Group Theory Methods in Physics

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Lecture – 49
Fundamental weight, Young diagram, Dimension of irreducible representation

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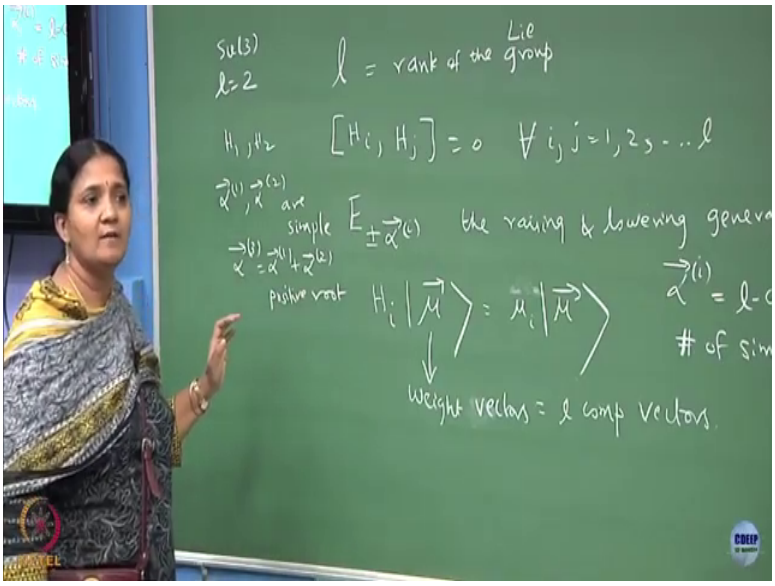

So, essentially I was trying to tell you that you have l which is denoting the rank of the group to be specific rank of the Lie group and then you have those generators which I call it as H i, H j which will be 0 for all i and j which is 1 to l right and then I introduced. So, the remaining generators will be those which are excluding them ok.

So, those you can try to make it as E plus or minus; E plus or minus alpha vector i. So, the i will be if the algebra had n as the number of generators then it will be the remaining one should be even. So, if l of them are diagonal that the remaining n minus l should be even and you can pair it up as plus and minus right which is the raising and lowering generators ok.

And, we write the state which is the simultaneous eigen states of these diagonal generators which we call formally as a mu vector and H i will pull out the i-th component of the mu vector. So, the vector is gone. So, this will be the notation that the every H i will pull out the i-th component of the new vector ok. So, these are what we call it as weight vectors weight vectors and they are l component vectors ok.

So, this one will be for every l of the H is you will have the corresponding that i-th component will be pulled out for the H i operator and there are l of them. So, this is going to give you a l component vector.

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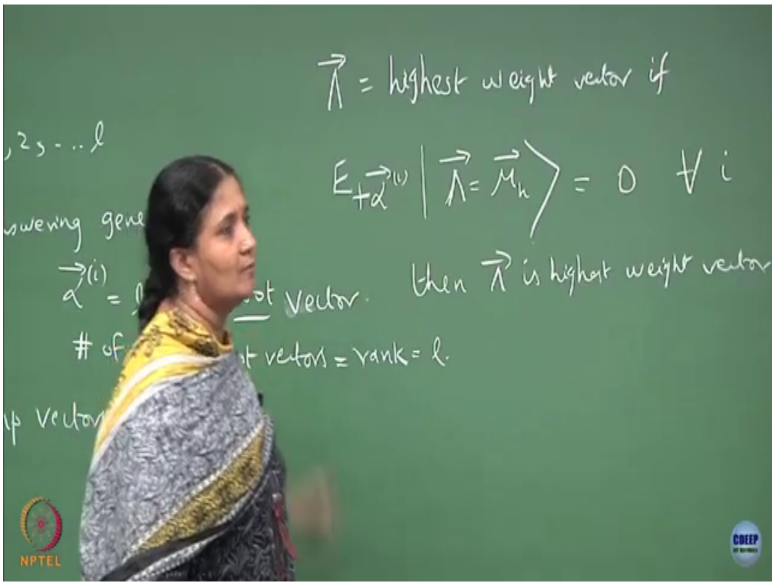

So, that is one and then I also taught you what is the highest weight vector is highest weight vector if E plus alpha vector on alpha vector I am calling it with the. So, this one when it acts on lambda which is one of the mu’s I do not know which one let me call it with mu subscript H ok.

One of these weight vectors suppose all the raising operators on that particular weight vector is going to give you 0 for all i ok, then we say that then lambda is highest weight vector and incidentally these alpha i's alpha i's there also l component root vector ok. These are l component root vectors and we can have l number of simple roots ok.

So, there will be number of simple roots equal to rank which will be l ok. In the context of su 3, this is 2 you have H 1, H 2 here you have alpha 1, alpha 2 are simple roots ok, they are simple and alpha 3 is alpha 1 plus alpha 2. So, this is called positive root ok. So, these are the simple roots ok. Is that clear?.

Abstract notation and a example which is a non-trivial example which you went through. So, you have one positive root here. There are three raising operators, three types of raising operators.

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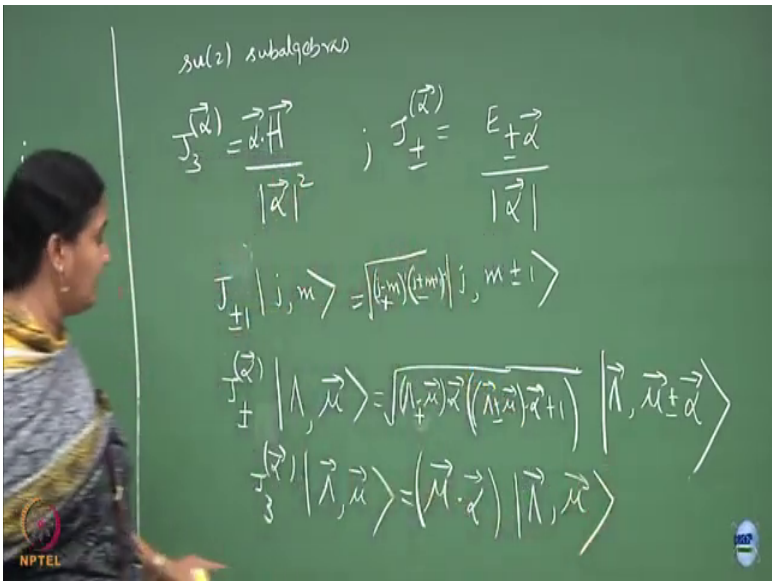

And, then I also said we can construct su 2 subalgebras for any Lie algebra ok. So, how do I define this? I am going to call the J 3 for a particular root vector, it could be simple root or it could be also a positive root. The way this is defined is alpha dot H H has l components alpha has l components l components. So, you can do a dot product divided by the magnitude of this vector squared as the su 2 z component generator associated with a specific root vector, is that clear? Put it in bracket.

Similarly, you will have a J plus J minus associated with the same root vector same root vector that will be E plus or minus alpha vector, I think it is divided by mod alpha is my memory. The way to check it is that you need to make sure that the algebra of su 2 is satisfied by them. The interesting thing about doing this is that then I know how to do the J plus J minus on any arbitrary state ok.

So, the way we will do it is typically if you had j, m J plus or minus you know how it does. How does it do? And, then you have a square root here with some coefficients. What does that coefficient? Is that right? I am just suppressing the H cross out of this. So, if this was if this m was the highest weight vector which means if m is j, then J plus has to be 0 which is happening, that is the way I am fixing it.

So, the same thing we could start doing it for these cases. So, let me put an alpha here. So, let me write another line maybe. See, interestingly you should see that these are all if you know these are all numbers. Finally, I have to get a number I should not work with vectors here right how do I get the numbers is the question you can ask. So, this m will get replaced by your weight vector, j will get replaced by your highest weight vector and then you have to take a dot product with the corresponding root vector.

So, in this particular case it is 1. So, it is dotted with 1 and dotted with 1 in this case. So, when you have to do J plus or minus alpha on lambda and mu; lambda is what I have defined here. So, this will become root of lambda minus mu dotted with alpha and then lambda minus plus mu dotted with alpha and then you add a plus 1 ok.

So, this is the modification times lambda mu someone plus or minus alpha vector for the plus minus yes, thank you it is. So, simple once you write it in this fashion whatever you know for Su 2 is what you extrapolate when you do the Su 3 and other things also follow for any abstract value ok, is this clear?.

So, those are the Su 2 subalgebras in the Lie algebra which I can refine this way and similarly you can find what is the J 3 eigenvalue also. What is the J 3 eigenvalue? Someone? U dot alpha incidentally there will be a mod alpha squared and a mod alpha here, but then those vectors I am taking it to be of unity. So, you can ignore it for this case, but in general you should put that denominator and so on. Then it depends on a matter of normalization sometimes there will be a by 2 here and then a half you know all those things have to be taken.

It is an eigenstate it is an eigenstate with the corresponding eigenvalue being written as the weight vector dotted with the root vector clear. So, this is the procedure of how to construct your states. For every alpha you will do this and you will generate those states ok. In the case of Su 2 you had only one alpha which was 1, so, you generated one state only along a line.

Now, in the case of Su 3 you have two simple roots and one positive roots. There will be three directions in which you can start generating the states, that is what is happening ok.

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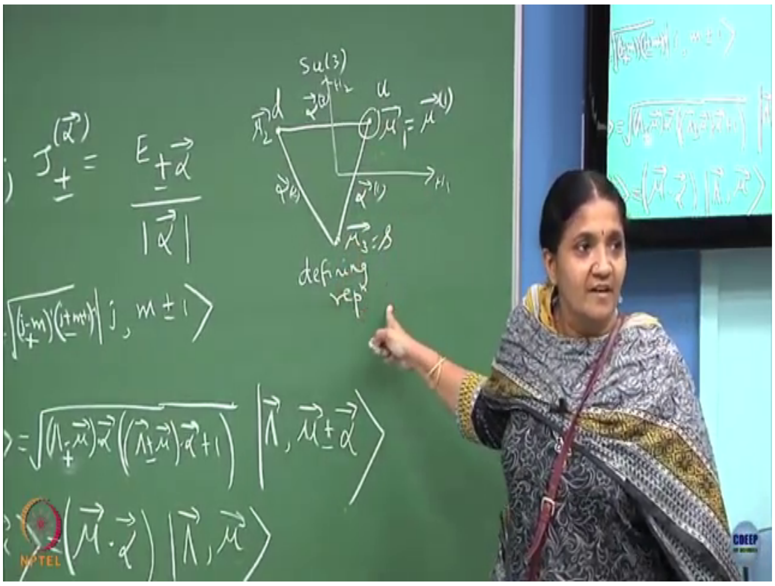

So, you can start with a highest weight state Su 3, you start with the highest weight state and then you have a direction which can give you let me call it as alpha 1 let me not worry what that is. You have a. So, it turns out that this will be the highest weight because any ladder operation which I try to do will not give you will not give you those states, these three states. So, this is the highest weight state. So, this is for the defining representation.

And, similarly you can start taking tensor products of defining representations and generate higher dimensional representations ok. So, the fundamental object can be taken to be the defining representation and you can start generating the higher dimension representation because in the case of Lie groups you allow irreducible representations to be of any dimensions, unlike your discrete group great orthogonality theorem told you the character table which fixed you that these are the irreps.

Now, when you compose two irreps it is reducible, but you can break it up into irreducible representations which can be of any dimensions. So, that is the difference between Lie groups and discrete, is that clear? We will do the tensor product. Any doubts on this?.

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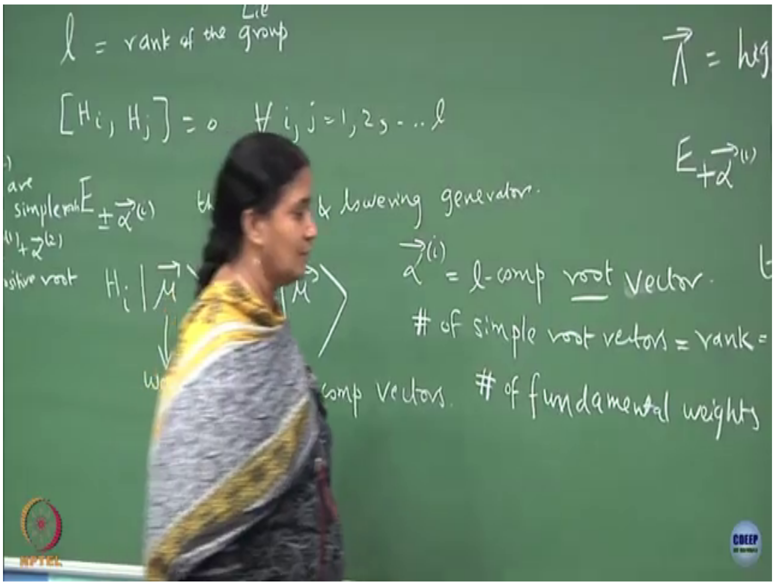

There is one more thing I wanted to introduce in the last lecture which I did not finish because the thing is just like simple roots you have fundamental weights ok. Fundamental weights are also l of them, there will be l fundamental weights number of fundamental weights it is l and we denote this by mu 1, mu 2, mu l ok. This has nothing to do with these three states.

These things I called it as probably with the subscript 1, mu 2 and mu 3 and this one was plotted in the x and y were H 1 eigenvalue and H 2 eigenvalue ok. Are you all with me? Those were the three states for the defining representation and this highest weight will be one of the fundamental weights ok.

So, this one will turn out to be one of the fundamental weights which is also the highest weight ok. So, now, we need to determine the remaining l fundamental weights in the Young diagram language ok.

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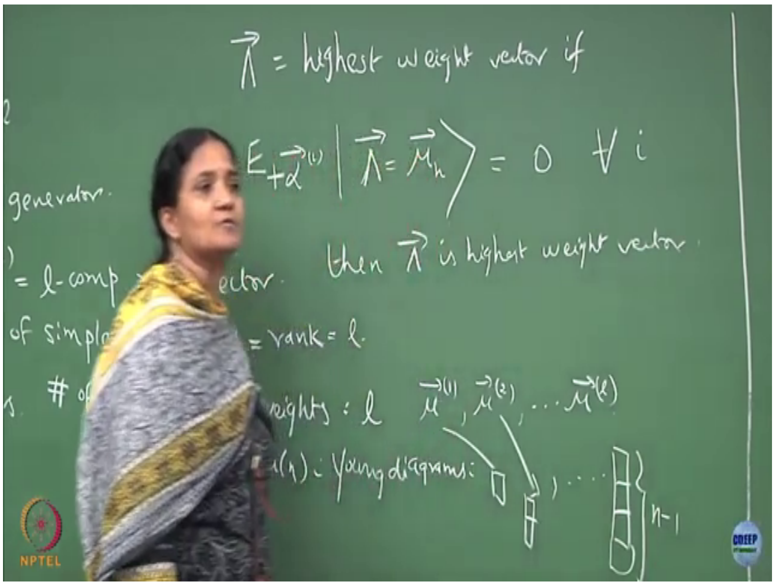

At least for su n Young diagram we will denote this, this one will denote it by single box, this one will denote it by double vertical box and so on. We will go on till the rank will be l will turn out to be n minus 1. So, you will go up to n minus 1 vertical box for the su n. Other cases let me not get on to it, but this is the procedure that you will have l fundamental weights.

Using these fundamental weights we can write any reducible representation of arbitrarily group in particular for su n, what will the procedure be? There can be some n number of single box, some other number of double box and so on ok. So, let me write that also.

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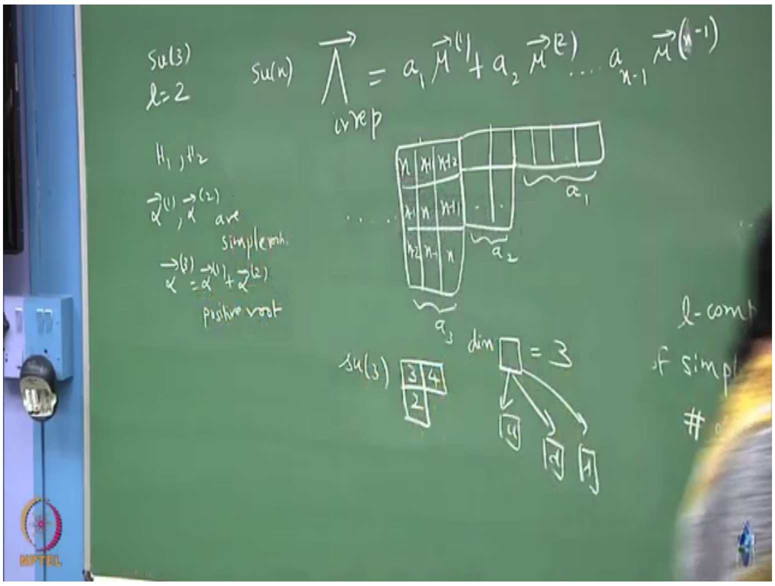

So, any arbitrary representation lambda highest weight for an arbitrary representation if I write it as a 1 times mu 1 plus a 2 times mu 2 so on. So, specifically let me look at it for the Su n group. If I am doing it for Su n it will go on till. Su 2 it will be only these two sorry, Su 3. It will be only these two for Su n it will be n minus 1 such term.

Diagrammatically what I will do is the convention is to put single boxes on the right hand side, and then double vertical boxes a 2 of them next to it and so on. This is the notation. This diagram is equivalent to talking about any arbitrary reducible representation where it has this highest weight is a linear combination of the fundamental weights, how many you are multiplying a 1 of the fundamental weight 1, a 2 of fundamental weight 2 and so on ok.

So, this is the arbitrary irrep I should call it irrep highest weight, lambda denotes the highest weight ok. Now, what is the next question? If I say it similar I need to know what is the dimension of the data just like what you did in your permutation group right. In permutation group we wrote the character table where on the column it was cycle structure and along the rows it was the irrep shown by the diagrams and we had a formula Hooks formula, Hooks number to define the dimensions right here I need to find out how to define the dimensions. This is also very close to the Hook number, but there is one slight modification ok.

So, the modification is that along the row we put integers which are increasing, along the column it is decreasing can you see it? It is decreasing n you start with n and increase along the row and you start along the column it should be decreasing. So, along the row it is increasing and so on ok. So, here it is decreasing and this will become n. So, it goes on like this. Whatever is the integers I have put in the box that will be the factor which will come on the numerator.

So, the numerator. So, let us take a simple example and let us look at Su 3. The way if the numbers will be put here is starting with 3 along the road increases by 1, along the column it decreases by 1 ok. So, that dimension of this representation or number of allowed states for this diagram just like if I had a single box number of allowed state is in Su 3 it can allow up entry to be up this can allow entry to be up, entry to be down and entry to be s. So, it was three states.

So, the dimension of this representation is 3 and those three states are what I have plotted here this is actually the up quark, this is the down quark and this is a strange quark ok. So, it is 3-dimensional; defining representation is 3-dimensional denoted by a Young diagram which is just a single box and explicitly you can put the entries on it and find that there are three possibilities because you are going to play around with three fundamental objects.

For the Su 3 it is three fundamental objects with which you should play around. For Su 2 what is the dimension of that of this diagram? that single diagram can have an up spin or a down spin. So, it becomes 2-dimensional right. So, the single box if you take SU 2 dimension of this is 2, this is for 3.

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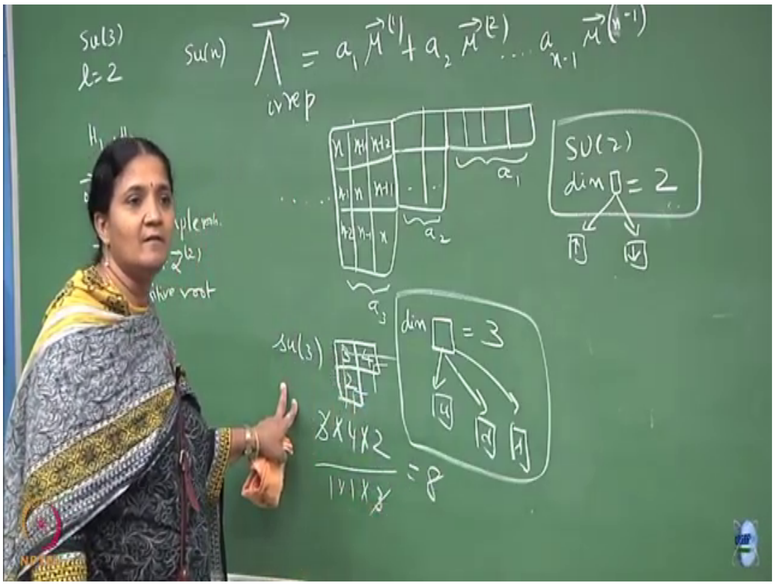

The reason is that you can either make the single box to have an up spin or it can have a down spin. Now, I want the formula this is simple to work it out, but I want a nice formula that if I give you an arbitrary irrep of SU n how many states are possible, how do I do that it is the question. One possibility is that I can take this diagram of SU 3.

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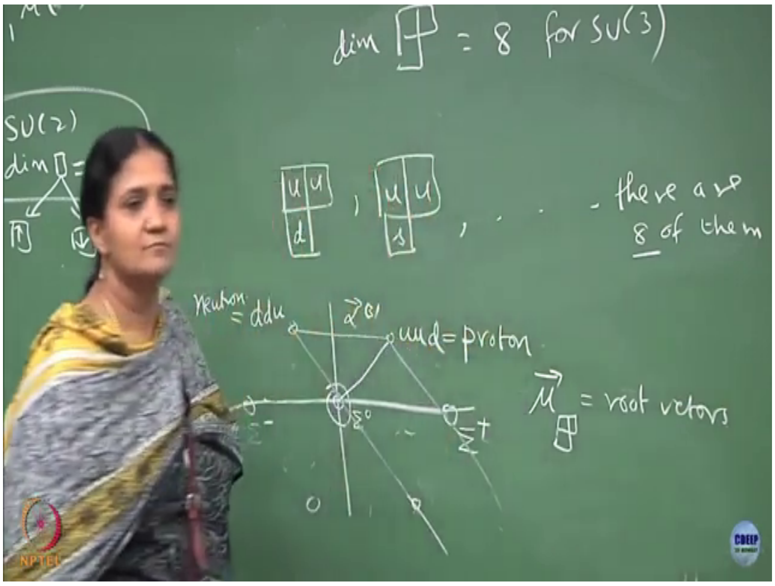

So, suppose I take this diagram of SU 3.

Student: (Refer Time: 26:23).

I will tell you, I am going to get to it. I am just saying these are the allowed ones and we will get to it ok. So, if suppose I have this diagram you can have can you have all things to be u is it allowed the Young diagram language tells you that along the horizontal it should be symmetric, along the vertical it should be anti-symmetric, this is not allowed ok. So, this is not allowed. So, it cannot put 3 possibilities here three possibilities here and three possibilities here to give you 3 into 3 into 3 that is not allowed. You all understand from this diagram right?

So, what are allowed you can start enumerating them. You can have along this same, but below it has to be either a d, this is one possibility; another possibility is it can be an s and so on you start listing it unless how many are there. That is one way of doing it. That is what we did even for the permutation group which is a symmetric group of degree 3, I took one diagram like this and I showed you that there are two independent possibilities linearly independent possible.

Similarly, you can list it out in the case of SU 3. The only advantage here is that it is not distinct objects. This box whatever object you put does not mean the other box should not have that object it can have a repetition that is the difference. In the symmetric group of degree 3 you had a object 1, object 2, object 3 which are distinct. If you put object 1 in one box, you cannot put the same object 1 in the other box the meaning is different.

Here the single box can allow any of the three possible states. The single box can allow any of the three possible states, it is not three distinct objects it is repetition is allowed, but repetition with symmetric and anti symmetric property of the diagram has to be fought ok. So, that is what is the Young diagram by just looking at it you get the information ok.

So, if you start listing it you can show that there are 8 of them ok. So, I am not going to say. So, there are 8 of them. So, which means dimension of this diagram is 8 for SU ok. How do I define such numbers without listing it out like this is the question for an arbitrary irrep is what is being dictated by this kind of a formula. A formula is whatever numbers are in these boxes, how to put these numbers I taught you put that in 3 into 4 into 2 and Hook number is what for this diagram? That is the same as what you have done earlier.

What is the Hook number 1, 1, 3. So, it is 1 into 1 into 3. What happens? You get exactly which is what I have tried to start listing it out and you will get this to be 8 states possible. Incidentally this one has 8 states, the number of generators of SU 3 is 8. So, all these 8 states should be what? Also identical like the elements of the root vector, it is a weight diagram for the there will be a weight diagram with 1, 2, 3, 4 and then one there will be a double circle and this one those are your 8 states ok.

So, what was this? It was uud will turn out to be the highest weight and the particle which is seen in nature that particle is actually the proton ok. So, this one is proton and this is obtained by a ladder operation which involves your alpha 3; alpha 3 will reduce one of the u to a d that is what is happening here u become a d, one u becomes a d ok.

So, what will happen here? It will become ddu and this is nothing, but the neutron ok. So, this is like your eightfold path and this pattern there are other particles here which are also here which is sigma plus proton is a positive particle sigma plus this, there is a sigma 0 here, there is a sigma minus, neutron is a 0 a neutral particle.

What is happening is that along this line should start drawing lines like this I am not very good at drawing, but you start drawing lines like this these are lines with same charge. So, you can see proton has positive charge, this particle has positive charge is very beautiful. And, these are the looking at it at these pattern of how the charges of these particles are GellMann came up with putting this SU 2 Lie algebra into action with appropriate normalization to fit in.

So, each one once I draw a diagram each one is an irrep. This irrep has an highest weight which is given by mu 1 plus mu 2, a 1 and a 2 will be 1 and 1. For that highest weight there are 8 states or the dimension of this irrep is 8-dimensional. This is another irrep the dimension of this irrep is 3 because there are three states.

What do I mean by three states? By the ladder operation I can go from one to the other. Similarly, by the ladder operation I can reach out to all the 8 states. The analog of this for SU 2 will be just up going to down or up spin going to down spin. What about here? It will be a spin 1 which is the analog of this will be just the spin 1 which will be highest weight then magnetic quantum number 0, magnetic quantum number minus one ok.

So, the SU 2 is sitting inside the SU 3 and this turns out to be this is a weight diagram, but because this weight diagram has number of weight vectors equal to the number of generators they are actually the root vectors. So, these weight vectors so, you can start plotting these weight vectors these weight vectors are for these diagrams there will be how many of them will be there? They are 8 of them and these we call it as the root vectors ok.

There will be 8 of them for this. Are you all clear? These are two different irreps of the same SU 3 group. This is the lowest non-trivial irrep which is 3-dimensional, this one will be 8-dimensional. So, the someone was asking me how do I find this mu 1, mu 2, mu l because I need to write this ok. So, that let me explain now how do you find that out and then we will get over here.

Student: (Refer Time: 35:59).

Completely.

Student: (Refer Time: 35:03).

Yes, yes. You can do that. I will come to it. I will explain some more examples and we will come to it ok.

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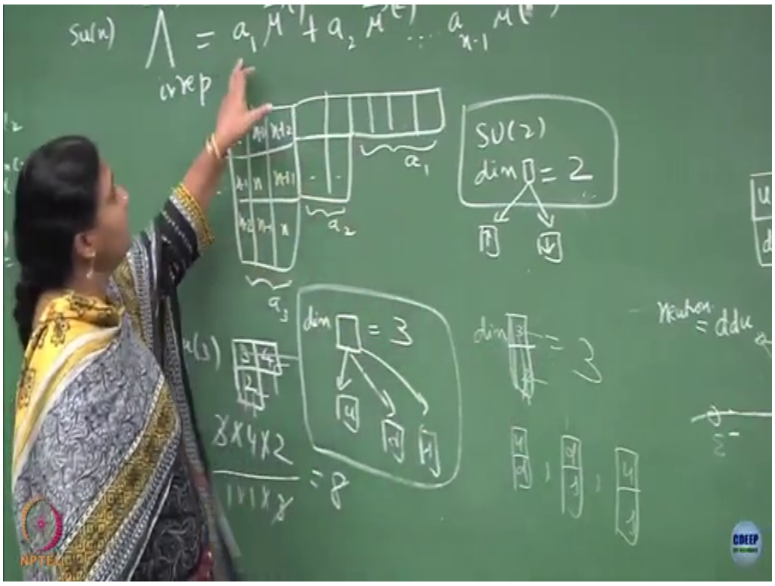

This is one example. But you could have a completely this one. Is that what he was asking?

Student: (Refer Time: 36:16).

This is not completely anti-symmetric. It is completely anti-symmetric only for SU 2, but for SU 3 2 states out of three can be anti-symmetric, you understand?

Student: Yeah.

So, this what is the dimension of this, can you work it out for SU 3? This is 3, this is 2, Hook number is 2. So, you will have?

Student: (Refer Time: 36:42).

3, dimension of this is 3 ok.

So, you will have anti-symmetric combinations. One anti-symmetric combination can be u d; second anti-symmetric combination can be d s; third anti-symmetric combination can be u s. These are the three states which corresponds to this. If I take this diagram then a 1 is 0, but a 2 is 1.