Group Theory Methods in Physics
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Lecture – 48
SU(2) sub-algebra, Dynkin diagram

Is this clear? Is the notation clear? This is what is the notation for the SU 2 sub algebras.

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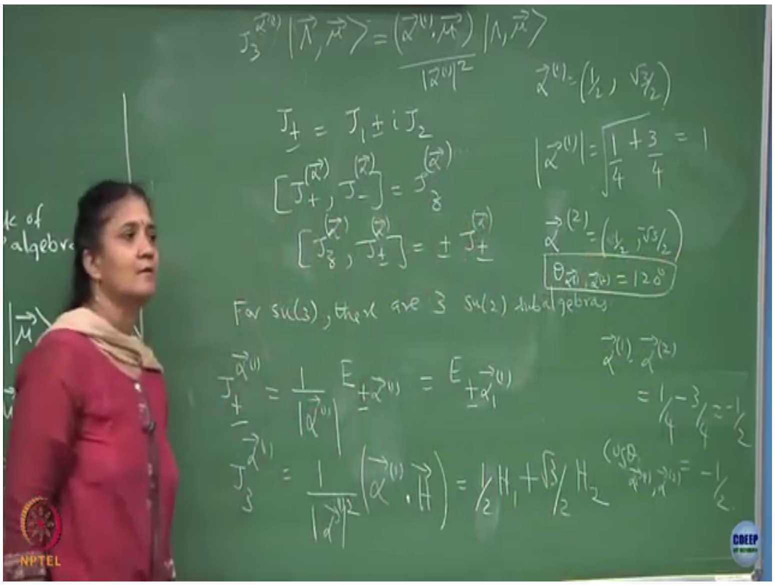

There are 3 independent positive roots. So, you will have 3 SU 2 sub algebras ok. So, what I am trying to say is that if I do J 3 alpha 1 or let say lambda and mu vector. What will that be? It will be. Why is that? The J 3 alpha 1 is this operator. So, it will be a dot product of that weight vector with the mu vector clear. So, that is where this will be useful ok.

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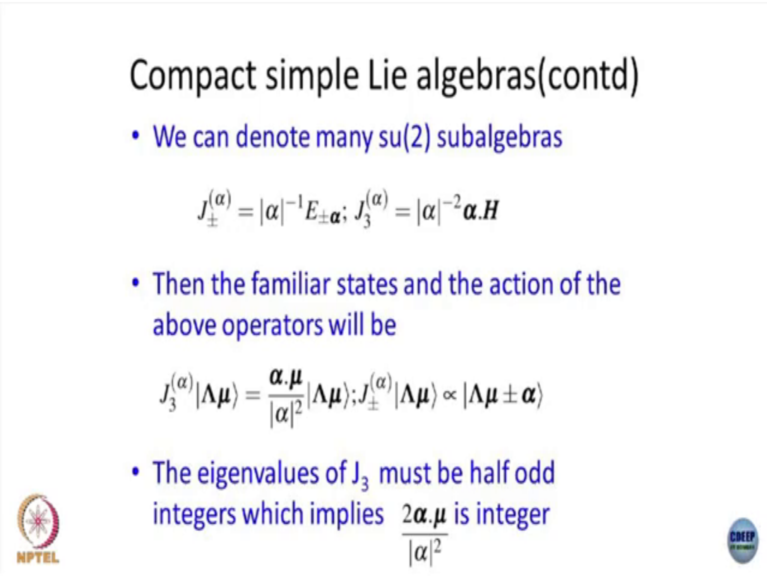

So, I am coming back to the screen again to just to make you get a feel of that you will have a calculational tool if you define SU 2 subalgebras and this is one definition of the SU 2 subalgebras. Only thing you have to remember here is formally I should have written this with this ok, but you know in SU 2 subalgebras what should be that coefficient.

This is a J 3 eigen value in SU 2. J 3 eigen values are always integers or half odd integers. It is a number. It is no longer a vector now. It is a number and this number has to be integers or half odd integers. I am putting h cross to be 1. This whole thing has to be always a either a integers or half odd integers. Can be half odd also because spin half state will have half integer if it is pin one it will be an integer.

So, these are things which I can put in as conditions here which is what I am showing it on the slide now that the eigen values of J 3 has to be half odd integers or integers or twice the eigen value should be an integer ok. So, you can check whether you all your twice the eigen values are also integers. You know what is the mu 1 vector mu 2 vector.

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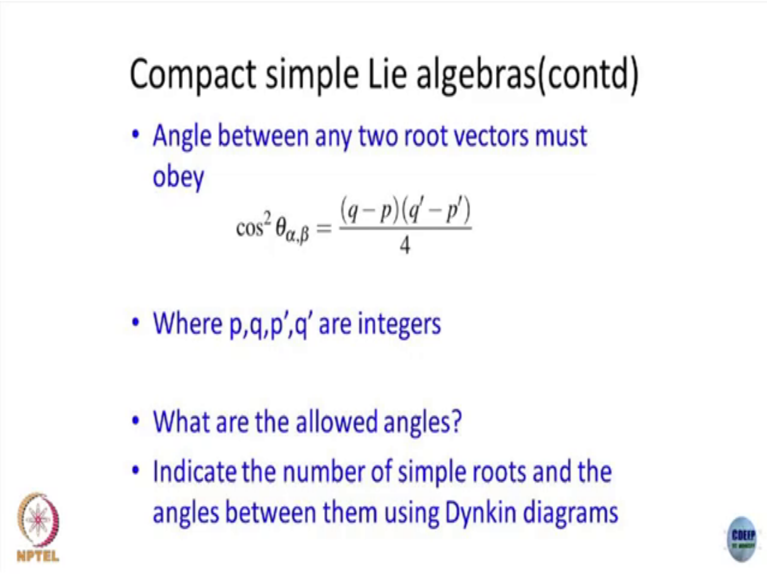

This can be exploited raising operator, lowering operator how many steps you can go from one states above p units below you can go q units and so on and you can get a condition on the angle between any 2 root vectors. I am not going to derive this anybody is interested can go and look it up Georgie talks about it, but what we need is that the angle between any 2 root vectors in any Lie algebra is constrained by these integers. q is an integer, p is an integer, q prime is an integer, p prime is an integer and there is a condition which you can derive.

This puts a lot of constraint. You cannot have any arbitrary Lie algebras ok. You have to have a Lie algebra with the root vector satisfying this condition. That is why you can determine what are the allowed angles with this. Please go and check what are the allowed angles using this. Theta equal to 0 is trivial 180 is fine, but you can get 120. Can you get some other angle like arbitrary angle like 10 degree 15 degree you can check it out.

You will find only the allowed angles for it are a list of them not every. Angle means the angle between 2 root vectors. So, one root vector I wrote here if you take the other root vector you can find what is the angle between those diagram ok. So, if you take this root vector and if you take alpha 2; alpha 2 was what half and minus root 3 by 2. So, if you take the dot product of these 2 what are you getting? What does that tell you about the angle between these 2 vectors? What is the angle between these 2 vectors? Mod alpha 1 mod alpha 2 cos of angle between these two. I am going to your.

Student: (Refer time: 06:14).

Huh.

Student: 120 degree.

120 degrees. So, this implies cos of angle between alpha 1 and alpha 2 has to be minus half. The angle between these two. Theta theta is 120 degrees everybody agrees. So, theta will be angle between alpha 1 and alpha 2 is 120 degrees in the SU 3 case ok.

So, there are only specific angles and you can check that integers q p are integers you can still get 1 by 2. Not arbitrary angles you cannot get arbitrary angles because q should be in all the 4 has to be integers. So, this is what you get and in the case of SU 3 you found that the angle between the 2 simple roots is 120 degrees cos of 120 is minus half clear.

So, now, what we are going to do is any abstract Lie algebra first of all the rank of the Lie algebra will tell you the number of simple roots and then you know the angle between the simple roots has to satisfy the condition levels. Using this there is a huge classification done using a diagram called Dynkin diagrams. So, I will just expose you to that what the Dynkin diagram is.

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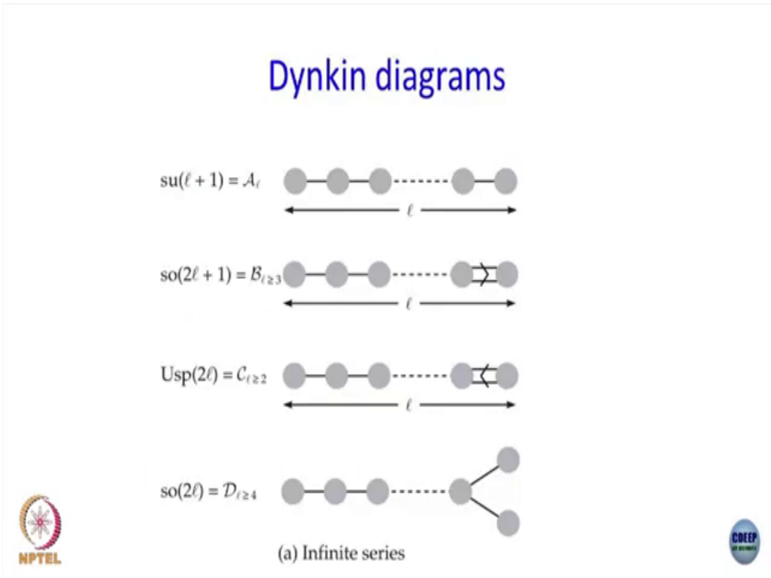

Number of simple roots is the rank of the Lie algebra the rank has l the each blob is a simple root and a single line is to denote that the angle is 120 degrees and so on. So, there is the double line then the angle is different. So, there is a notation for orthogonal group symplectic group. Even in orthogonal group you have 12 plus 1 is different from 12 and this has been classified and these are the only possibilities which you can play around ok. This has been it is call the complete classification of simple Lie algebras.

So, it is it is actually an ocean. If you want to get into it you can sit and try and look at the proofs, but for me in this course I want you to get a feel of applications of some simple physics problems which you are doing. So, my focus will be only on SU 2 and SU 3 of course, orthogonal groups I have discussed and I have given assignments on Lorentz groups and rotation groups.

This beyond this I am not getting on to in one semester I cannot do everything, but if anybody is interested can get on to and it is a compact way of seeing at the site what is the rank by looking at the diagram Dynkin diagram you can say what does an abstract Lie algebra it has rank (Refer time: 09:49) so on and then you have these notations. So, I am not going to go further on this, but at least for SU 3 what will be the Dynkin diagram? SU 3 the Dynkin diagram will be 2 simple roots.

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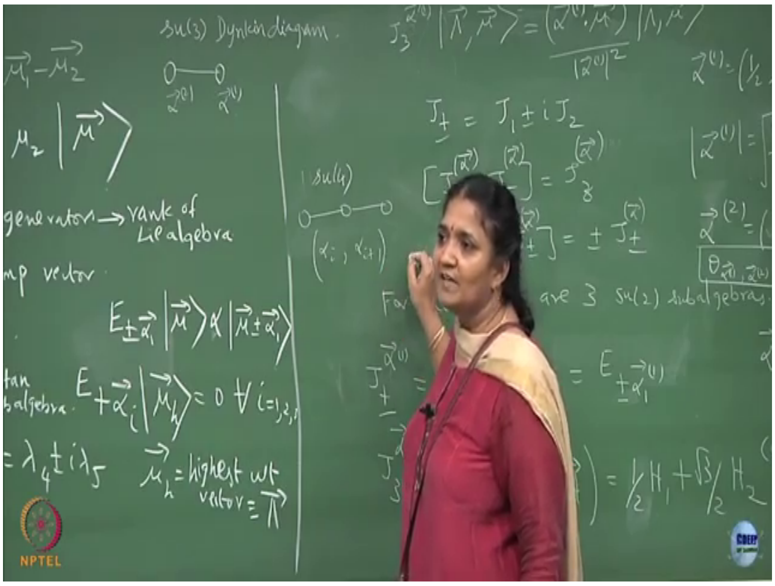

Which I am calling it as alpha 1 and alpha 2 and they have an angle; angle is 120 degrees and 120 degree angle is denoted by a single line and this is the Dynkin diagram for SU 3 ok. SU 2 what is the Dynkin diagram?

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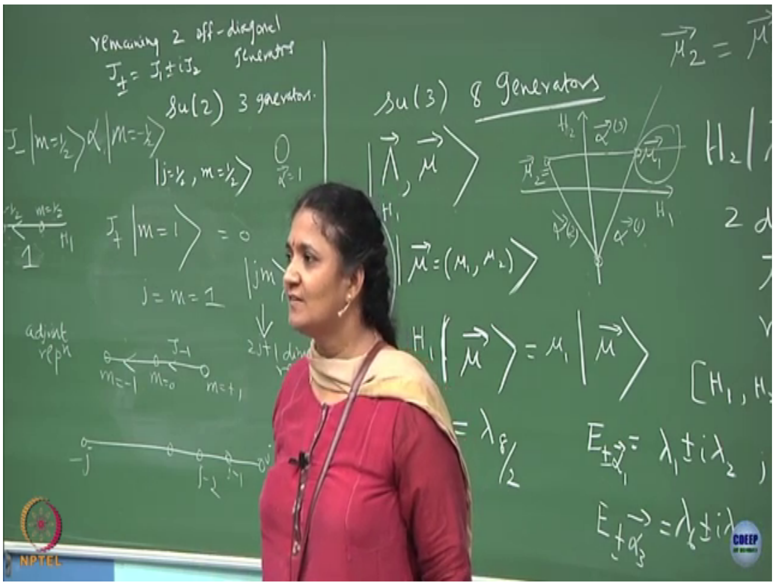

Just a single one with alpha vector to be just 1. That is it. So, this will be the what does it mean by talking about angle between 2 root vectors there is only one simple root right. So, go to SU 3 SU 4 you will have 3 simple roots and it will turn out that if you try and do it for SU 3 sorry SU 4 you will have 3 simple roots, but only the consecutive ones are connected. Angle between the third one and the first one is 0. This will be for SU 4 ok.

So, just looking at the diagram you can say that there are 3 simple roots and the angle between the consecutive roots alpha i and alpha i plus 1 the angle is 120 degrees and so on ok. This is where the Dynkin diagram is really you know just at a site you can look at an abstract algebra in this language fine ok.

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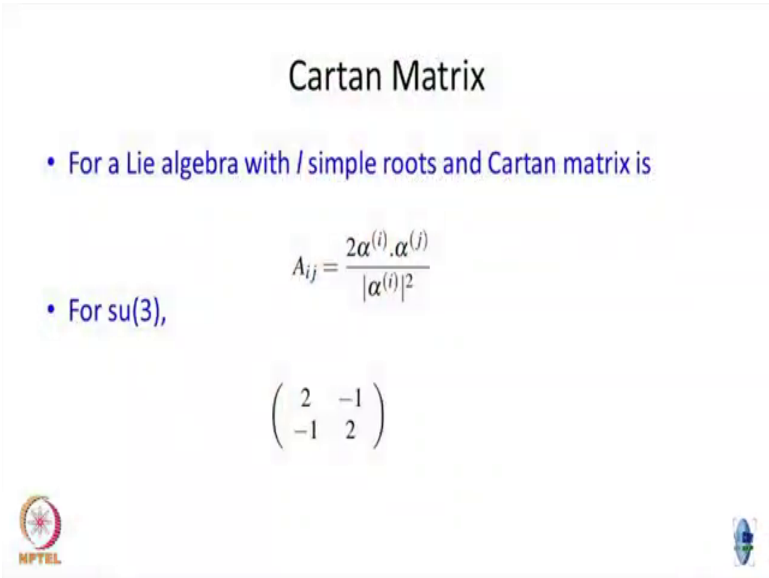

So, I am not going to get on to this is what I said the angle is given by different simple roots. You can write the matrix form of it and so on. So, I am not going to get into the, but the you can construct this Cartan matrix for the SU 3 for completeness you can sit down and write down the Cartan matrix between simple roots. Simple roots are alpha 1 and alpha 2.

We have already seen alpha 1 dot alpha 2 is minus half. So, 2 into minus half is minus 1. The off diagonal elements are minus 1 diagonal elements are numerator and denominator will cancel, but the 2 factor will remain that is why you have a 2 here. So, this is what is the Cartan matrix sometimes they write just the Cartan matrix sometimes they show it by a Dynkin diagram.

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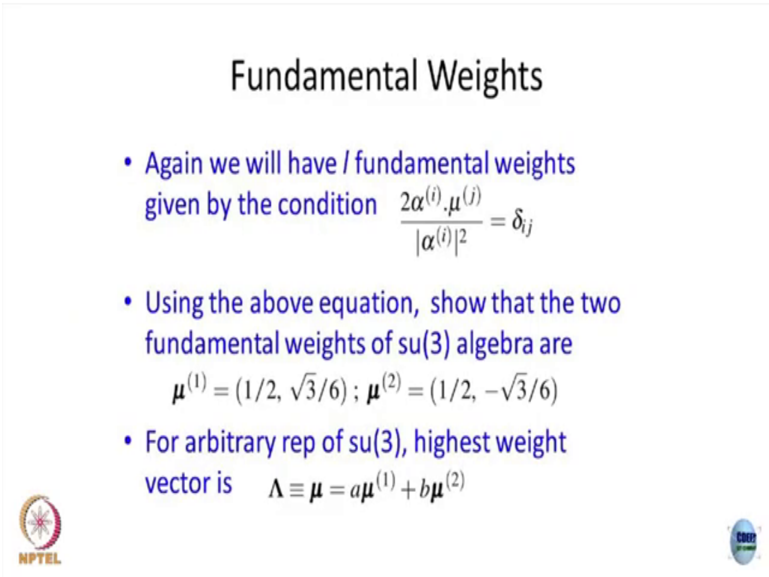

So, these are various ways of trying to convey what are you looking at an abstract Lie algebra ok. This is something which I will also need to explain for SU 2 and SU 3. Just like you had simple roots we also will have fundamental weight vectors ok. The number of fundamental weight vectors will always be equal to the rank of the Lie algebra. So, let me get to the fundamental weights now.

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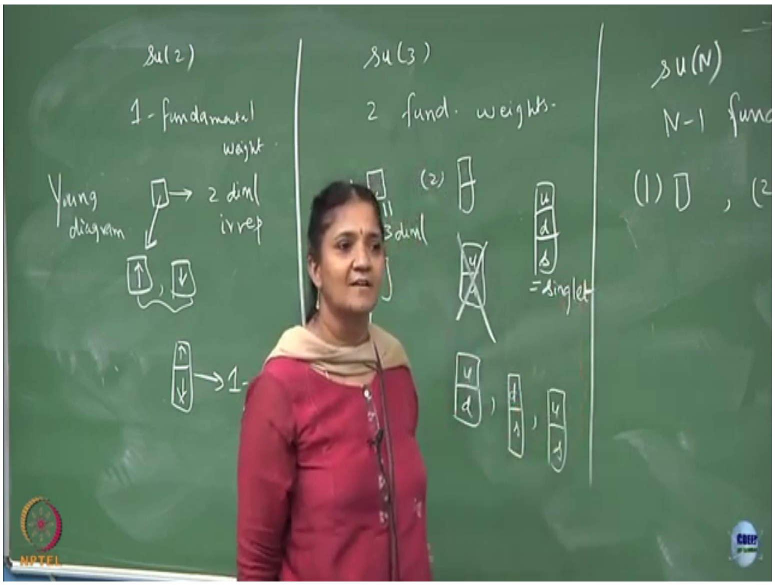

SU 2 1 fundamental weight, SU 3 there are 2 fundamental weights. I will also denote it by Young diagram now ok. This one fundamental weight I will denote it by a single box. The 2 fundamental weights you will have a single box that is 1, the second one will be 2 vertical box.

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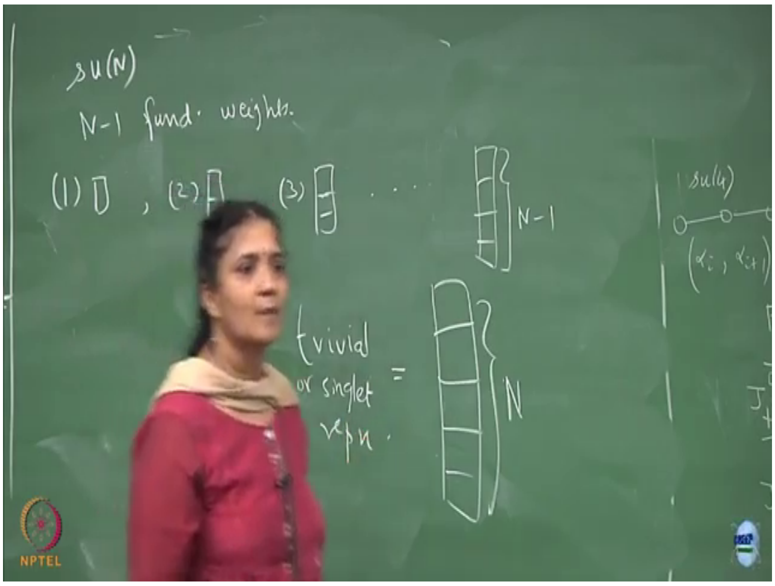

SU N what is the rank? N minus 1 is the rank. So, that will be the N minus 1 fundamental weights. I am going to denote it by so on. What will this be? You remember these diagrams I kind of when I was looking at the discrete group I was giving some definitions to these diagrams.

These diagrams when I put below what is that called antisymmetrizer right antisymmetrizer. If I put 2 vertical it is antisymmetrizer. This single box denotes my fundamental representation. In the single box I can put an up spin in the box or down spin in the box. So, this is what means it is 2 dimensions 2 dimensional irrep which is also a fundamental representation or defining representation shown in the Young diagram (Refer time: 16:42).

There are only 2 fundamental states with which I can play around. Here this means I can allow an up quark, I can allow a down quark, I can allow a strange quark. So, which means this is 3 dimensional or 3 states 3 independent state. I showed you by weight diagram also for the defining representation or the fundamental this diagram you can have 3 possible states clear.

Now, when I take this now when I take this I have to make sure that each box can be u d s and whatever is the entry on this box the entry on the other box should not be same. You agree. I cannot put is this allowed not allowed why because it is anti symmetric. What happens in the case of SU 2. Suppose I say this is allowed what does the state I can put here. If I put up here this has to be necessarily down. What is this?

This is a 1 dimensional irrep which is analogue of your unit representation which is an analogue of your unit representation. Singlet is what we call anti symmetric between the 2 boxes ok. So, this is 1 dimensional it is not really relevant. You can add this like any number of such singlets to a collection of non trivial spin. What will happen if you add a singlet to another spin half anything happens it will still be a spin half system ok. So, this state is not important in the context of SU 2. That is why when I write the fundamental weights.

What about SU 3 is that non trivial or it is also a singlet? In SU 3 what are the possibilities here? You can have up down. What are the possibilities somebody? This is one possible then down strange, u strange, but this ordering is unimportant. Why is it an important which is already antisymmetrized you do not need to worry about the order.

This turns out to be a 3 dimensional vector space. 3 states are possible for it, but that is nontrivial it is not like a unit representation. If I want to look at this what happens, if I put up here I have to put d here, I have to put s here. This is again a unit representation or a singlet that is why it is trivial. So, in this case what will be trivial that will be ok. So, I am just giving you some kind of a diagrammatic representation for the irreps of SU 2 SU 3 and SU N. Fundamental ones fundamental means what I will say is an any arbitrary state.

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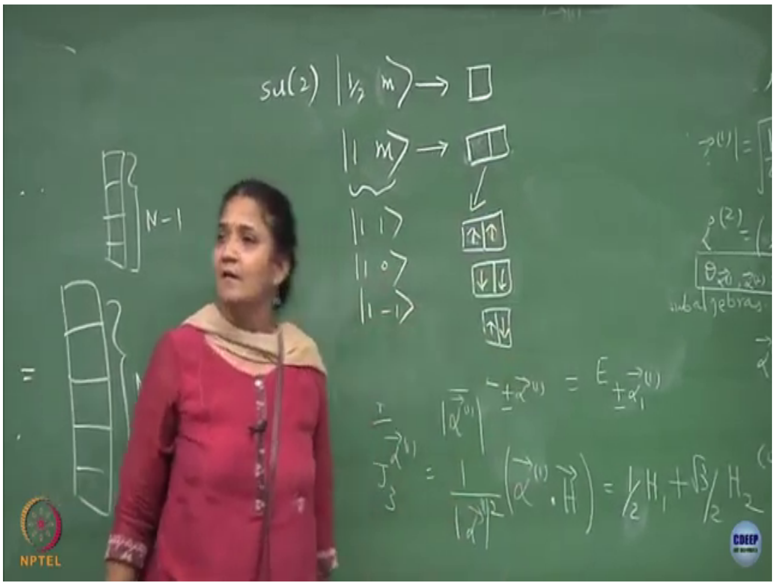

So, suppose I want to write. So, half or half m this I am going to associate with single box in SU 2. You understand what I mean it is a 2 state system I will represented by this. Suppose I take 1 m this is 3 dimensional. You all know that right this will be having 1 1 1 0 1 minus 1 clear. This I need to represent using the fundamental weight. What I will do is I will say that it is 2 times the fundamental weight which I will diagrammatically denoted by 2 single box.

This means it is 2 times the fundamental weight by fundamental weight its a single box. So, 2 times the fundamental weight and this diagram means what symmetric. This diagram means it is symmetric which means what are the allowed here. I can make it u u, I can make it. I am working with SU 2 sorry I have to put up up.

Student: (Refer time: 22:58).

What else. Down down is allowed and what about this. These are the 3 states which gives you this diagram ok. So, that is why this diagram is the Young diagram way of looking at the irreducible representation which is 3 dimensional of SU 2.

Now, you tell me if I have to put an arbitrary spin what should be the diagram? Spin half it is 1 box, spin 1 it is 2 box. So, tell me how is it defined for spin n by 2 it will be n boxes. And everything has to be horizontal because you have only this fundamental way to play around and n by 2 times that single box which has to be concatenated. Now the same thing you play on the SU 3. What are the fundamental rates you have you have this and this. Now you can play around.