Group Theory Methods in Physics
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Lecture – 51
Tensor product, Wigner – Eckart theorem

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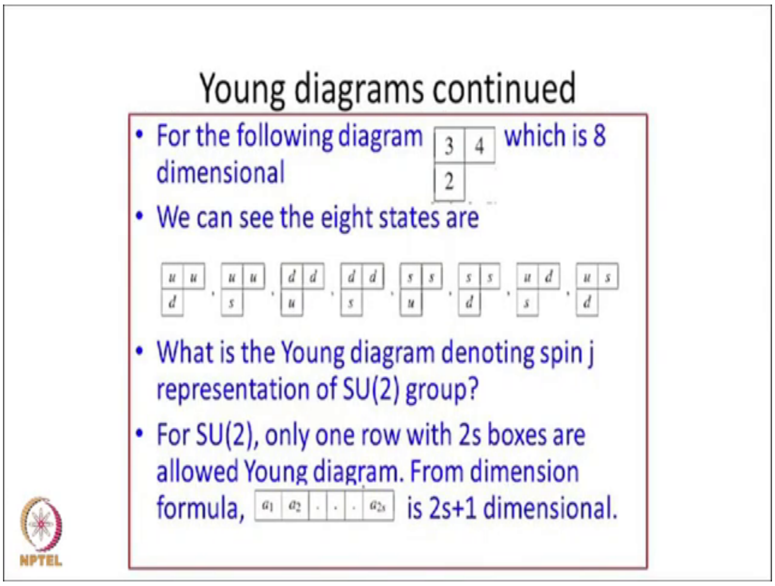

Coming to the slides for SU 2 only one row with 2 s boxes are allowed Young diagram. From dimension formula you can show that this one is 2 s plus 1, incidentally the 2 s could be an integer even integer or odd integer s is not giving you the j or s could be half odd integers also. So, 2 s could be not necessarily even integer it is not even number of boxes. It is even number of boxes for s being integer, but if s is half integer then 2 s can give you half odd integer 2 s can give you an odd number ok.

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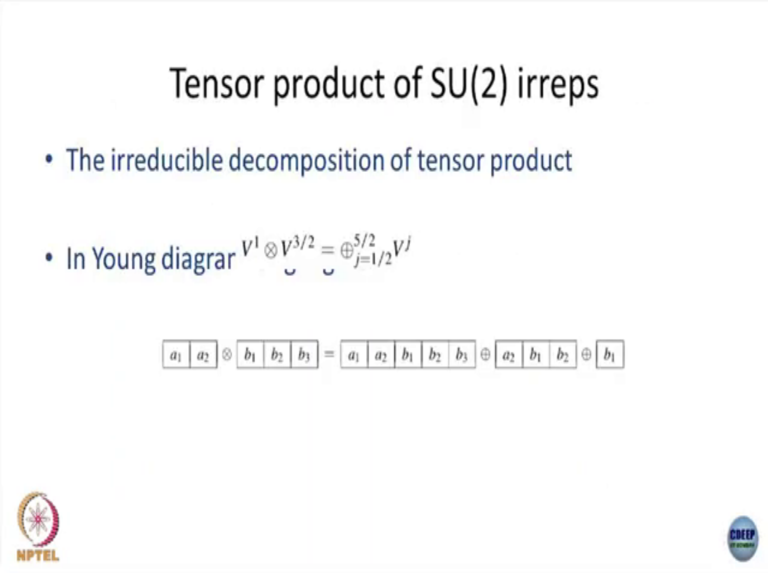

Come back to your familiar notation, if you take a tensor product of spin 1 with spin 3 by 2, you all know this even in the middle of the sleep if I wake you up, you know how to compose spin 1 with spin 3 by 2. You will start saying j 1 minus j 2 mod and go on right that is what I have written it on the right hand side j runs from j 1 minus j 2 mod which is half in steps of 1 goes up to 5 by 2 v j. So now, I am putting it in this notation, so that you can map to your tensor product direct sum right hand side is irrep left hand side is a reducible representation.

What are the dimensions? Spin 1 is dimension is 3 spin 3 by 2 dimension is 4, 3 into 4 is 12 and it has to split up into so many irreps, so that the dimensions adds up to 12. Is it happening for a spin 1 you will put two box diagram, for spin 3 by 2 you will put 3 box diagram, because number of box plus 1 is the dimension 2 plus 1 is the dimension 3 plus 1 is the dimension, now you combine and write the diagrams.

When you start combining it a something more you have to remember, even here I forgot to mention that if you are looking at SU 3 something probably I said in the earlier lecture, but let me assert it now. We did this know the I wrote this ok.

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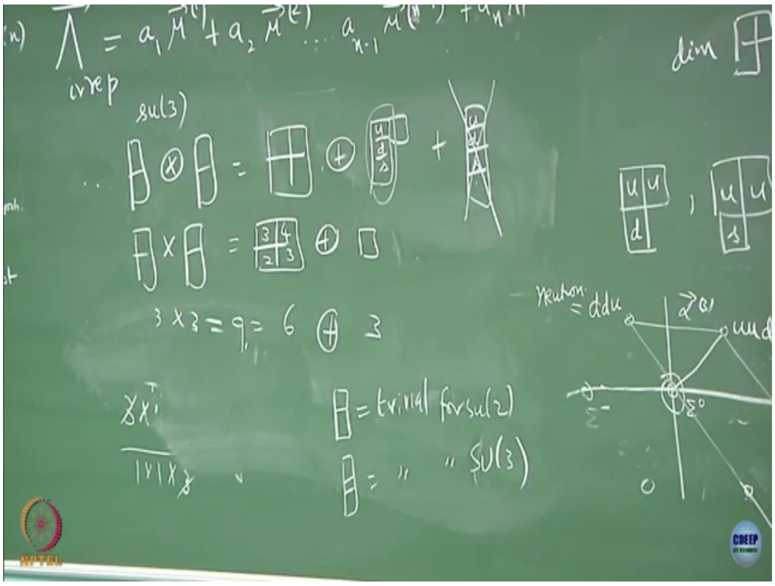

This is in general true as a diagram, if I say that I want to look at SU 3 there are only three basic objects with which you can play around. If you put u here b here and an s here this box you are forced to put one of the three, this is not allowed of SU 3. But it is allowed for SU 4 if you allow four fundamental objects ok.

Up spin down spin was for SU 2 SU 3 up quark down quark s quark SU 4 will be up quark down quark s quark under a charm quark and so on. If you had four objects totally anti symmetric with four objects is possible with three objects making a totally anti symmetric of four objects is impossible ok. So, this diagram will not be there for SU 3, what about this u d s is forced here and this is just one dimensional this I explained it that any object with vertical column of n boxes is like, as if it is a trivial representation added to things ok.

So, I can add any number of them with a trivial unit representation. So, I can call that as a n with A 1, I do not know let me call it by our conventional one this is like your trivial unit representation it does no harm, because it is like a identity state anything you multiply with it nothing happens. So, this one also I can remove out and write this as, so this is like the A1 which I can ignore and write this as a it is this these are the two possibilities for SU 3.

What is the dimension of this? This is 3 and 3 which is 9. What about this? This is probably 6 and this is 3 ok. You can work it out and see what is the dimension 3 4 2 3 and then a hook number ok. So, this is something which you have to remember, that whenever you have a vertical three box for SU 3 you can ignore it. Whenever you have a vertical two box for SU 2 you can ignore it right this is trivial for SU 2 this is trivial for SU 3 clear.

So, that is what I have done on the screen, that I have removed those trivial representations and written in the young diagram going back to the screen. You see that spin 1 s 2 box spin 3 by 2 s 3 box, you take the product keeping track of things. Whenever I have a vertical two box I can ignore vertical three box is not allowed for SU 2 doing it for SU 2.

So, these are the three diagrams allowed for SU 2 and which one is spin 5 by 2 in this diagram. The one which has five boxes corresponds to the spin 5 by 2 representation, the one with three boxes after removing the trivial representation two box became trivial here you can make four box trivial and remove it out and you get this is this clear ok. So, this is the tensor product decomposition ok, it is very beautiful in the context of SU n that this you rigorously.

I am sure you would have rigorously proved in your quantum mechanics course. But here I am diagrammatically showing that every diagram is an irrep for the SU 2, the diagrams can have only a single row with so many number of boxes each one is an irrep. But you can compose two irreps and they can undergo a decomposition into irreducible representation in a systematic way clear ok.

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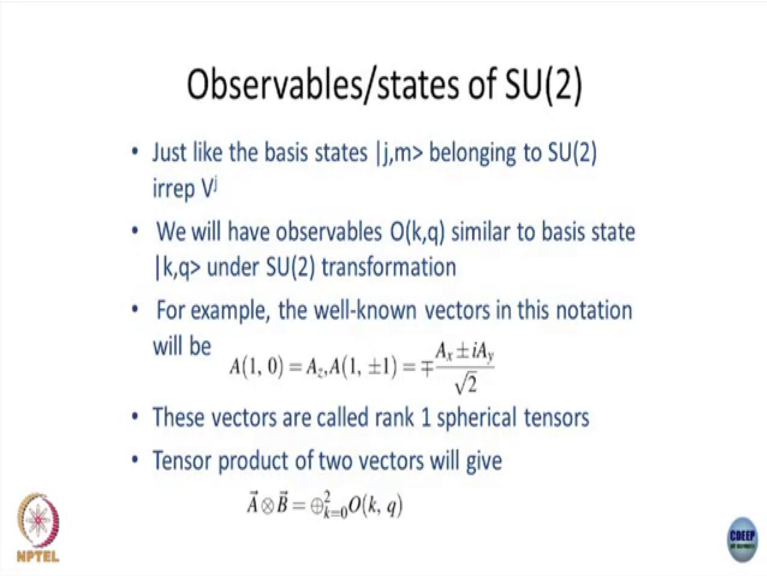

The next question is about Observables, remember when we did the selection rules. I said that every irrep is not only applicable to the states or the basis states primary basis binary basis or tertiary basis. But, you can also apply to electric dipole moment vector magnetic dipole moment vector quadrupole moment vector and so on. So, those are the observables and our aim is to understand selection rules even in the context of the continuous group symmetries and that is given by your Wigner Eckert theorem, just not very different from what you saw in your discrete groups.

The product of all the three states should allow for you a trivial representation, we have all done this right we did this. What is the trivial representation it should allow me something which gives me this representation for SU 2. If it gives me that then I know that such a transition from an initial state to a final state triggered by an operator associated with an observable is allowed. Otherwise it is forbidden and I am going to assume that the system is respecting such a SU 2 symmetry.

If the system is not respecting that symmetry, there is no point in talking about it if the system has a rotational symmetry a rotational symmetry of spin particles is incorporated in the SU 2 group. For such a group I want to look at whether certain matrix elements are 0 or non-zero. I will not be able to say what is the value, but I can at least say that these matrix elements will be 0 or non-zero and that is Wigner Eckert theory. I am sure you have all seen Wigner Eckert theorem in your quantum mechanics course I am just rewriting it in the group theory language.

So, associated with the states j m in the context of SU 2 you will have observables which I am going to write it as O k q which is similar to writing a basis state like k comma q. So, remember your electric dipole moment will be like a vector and if this one is pin j is one, then the three components of that vector should be related to the three components of that vector which is what I have shown you here as an example ok.

There is a slight modification writing it similar to your spherical harmonics which you might have seen that you rewrite things; so, that they are like complex conjugates of each other, the z component is like j is 1 spin 1. But the magnetic quantum number 0 is associated with the z component of the vector. The other components A x and A y you take a linear combination of A x plus or minus I A y those are similar to your j equal to 1 and magnetic quantum number plus or minus 1 ok. This is not new to you write e to the I m phi times cos theta p l ms right.

So, this is why it is called sometimes as a spherical tensors. So, if you just write Cartesian tensors I should have written it as A x A y and A z to match with the states you convert them into spherical tensors ok. So, if you compose to rank one spherical tensors rank one means the first k value is one ok, take two vectors it is what I was saying take a tensor product of two vectors. In this notation then you will end up getting what are the possibilities this one has j equal to 1, this one has j equal to 1 1 cross one will give you two spin 2 spin 1 and spin 0 and associated with that you will also have spherical tensors of that.

So, this is exactly similar to my previous slide, if I replace this three horizontal box by only two boxes. So, whatever is applicable to the states is applicable to the observables also, this double box diagram is for a vector which you are all familiar electric dipole moment is a vector, you can start doing the compositions of two vectors and you can find what are the irreps using this diagrammatic language. There you use this great orthogonality theorem to write the decomposition here I am not doing that ok.

The next question is can I find those explicit states from those fundamental objects, which is what you did also. If you remember I took a 1 b 1 a 2 b 2 I compose them and then I said that the one which corresponds to a trivial representation is a 1 b 1 plus a 2 b 2 that is the dot product and we derive that using a projector. you wrote a projector and then you apply the projector on a arbitrary reducible representation and you found what you get and then you wrote the corresponding binary basis right.

Similarly, you will have if you compose two fundamental objects; if you take these two as one of those fundamental objects, when you compose it what combinations of it. We will decide the state for this that is not clear here for that you need a projector and that projector is nothing but your Clebsch Gordan coefficient, Clebsch Gordon coefficient formally if you see here you have a tensor product of the basis state.

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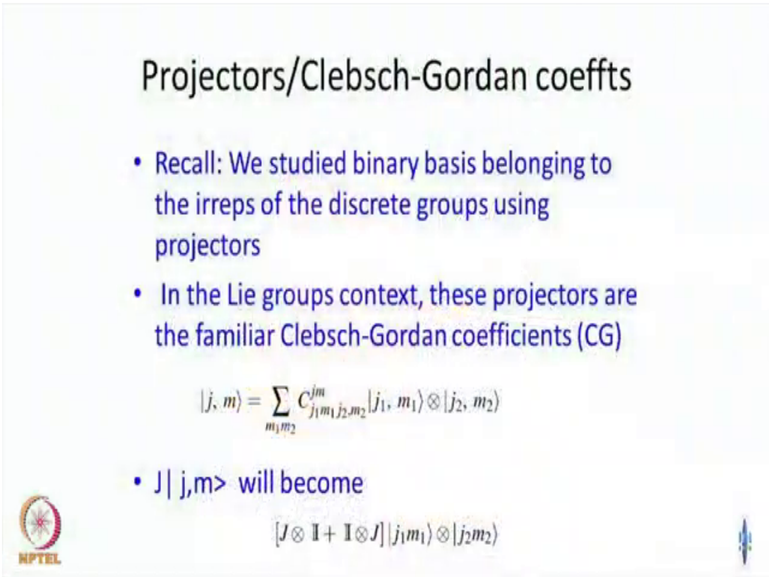

So, this right hand side is a reducible representation the projector is your Clebsch Gordan matrix. Projector on to this will give you a particular state which belongs to an irrep given by spin j exactly like what you did. So, this is you can call the state which you write here in terms of this as a binary basis is this clear. This is these are fundamental basis primary basis you compose the two primary basis and get a binary basis to get the binary basis usually you will take a projector. Now the projector role is played by the Clebsch Gordan matrix and these are the elements of the Clebsch Gordan ok.

Similarly, you can show the generators which was j plus j minus and j z, we will now for this combined state obtained from composition of two primary states will involve j cross identity plus identity cross j that should operate on this state this also you can show. You do this mechanically, but this is the way the derivation will go in ok.

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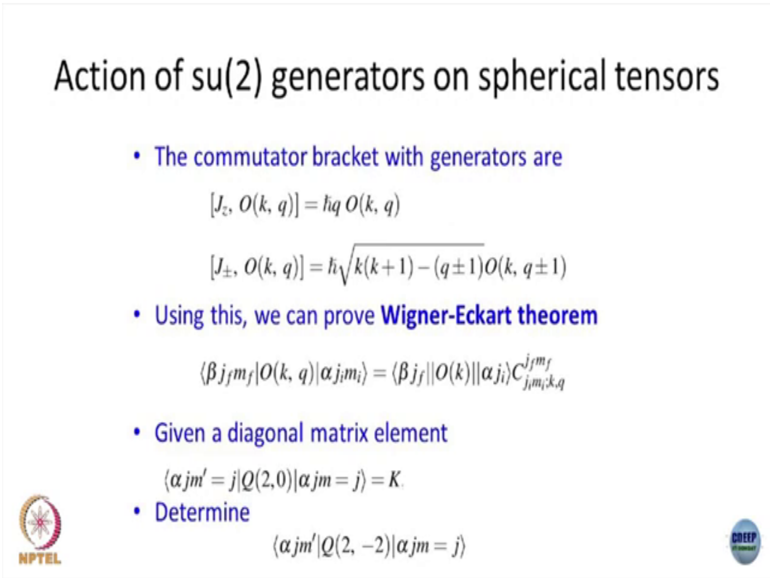

Similarly, the observable there is always a rhythm between the observables and the states. So, whenever you have an observable you have to look at the commutator bracket with the generators and they have the similar properties it is like an Eigen state of j z with j plus j minus they increase or decrease by plus or minus 1. Now, comes this matrix element, earlier we did electric dipole moment we took an initial state which is in one of the irreps of a discrete group. And, we took a final state which is another irrep of the same discrete group symmetry and then we ask the question whether that matrix element is 0 or non-zero.

Now, in this case SU 2 symmetry tells you that the states have to be j i m i other quantum numbers, I put it as alpha we do not need to worry about it. Similarly the final state is j f m f; the question is whether such an observable the operator associated with it. Will it be non-zero or 0 and I am not going to give you the proof there is a rigorous proof of showing it.

What I want to stress on this is that to get a unit representation out of it. This will be proportional to the Clebsch Gordan coefficient, what does the Clebsch Gordan coefficient you compose k q multiplied with tensor product with j i m i that is what is this composition and you have to get the final state you all know. The properties of the Clebsch Gordon coefficient when will it be 0, if m f is not equal to m I plus q it is going to be 0 and similarly j i composition with k has to have a j f otherwise it is 0.

So, you can trivially rule out which matrix element is 0 which matrix element is nonzero ok. There is the rigorous proof to show this we cannot determine this element, this element is called reduced matrix element ok. I cannot find the number, but I can definitely say whether this matrix element is 0 or non-zero by looking at the Clebsch Gordon coefficient and that is purely from SU 2 symmetry ok, so that is the selection rule for your.

Suppose an experimentalist comes back to me and says that the quadrupole moment tensor he can give me the matrix element for the 2 0 component. Suppose if he gives me this component k, then I know the CG coefficients I can determine this matrix element for the quadrupole moment tensor with minus 2 for the q. That Wigner Eckert theorem is the selection rule in the context of SU 2 symmetry and in particle physics many processes decay processes.

You can rule out such a decay is impossible, you know you can tell from using just the SU 2 symmetry. The SU 2 symmetry is what we call it as an isospin symmetry in particle physics, incidentally you can use the isospin of few particles and look at a process like this.

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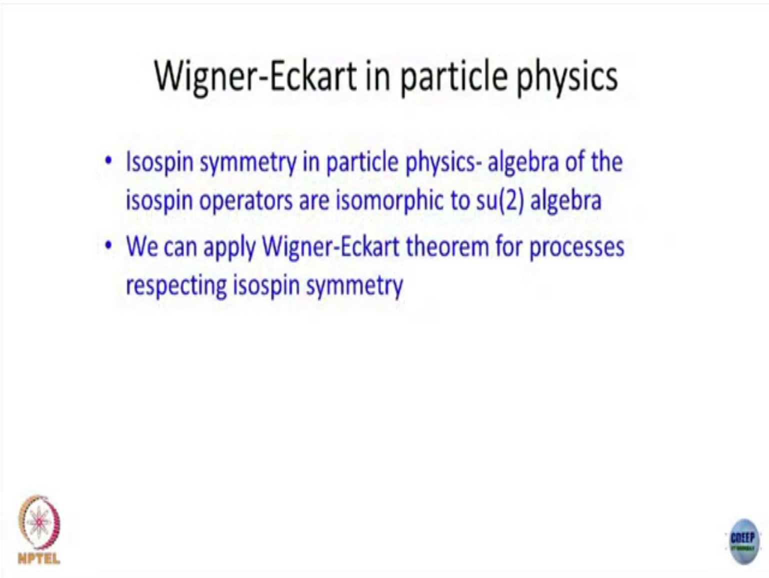

Where the operator which is the s matrix is a it is like spin 0 ok. So, you need to see whether the isospin initial state and the final state are allowed. So, here it is a combination of isospin of proton and a pion and these can be worked out using your Wigner Eckert theorem in the context of particle physics.

And this is a very strong indication experimentalist do see that such a decay process. If it is the decay ratio between these two is two is to one you know this is being seen and we can account from purely from our group theory argument.