Group Theory Methods in Physics
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Lecture – 45
Rank, Weight and Weight vector

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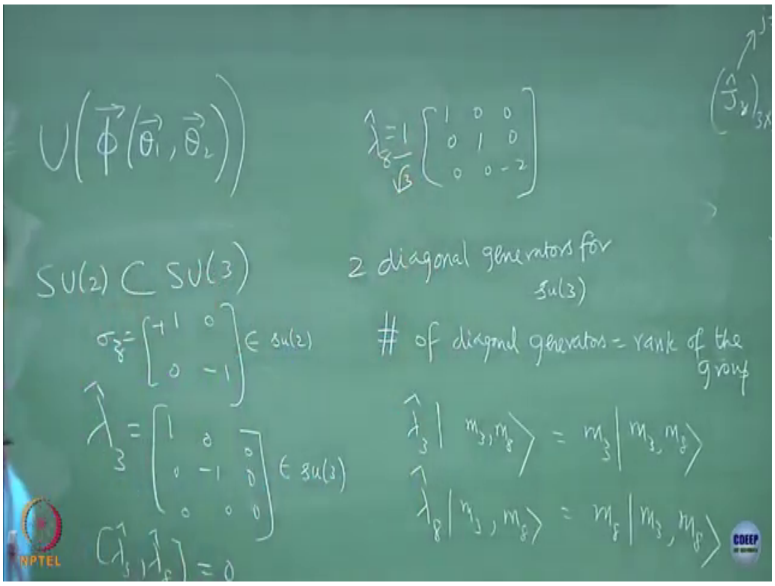

I have just given you a comparison between SU 2 and SU 3. So, SU 2 had 3 generators one of them is diagonal; SU 3 will have 8 generators - the lowest non trivial dimensions of those generators has to be 3 cross 3, inside that the SU 2 algebra should sit in. So, you still have one diagonal generator which is coming from the SU 2 because this part is sitting inside that, and you also have one more which you can construct.

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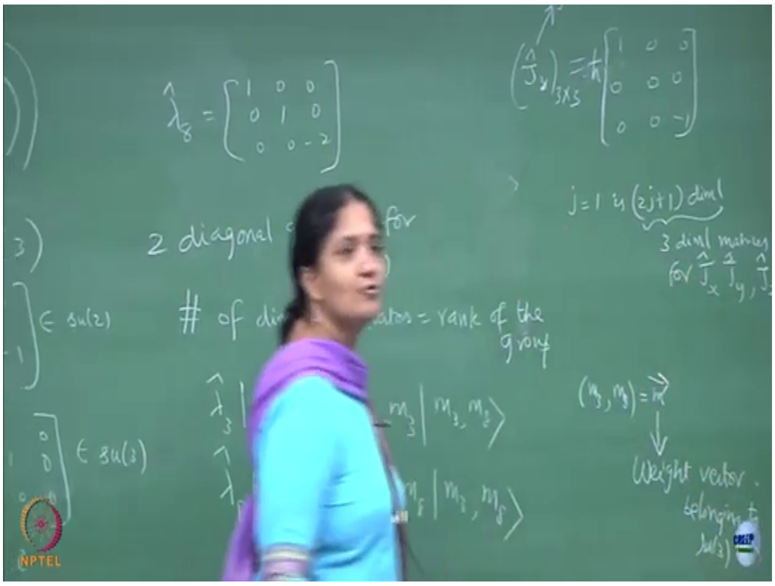

So, there are two diagonal generators for SU 3 algebra, ok.

Student: (Refer Time: 01:10).

Yeah.

Student: (Refer Time: 01:13).

I did find out lambda 8, I just said I want to construct with integers put on it real numbers, in such a way that the trace is 0 and Hermitian.

Student: (Refer Time: 01:29) multiple ways.

What are the multiple ways?

Student: (Refer Time: 01:31).

You should not increase just arbitrarily high numbers; I want the numbers to also be low. You are saying, I will put 100, 50, yeah not that way. You have to put in with.

Student: (Refer Time: 01:43).

Lower number of integers to start with and see whether you can construct it ok. With lower integers I do not think you can do any other way. Can you try it out? Over all negative sign is not a new one, relative numbers if you can put something else but they have to be integers.

Student: (Refer Time: 02:05) yes.

Because they are some things which is capturing for me some kind of quantum numbers in some fundamental units, and I want them to be integers. If you put in this constraint and you do not want to.

Student: (Refer Time: 02:21).

The way you are saying is I will put 500, 500 and minus something right, yeah, but then you can still see that there will be just a 500 scaling factor out of it. Scaling factor is unimportant. What you have to get is a non-trivial matrix. So, you have only two diagonal generators in that sense ok, and the number of diagonal generators for the lie algebra is what we call it as a, this is what we call it as a rank of the lie algebras ok, rank of the group is also another way of saying.

So, if I say a rank of the group is 2, then you should know that the lie algebra should have two diagonal generators. What does that also mean if I write a state in the case of SU 3 just like I wrote it for SU 2 I should be able to write the eigenvalue corresponding to lambda 3, let me call it as m 3 eigen value corresponding to lambda 8, ok. And you will have some analog of the Casimir operator let me not right to say what it is, but definitely the magnetic quantum number will have two values depending on lambda 3 acts it will give you m 3 right, lambda 3 will give you m 3 times m 3 m 8, lambda 8 will give you m 8 times m 3 m 8. You agree?

This is the slight variant it starts happening once you go to the states in the case of SU 3, ok. So, many times this m 3 and m 8 together they write this as a is denoted as a m vector. So, it is m 3 and m 8 are the two components of the m vector. So, if you operate lambda 3 on the m vector, it pulls out the first component.

If lambda 8 acts on the m vector, it is a two-dimensional vector, you get the second component. This is a shorthand notation of right, and this is what we call it as a weight vector, ok. So, the m vector is called as a weight vector. What happens in SU 2, SU 2 it is just a one component vector, because it is only picking up one of the diagonal generator, it has only one diagonal generator, so that is one component vector.

If you had two diagonal generators, the weight vector will be two components; if there are three diagonal generators, you will have weight vectors to be three components and so on, ok. So, this is the formal notation the first non-trivial group is SU 3 where you will start seeing that you can represent the analog of your magnetic quantum number by your magnetic vector, where the first component will be like your magnetic quantum number because that is the lambda 3 eigenvalue. The second component will be like the new diagonal generator for SU 3, clear.

Student: (Refer Time: 06:38), diagonal generator (Refer Time: 06:42).

Diagonal generator because the diagonal generators commutator bracket will be 0 right, because they are diagonal you can show that lambda 3, lambda 8 is 0. So, which means you can write a state to be a simultaneous eigen states of both lambda 3 and lambda 8, and I am compactly trying to write the lambda 3 lambda 8 eigen values as a two component vector, ok.

So, this is just a two component vector. Weight vector belonging to a SU 3 belongs to SU 3, the two component, ok.

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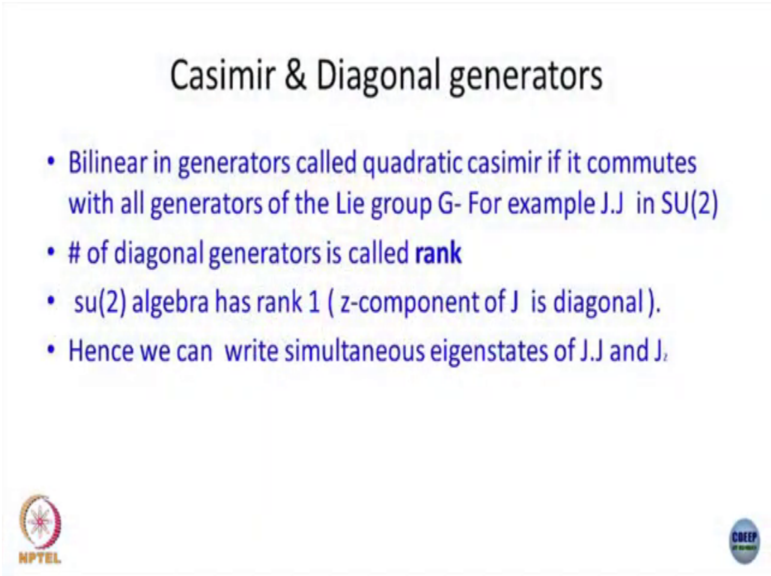

So, let me come to the things which I am saying on the slide here. So, number of diagonal generators is called rank SU 2 has rank 1 z component of J is diagonal and hence we can write the simultaneous eigen states of casimir operator and J z in the case of SU 2, there is an analog of casimir operators. But let me not get into it for SU 3. So, again you can construct bilinears here also, you can take lambda 1 square, lambda 2 square up to lambda 8 squared, and you can show it to be commuting with lambda 3 and lambda 8, but I am just going to confine myself to the magnetic quantum numbers, ok.

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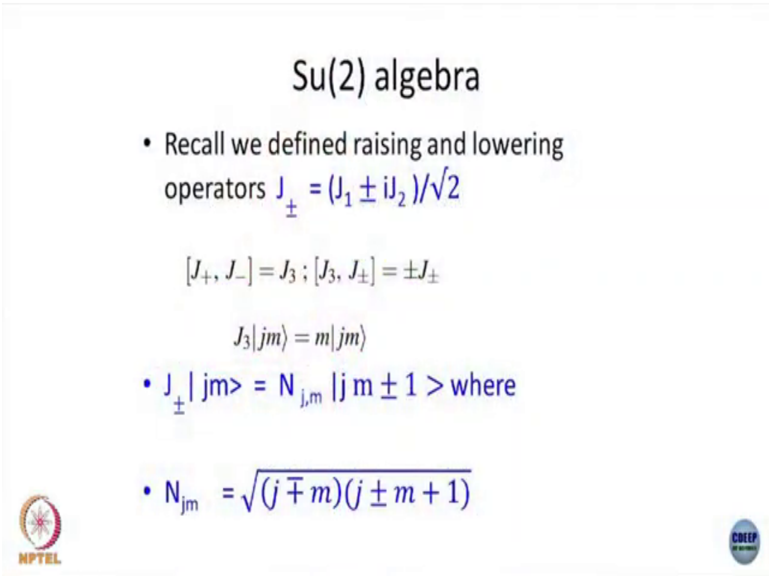

So, recall your SU 2 algebra where for comfort you define raising and lowering operators which is Hermitian conjugates of each other. So, if a J 1 is Hermitian, J 2 is Hermitian, but J 1 plus i J 2 is not Hermitian, but J 1 plus J I J 2 the Hermitian conjugate of it is J 1 minus I J 2, these things you know. And we could rewrite the same algebra which I wrote in the beginning for J x, J y, J z, right.

You can rewrite it in terms of J plus J minus and you know this is a closed algebra again all this equations together. There is some slight probably a different in notation from quantum mechanics there is a 1 over root 2 here, some books follow without a root 2, but then you will get a twice J 3 and so on, so that is a matter of normalization, ok.

This is a normalization I am following and J 3 anyway I have said that J 3 will give you I am suppressing the h cross I am putting h cross to be 1. So, this gives you some m which is some value which could be half odd integers in the case of SU 2, right. For spin j s spin half, it is half; if it is spin 1, it is 1, so it could be integers or half integers. And then the ladder operation will take you from that is why the definition of a ladder operation, you can prove all these things using this algebra.

I am sure you would have done it as a quantum mechanics course; here, I am just trying to give you the final result which you have learnt. Basically if you take this magnetic quantum number m, the plus raising operator will take it to m plus 1; minus we will take the magnetic quantum number m to m minus 1, ok. So, there are two operators which helps you to go from one state with magnetic quantum number m to m plus 1 or m minus 1. And then this coefficient's also can be determined purely from this algebra and they turn out to be related to j minus m, j plus m, ok.

So, this is something which you all know not going to derive this. My only requirement is that I want you to understand the fact that you had a diagonal generator, and then the remaining generators you made it into a complex conjugates of each other ok, which are off diagonal, but you try to make it into a complex conjugates. So, this is the theme which is very important in group theory, ok.

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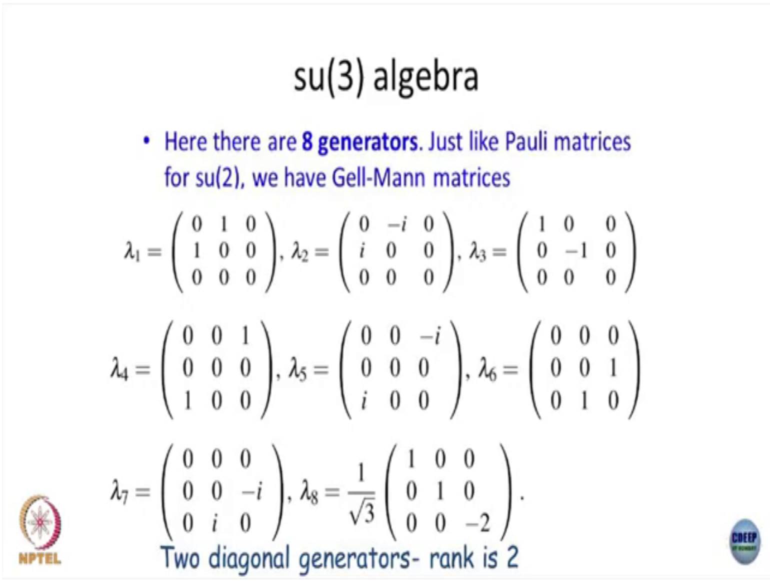

So, let me try and say for the su 3 algebra whatever I said for su 2, there are 8 generators. The explicit form you do not need to memorize, you can write it down on a sheet you know to see that lambda 1, lambda 2, lambda 3 is diagonal and similarly lambda 8 is diagonal. These coefficients are you know put on purpose to take care of some of the quantum numbers which are seen experimentally, ok. Just like I wrote for the Pauli matrices half h cross just to take care that the experimental value of the J z quantum number is half h cross or minus half h cross.

Similarly, these coefficients come because of some experimental detail, but as of now even if you keep this to be you know some kind of a constant these are the eight matrices where lambda 8 and lambda 3 are diagonal matrices, so the rank of the su 3 algebra su 2. So, I have confined to su 2 and su 3, but whatever I am saying for su 2 and su 3 goes systematically to su

4, su 5 and so on. I hope you appreciate that fact. So, the su 4 you will have how many generators, somebody?

Student: 15.

15 generators ok. You can check it out, 1 5 generators, 15 of them and then how many diagonal will be 3 of them. So, you will always have for any su n and square minus 1 generator and you will have n minus 1 diagonal generators. So, the rank of the su n will be n minus 1, ok.

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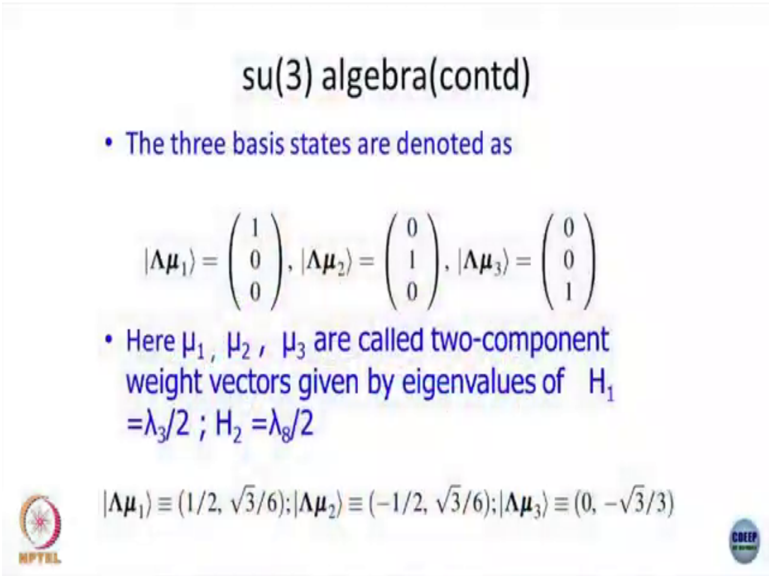

So, just like I wrote the up spin and a down spin ok, so I am going to do this here for you. The lowest dimension which I am going to write is called sometimes as the defining representations or fundamental representations. So, let me put both the cases here.

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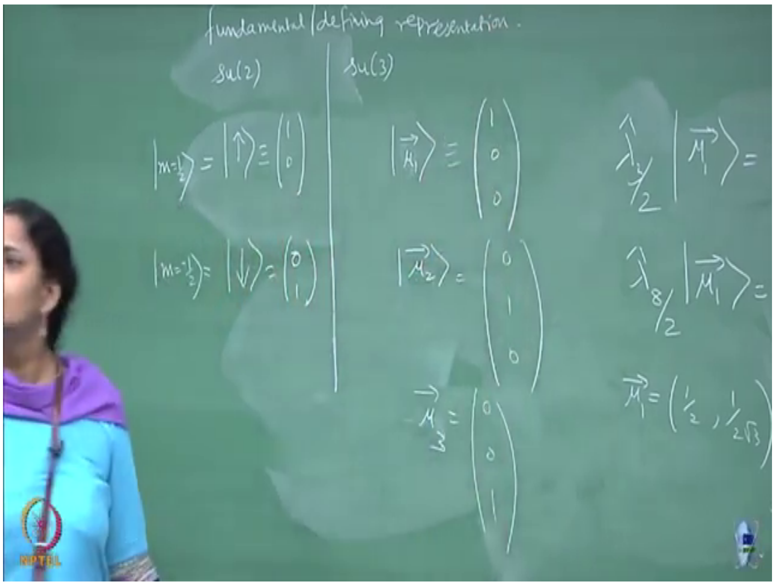

su 2, su 3, you can have states which is up spin which you denote it as 1 0 and down spin which you denote it as 0 1, ok. And this is also equal to m equal to half, I am suppressing h cross this is equal to m equal to minus half ok, this is su 2. Su 3, you will have m 1 vector which I am going to denote it as what is m 1 vector the two components of the m 1, it has two components; first component will be the eigen value of lambda 3, second component will be the eigenvalue of lambda 8, ok. Here this m was an eigen value of jz alone or sigma z.

Now, it will be that it is a simultaneous eigen value of lambda 3 and lambda 8, clear. So, m 1, m 2 and m 3 ok, this is what I am calling it as mu 1 in the slide, I am calling it as mu 1. Let me just follow the notes which I have mu 1, mu 2 and mu 3.

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Now, try to find operating your lambda 3 by 2 on mu 1, do that matrix. What will that give you? Half on mu 1, right. Are you all with me? mu 1 denotes 1 0 0, so the mu 1 on the mu 1, if you operate this, I am going to use lambda 3 by 2, so you will get a half.

Student: (Refer Time: 16:23).

Just the convention this is because of su 2 sits in that and I am looking at the fundamental representation which should also have the same half eigenvalue that is the reason nothing more. In fact, all the generators I will scale it like that ok.

Student: (Refer Time: 16:44).

This is just an overall scheme should not really matter. What is this going to be, lambda 8 I wrote somewhere, but then there was also this convention of putting a normalization here, let us follow the normalization. So, tell me what happens here that will give you, ok. So, what is the mu 1 vector explicitly is that ok. So, this is nothing but half, it is half root 3 is what I get right, but this is just a matter of normalization. We fix the normalization and we will get coefficients.

What I am trying to give you the fact is that formally I am writing it as a weight vectors, the explicit two components of the weight vectors are the eigen values of lambda 3 and lambda 8 up to some normalization. If you use that, then you would define your mu 1 vector to be this. So, this is all I am trying to do, yeah.

Student: (Refer Time: 18:06).

Yeah, any questions on this, it is fine.

Student: (Refer Time: 18:15).

Yeah. So, one way of seeing is that if you square this, you do get 1 plus 1 plus 4 which is 6, right. So, I have to put a root 6, and then I also have another half is coming up. So, there is some reason which is put in we will come to it or another way of saying is I want to get my proton charge to be plus 1. You understand what I am saying. If I want the proton or an electron, electron of course is fundamental; it is not going to be a composite which is going to be by tensor product.

If I want my proton charge to be convention with my experimental evident, I need to fiddle around with these normalization constant and Gell-Mann has put in these things, ok. So, I am not doing anything, ok. This is good, ok. So, the three basis states of the lowest non-trivial representation of su 3 algebra which is called also I said defining representations, ok. All these things which I am writing are sometimes called in the literature as fundamental or defining representation, ok. So, fundamental or defining that is the lowest non-trivial representation for su 2, it is 2 cross 2; su 3, it is 3 cross 3, ok.

So, let me just try to say what I am trying to put in here. This lambda is analog of your quadratic Casimir will come to it at some point; mu 1 is bold faciar which refers to two component vector in this context, and explicitly the corresponding basis state in the three-dimensional vector space which I take is just the usual 1 1 0, 0 0 1 and 0 0 1.

Corresponding to this you try and find out what are the eigen values of lambda 3 by 2 and lambda 8 by 2, and you can show that the mu 1 vector is nothing but half comma root 3 by 6, which I have explained it now. But, you can show mu two will be minus half, the minus half is coming because of this, and root 3 by 6. And the last one mu 3 has 0 as the first component, and the second component is minus 2 by 3 by half, so it is minus 1 by root 3, so that is what we get as minus root 3 by 3.

So, these are the analog of your magnetic quantum number in the case of spin half particle, the magnetic quantum number there are two values minus half or plus half or my downstate and upstate. Now, you have three fundamental states.

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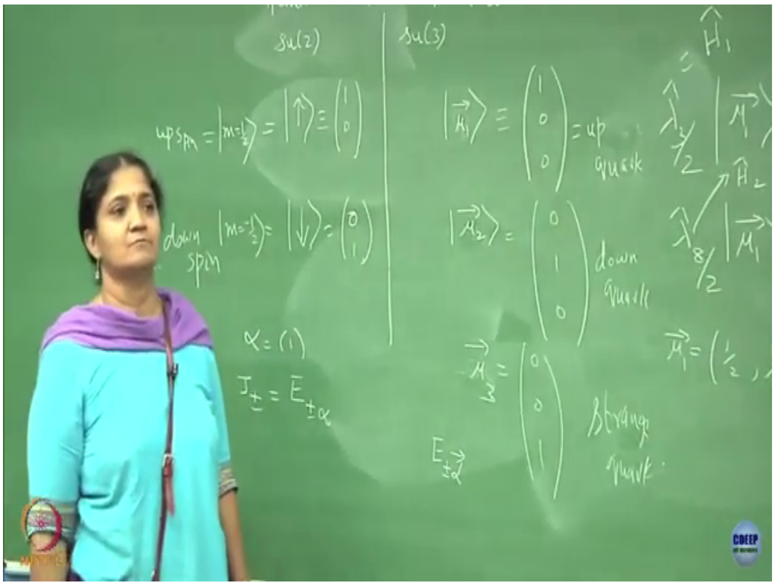

There are three fundamental states, and this is what he attributed to saying that nature has three fundamental quarks, he called it as u quark, just like this is called as up spin and down spin.

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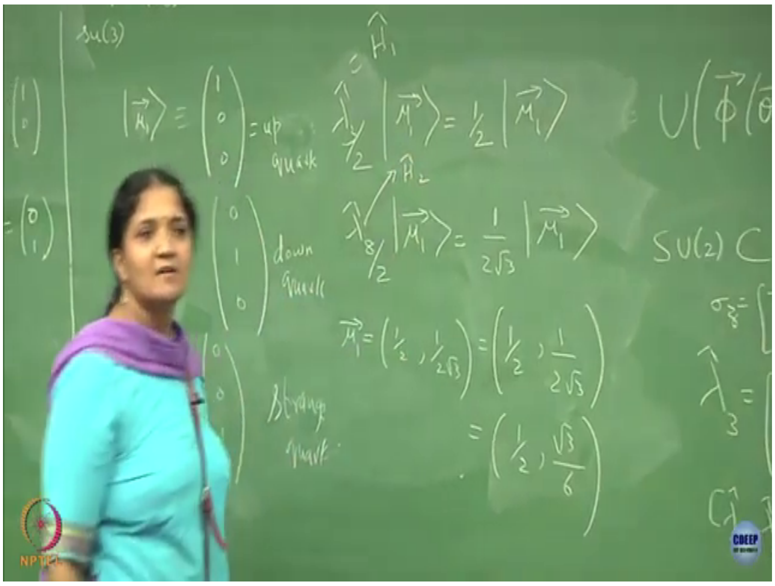

He called it as u quark which is also up quark, down quark and strange quark, strange quark. So, each one is a, is that clear. So, there are three fundamental states of su 3 which is attributed to and the corresponding values are going to give you something which is physical, we will come to it, but there are two diagonal elements. So, you will have two eigenvalues which is put together which is called as a weight vector, ok.

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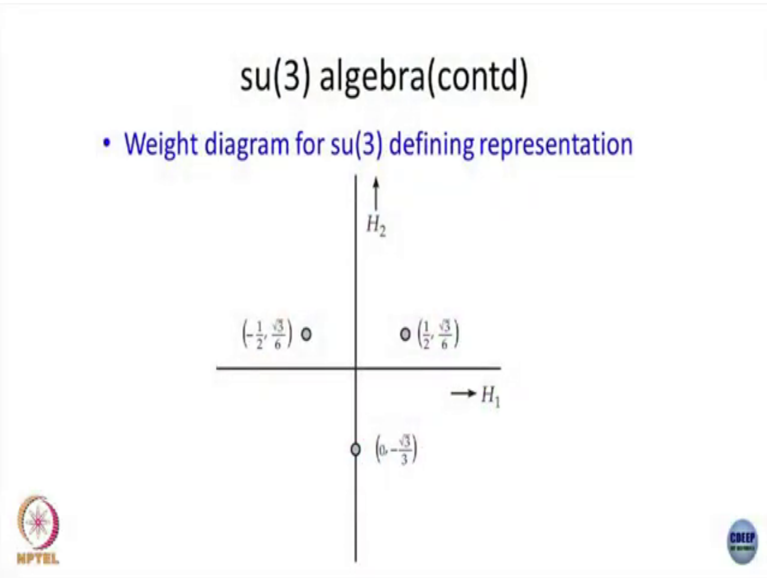

So, I have drawn it also for as a diagram the three fundamental states. This blob the circle is what we call it as a up quark state. Another one it is called as a down quark; basically, it is a coordinate in the weight vector space, ok. Weight vectors has two components that is why I am drawing a two-dimensional plane. If it was su 2, what is the situation, su 2 does not have the second one.

So, this one is not there, and only half and minus half, the second component will not be there. This is clear, so it is a weight vector concept here that as you increase the number of diagonal generators that I weight vector diagram will become three-dimensional, four-dimensional and so on, but at least algebraically I can write down the plotting I could do this way definitely for su 3 in a two-dimensional plane. The axis are the H 1 eigenvalues and the H 2 eigen values, and H 1 and H 2 I am calling lambda 3 by 2 as H 1, and lambda 8 by 2, I am calling it as H 2 ok, this is the notation fine.

Student: That is (Refer Time: 24:29).

Yeah. So, see the thing is most of your protons and neutrons were conventional, then they started seeing new particles, and they wanted to give some name to it. And then they said oh, it will involve some new exotic particles, and they probably called it strange, ok.

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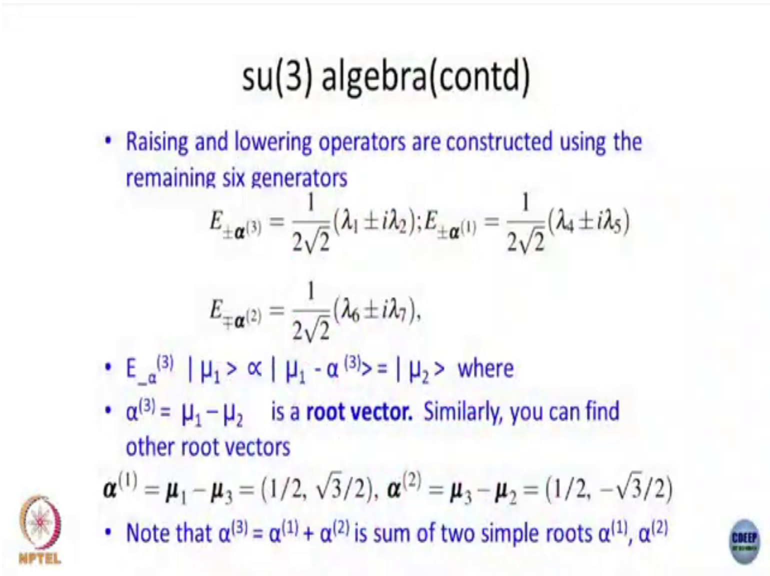

So that brings me to a summary of what I want to say for the su 3, ok. Just like you had J plus and J minus, J plus and J minus the role is what. If I do a J minus on this the m quantum number shifts by 1 unit, plus or minus 1 unit depending on whether I do J plus or J minus, you all know that. So, I am going to call that as some kind of a shift which is just a single number 1 and J plus and J minus, I will call it as plus or minus alpha, this is a single number in the case of su 2.

In the case of su 3, what should J plus J minus do, it should take from here to here or here to here, right. And you know that this is a weight vector, I would like to write that as a plus or minus corresponding number which is a two component vector or in other words when this acts on mu 1; so, let us do that.

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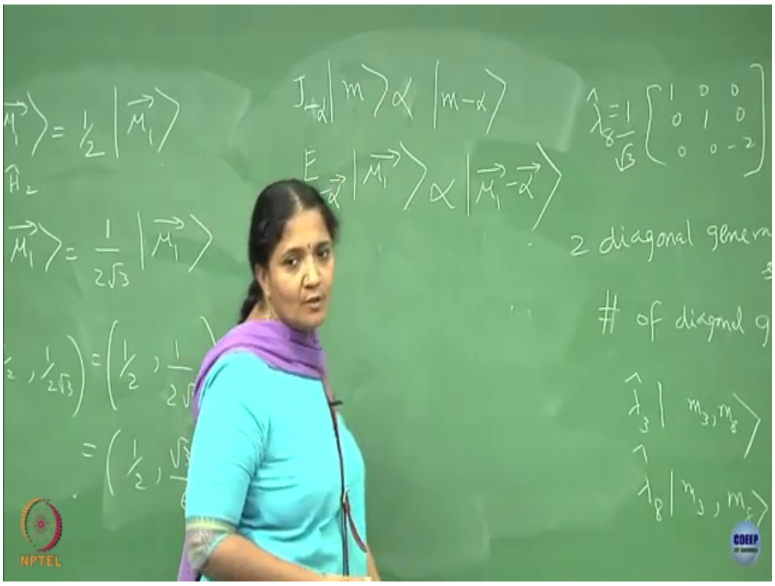

So, if you have J plus minus on, let me do it J minus, J minus alpha on m will give you m minus alpha, is that clear. Now, I am going to say that E minus alpha on mu 1 is going to give me fair enough, a single number is promoted to a two component vector for su 3, and then these operations then we just know J plus minus because it was just increasing or decreasing by 1, we did not even need to put a plus 1.

But in general here, it would be a non trivial vector which will take you from mu 1 to mu 1 minus alpha, and this should be some definite one of these three states. Because J minus 1, this will give you this similarly J minus 1 that should give you one of those states I will come to it. But, this is the formal ladder operator notation where the weight vector decreases by some unit. And what does this vector we need to figure it out, ok.

Student: (Refer Time: 28:11).

Two component, how many alphas are there is what, there are remaining how many generators are there out of 8, lambda 3 and lambda 8 are diagonal. The other generators are 6 of them are there. Now, I have to make just like I take J 1 and J 2 which was remaining, I made J 1 plus 1 minus i J 2. Now, out of the 6, I have to take lambda 1 plus or minus i, lambda 2 lambda 4 plus or minus i lambda 5, then you will have lambda 6 plus or minus i lambda 7.

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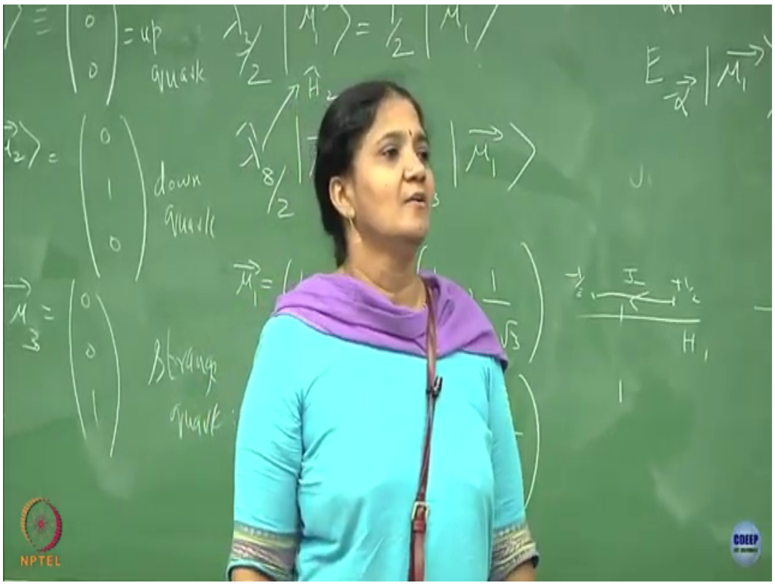

So, there will be three such ladder operators going only in this direction clear you can only go in this direction. If you are here with plus half, the ladder operation J minus takes you to minus half clear.

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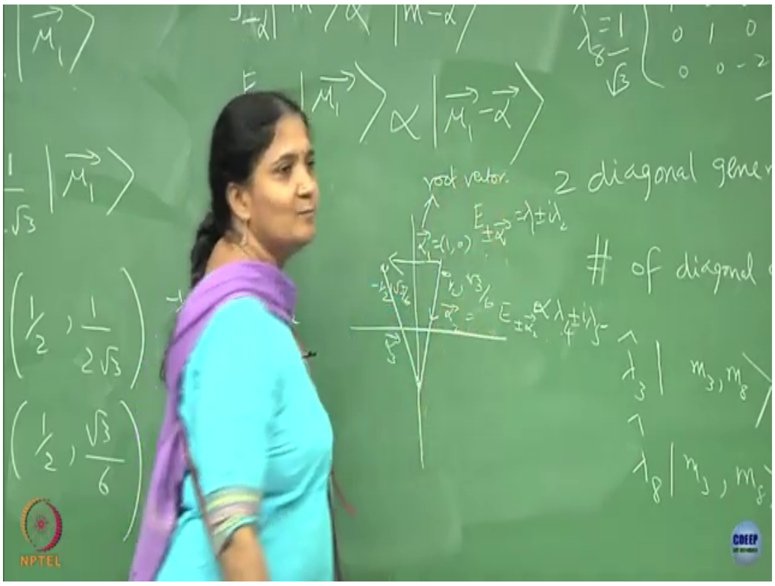

But now su 3, you saw that there are three states. You can have one ladder operation this way which is the conventional su 2 1, which takes you from half to minus half. So, you can see that this vector alpha has to be 1 comma 0, you understand?

So, this decreases by 1, but the other one remains the same, the root 3 by 6, there could be another ladder operation which is going this way which is your alpha let me call it as alpha 1, this is alpha 2. And another one which can go this which is alpha 3; so, one of them here is going to be the corresponding E plus or minus alpha 1 is nothing but lambda 1 plus or minus i lambda 2.

This will turn out to be E plus or minus alpha 2 will turn out to be proportional to lambda 3, sorry lambda 3 is not there lambda 4 plus or minus i lambda 5, and one more which is the; does not involve lambda 3 and lambda 8, but the remaining six generators will form

Hermitian conjugates. And it is very beautiful seeing it in a diagram ok, is this just clear. We will come to it, I will repeat it again, but I am not going to redo the lambda 2 and lambda 3 what I did here, but do not forget this, there is one more week next Thursday only, but this is what we will continue from how to do this, and then generate for a general lie algebra how people look at these things.

So, these are also in some sense, some kind of a vector, and this vector generates for you to go from one weight vector to another weight vector. So, the one which generates for that is called as a root vector; so, this is what we call it as a root vector, ok. I explained today weight vector. I am also saying that to go from one way to another, the ladder operation or these Hermitian conjugate operators which are raising and lowering operators can be constructed for su 3, and there will be three of them and they are going to give me given by these root vectors. So, I have to give you what is a root vector, what is a weight vector and then you can play around in doing all the matrix representations and so on.