitrary rotation is exponentiating the generators. I use L i j; here i and j will run from 1 to 4 ok, there are six generators, anti-symmetric, clear. So, now, we want to exploit the symmetry where the fourth component gives me the Runge Lenz vector component up to scaling; I want to exploit this S O 4 symmetry to see whether we can obtain the energy eigen values of this type and degeneracy of this type, I want to achieve that.

Now, I put in the setting; I have showed you how the algebra of Runge Lenz vector and angular momentum with Runge Lenz vector with an appropriate scaling is exactly like an S O 4 algebra, that I have already convinced. Now I am going to use that S O 4 algebra and see whether I can extract information about the energy spectrum. The reason why you can extract is because, the rescaled Runge Lenz vector has a 1 over 2 E; the E is showing up there that is why you are going to use that fact.

I can find the energy from information on M prime and L vector L prime, where the L angular moment, orbital angle; because there is a 1 over 2 E in the scaling factor. So, just remember that, it is not that it has got in that scaling to get you to the S O 4 algebra where the scaling has an information about the energy eigen value of a state, is that right, it is clear?