Group Theory Methods in Physics
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Lecture - 05
Factor group, Homomorphism, Isomorphism

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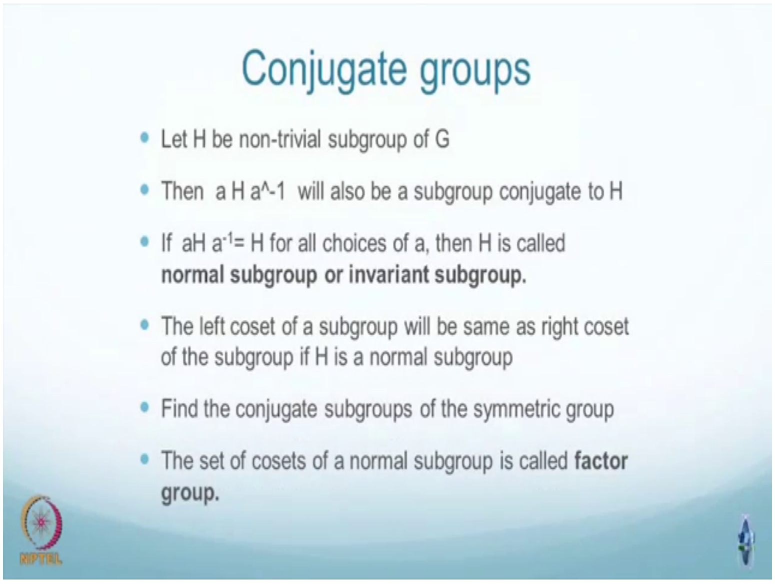

So, that is what I am trying to say here. So, find a H a inverse it will give you a subgroup conjugate to H start with a group with a subgroup and generate other conjugate groups out of this and then if you also find that you do not get anything new the conjugation gives you the same subgroup in the context in this context it is H 2. Then you say that that subgroup is a normal subgroup or an invariant subgroup ok.

So, the left coset of the say subgroup will be same as the right coset if H is a normal subgroup. So, so we have already done this find the conjugate subgroups of the symmetric group I have explicitly done it for you ok. This is something which I want you to check the set of cosets ok. So, let us write it out of a normal subgroup. What is the set of cosets for a normal subgroup? This was multiplied by e.

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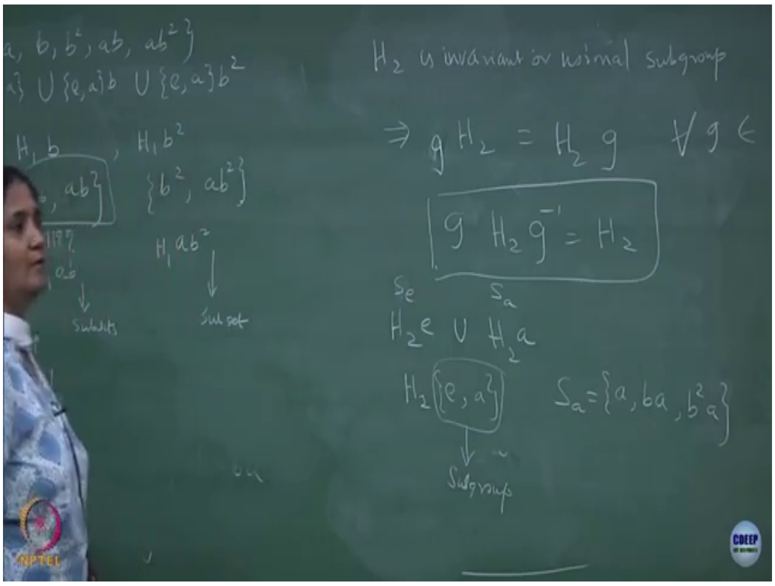

Normal subgroup was H 2 I just want to look at the set. So, this is one another one will be somebody with a right. So, what you can do is you can write H 2 multiplying a set which is e and a. So, this one this set is also a. So, e is a candidate for the first coset, a is a candidate for the second coset by that I mean you multiply a with all the elements of H 2 you can equivalently call this as some kind of a set S e and you can call it as an S a ok. So, this S e is also a group because identity element does not change the subgroup nature, but this is not a group right you get what in the H 2 of a.

S a set is what someone H 2 with a. So, you will have an a, ab, ab squared right or the other way round, the way I have written it should be it is a same because it is an invariant subgroup ok. So, the corresponding candidate is S a the a. So, you can see that if you take a group this one is actually a subgroup are you all agree e a is a subgroup another way in which we write this.

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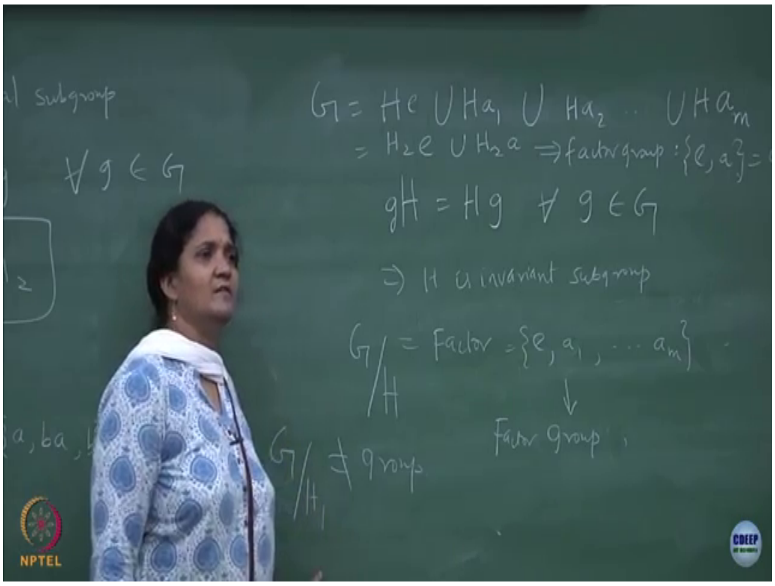

So, we take a group G we take a subgroup in general if you take a disjoint union of cosets and if H ok. So, this implies H is invariant subgroup ok, this happens. Then you can show that G mod H this is what we call it as a factor group factor the factor is going to be candidates of the cosets and this will be satisfying the group properties it will also be a group called factor group.

We can always factor by any arbitrary subgroup it is not required that it should be an invariant subgroup that will not be a group only if H is an invariant subgroup you can show that this factor has group properties that is what is happening here ok. So, the factor so, this is S a and this is S e in this particular example you have a H 2 e union H 2 a in this particular example. So, you find that factor group is e and a which is nothing, but G mod H ok.

Yeah is that clear, any questions on this? So, the factor will not be a group unless this subgroup is an invariant subgroup you can do the other way around you can do G, but H 1 and check it out, what happens and this will not be a group H 1 is not an invariant subgroup ok.

So, with a simple example of the first non-trivial non commuting nature with 6 elements I have tried to give you so much of data right. So, we started with I initially made a simple abelian group then I made it non abelian. Abelian group if you take if you take an abelian group, what are it is subgroups? What are it is conjugate elements or conjugate subgroups? These are questions which you should see for the cyclic group.

If you take a cyclic group with 2 pi by n which is non-trivial so, let us take n to be something like a prime number ok, you can take C 5 will it have subgroups. Why did I take a prime; why did I take a prime? Lagrange theorem says that if you have a prime number then you cannot find a subgroup, am I right. If you have C 5 order of C 5 is what 5 subgroups order should divide the 5, the only possibility is 1 or 5 which is trivial identity element is a subgroup total group is a subgroup right.

So, here you can start seeing what is going to be your every element will be like a coset you cannot further break it up, but if you do let us say C 6; C 6 is what? It is generated by rotation by 2 pi by 6, 2 pi by n is for general. So, now, you can start seeing whether there are subgroups order 6 group, but that order 6 group has nothing to do with the symmetric group, why? That order 6 group is a cyclic group which is abelian right. So, just think over these concepts.

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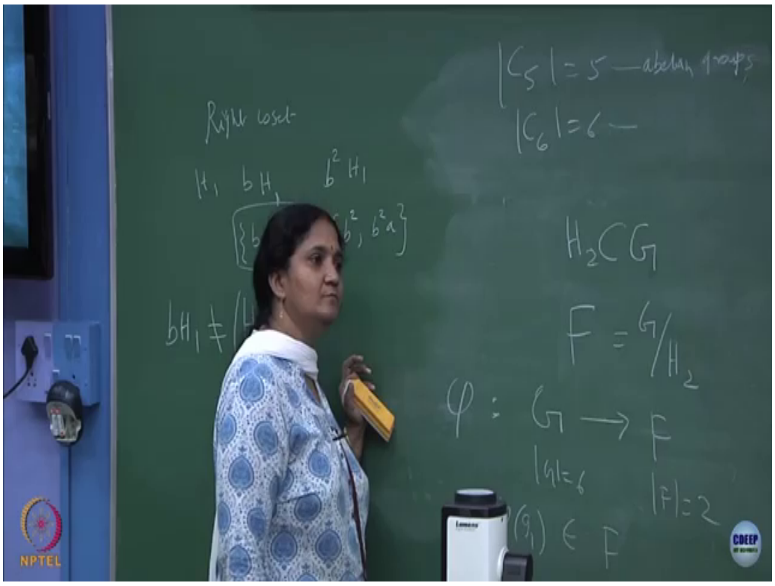

So, look at this group C 5 look at C 6 check for you know these are all abelian groups, if you have abelian groups you do not have these conjugate elements, why? Abelian means that this can exchange with this trivially whether it is invariant or not invariant, it can exchange.

So, everything will be self-conjugate even if you find subgroups in abelian groups you will find them to be self-conjugate you will not get conjugate elements. So, do the simple exercise for C 6 I will give an assignment on Thursday. So, do it for C 6 and compare in contrast is what we did for the symmetric ok. So, that brings me to a concept called. So, what are we seen? We have seen that we have a symmetric group G we have a subgroup which is H 2 which is a subgroup of G right, you can try to write the G mod H 2 to be a new group K ok, let us call it as a new group K factor group, if you want we can call it as a factor group F.

We have seen that it is a group right e a is a group subgroup it has all the group properties for this particular example. Suppose I write a map from G to F using a map file let us take a map which takes you from the group G order of G is 6 order of F is 2 you are all with me, e and a order of that is 2. If you try to write this map then we call this map, what is the meaning of this? phi of G 1 should be an element of F correct.

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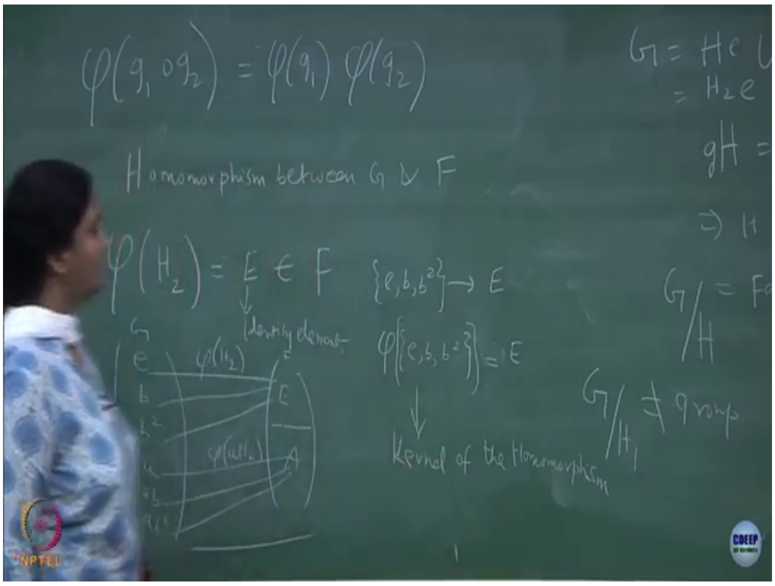

And what else is required, phi of g 1 combined with g 2 should be same as is that right it is true for identity element. So, we need it to be these are all elements of F and this is element of G you multiply find a new g and then you should get a element of F right ok.

So, now this properties is what we call it as a homomorphism. E is an identity element ok, so I am calling this as identity element. Can somebody tell me what this means what is this phi called? So, if you have a subset in G which gets mapped to an identity. So, let us draw it here let me draw the group G, let me draw the group F, group F has only 2 elements identity and A I am just calling it as capital just to distinguish this from here nothing more.

Here you will have e b b squared, then what else? Trying to say that there are 3 elements here which gets mapped to the identity element. So, this is not 1 to 1 on 2 right. What else happens?

Student: ma'am.

Yeah.

Student: How is it phi of (Refer Time: 15:21)?

Phi of?

Student: H 1.

Where I do sorry phi of H 2 sorry thank you this has to be phi of H 2 I am looking at specifically the symmetric that is what I have shown here ok. So, that is just. So, this is what we call it as a map where when I compose this element with any of these elements e times a b will give me a b, e times a will give me a and this map is correct ok.

So, this is what we call it as a homomorphism between G and F and the set of elements which maps to the identity element that set is called as a anybody was done Kernel of the ok. So, this e, b, b squared is mapped to identity anything phi of this set is mapped to identity equal to we call this as a Kernel.

Given a homomorphism between 2 groups if the kernel is non-trivial the set of elements of the group G which gets mapped to the identity is called the kernel of this map ok. So, this is what is the definition of a homomorphism sometimes you would have seen isomorphism's, when will it be isomorphic.

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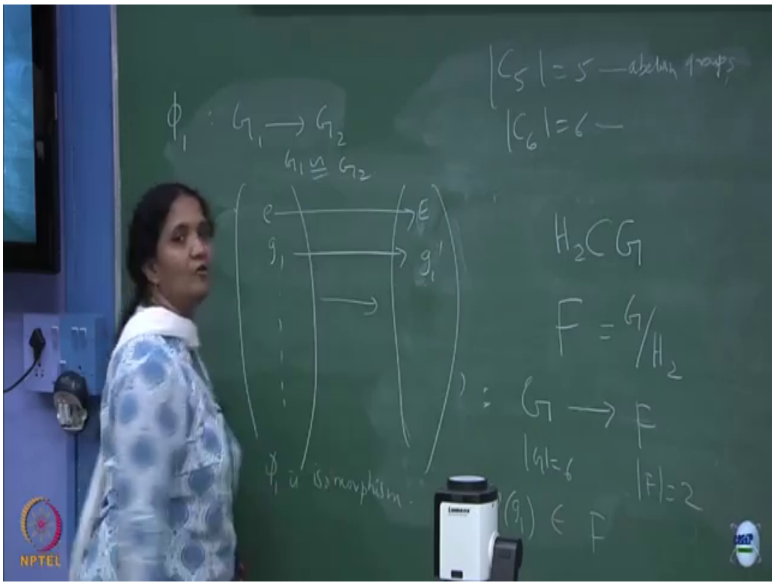

The kernel is just having only one element which is identity. So, suppose I have a group G 1 and G 2, you have a set here and a set here, identity element gets mapped to identity element and so on. Then we say that phi 1 is isomorphism and you can sometimes denote it as G 1 as isomorphic to G 2 or G 1 and G 2 are actually one you can see it like a rotation by 90 degrees another one you can see it like a cube root of unity sorry fourth root of unity and both are one and the same, it may look meaning maybe looking different, but those 2 are related by a map ok.

So, 1 to 1 if you get then you call it to be an isomorphism if there are many elements which gets mapped still it is the group G 1 is homomorphic to this one G is homomorphic to F and the kernel is given by the invariant subgroup.