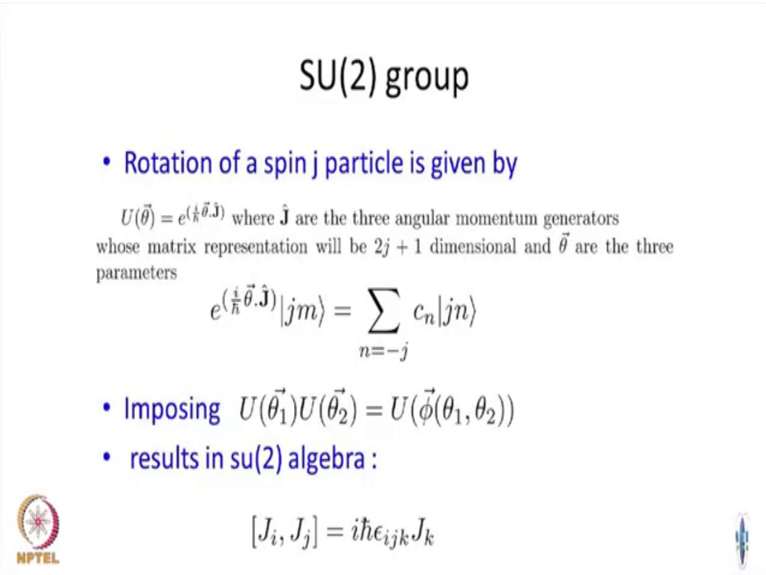
Group Theory Methods in Physics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 44
SU(2) and SU(3) groups

(Refer Slide Time: 00:19)

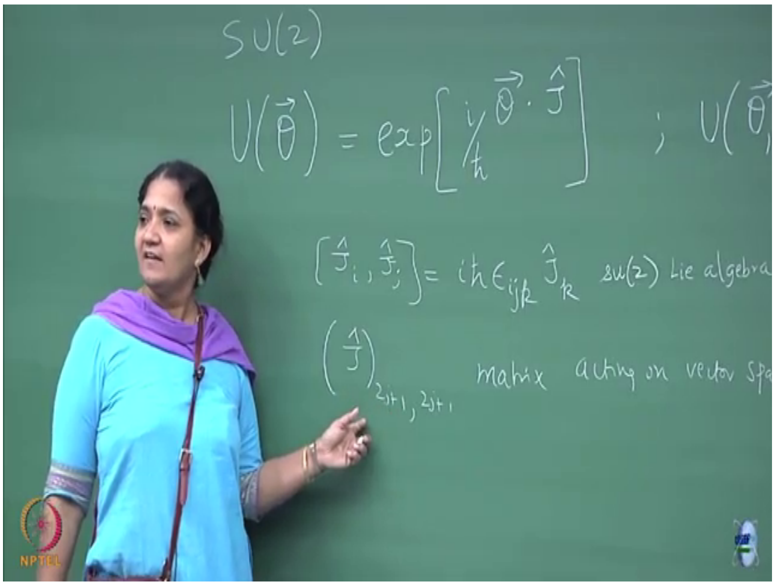


So let us start. So, I am going to confine now to S U 2 group. S U 2 group is now by now you should all know the usual notation of group notation; S denotes the group element should have determinant plus 1, U is for the unitary groups, 2 is to denote the first lowest non trivial dimension of the representation of the vector spaces two dimensional.

The corresponding matrices will be 2 cross 2 matrices and the number of constraints if you subtract 2 cross 2 matrices, unitary matrices; means it should satisfy unitarity condition and determinant equal to 1. So, you will get the number of parameters which is number of real elements in the matrix. So, they are only 3; that is why you have theta vector is formally written to remember that there are three components theta 1, theta 2, theta 3.

So, U of theta this is for S U 2, group once it is a group I try and put it to be capital S capital U and 2. So, this will have an exponential form it will be an exponential form, i by h cross theta dot the generators; I said know number of parameters will always be equal to the number of generators.

(Refer Slide Time: 02:07)



In the case of rotation, I wrote it a orbital angular momentum, here I am going to write it as a total angular momentum; which includes spin angular momentum and orbital angular momentum of a particle. So, this will be your general element where J's are the generators; this cap is generally written for a linear operator or you can even write the matrix representations for it, depending on which vector space it is going to operate, ok.

So, this is the familiar algebra of the generators; algebra is denoted by small s small u and 2 ok, this is s u 2 Lie algebra. So, how do you define this explicitly, the way to see it is that; you could take to multiply two such matrices.

(Refer Slide Time: 03:17)

I am not going to do this for you, take two such matrices ok. What is the group property required, if you take a product of two such matrices? It should again be an s u 2 matrix, but parameters are not just going to be additive. When will it be additive? If you do a rotation about an axis which is the same, then you can add it up.

But if you do rotation about theta 1 vector which is in a different direction and theta 2 vector in a different direction; it will give you again a unitary matrix, it depends on phi, ok. This is also a vector, but it will depend on theta 1 and theta 2; it will be from what exact functional form I will not know, but depending on what is the value I can find it, ok.

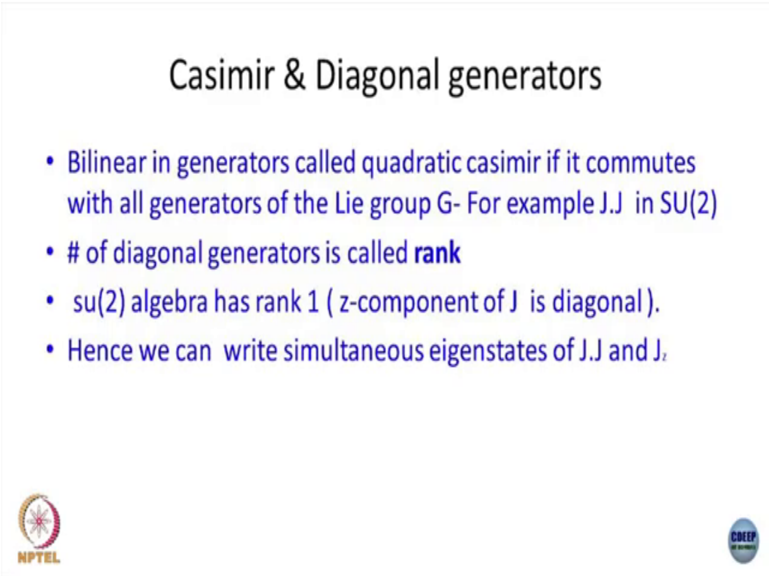
So, use this, use the infinitesimal P's and keep up to order theta 1 theta 2 and see what you get and you will end up getting this algebra for the generator. Just like I you did for the orbital angular momentum, you can do this for this case. You see that, the dimensionality of the vector space depends on 2 j plus 1 dimension like this way. So, I said that J hat is 2 j plus 1 cross 2 j plus 1 matrix, acting on vector space which is given formally as you all know this. You write it as j and m; we will come back to why we add two numbers here.

But this m value, so this m takes values from minus j, minus j plus 1 up to j. So, there are 2 j plus 1 states on which this 2 j plus 1 cross 2 j plus 1 matrix acts, you all know this.

So, those are if j is half which is your lowest two dimensional vector space, which is up spin and down spin. So, someone interestingly pointed it out that in your discrete groups the character tables tells you that, you have only up to certain dimension irreducible representations, right. So, like if you take a tetrahedral group, you cannot get a five dimensional irreducible vector space. In the case of continuous groups for i j you still get a irreducible representation; but that irreducible representation acts on the corresponding the vector spaces.

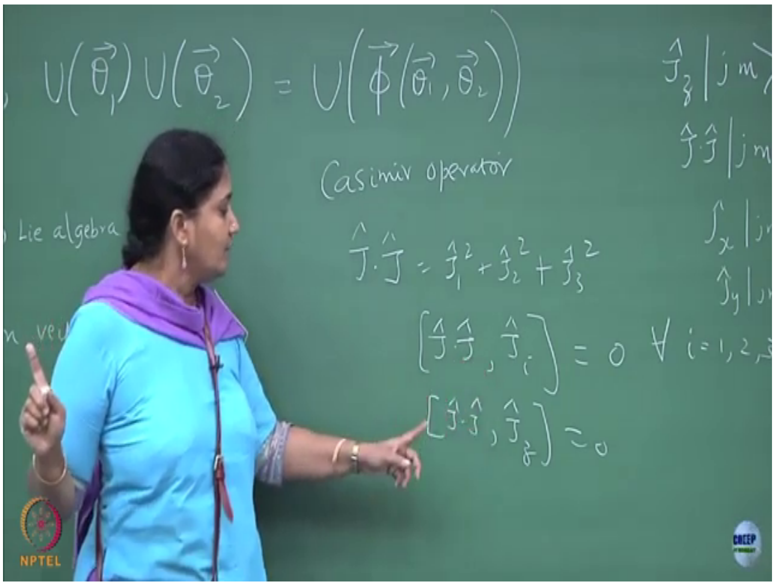
There is no limit to, there is no analog of character table here; but we will see how to get irreducible representation. Just because I write a 3 cross 3 matrix it need not be an irreducible representation. You can always, if you are able to find a matrix with block diagnosis then it is not an irreducible; but analog of the character table you can allow all possible dimension irreducible representations for the continuous.

(Refer Slide Time: 07:19)



So, why are we using this state is something which you have learnt in quantum mechanics, that is what is called as a Casimir operator. What is the Casimir operator? Casimir is bilinear in generators and it should commute with all the generators of the Lie group. So, for example, this J dot J is bilinear in generators, right. I can write it as J 1 squared plus J 2 squared plus J 3 squared.

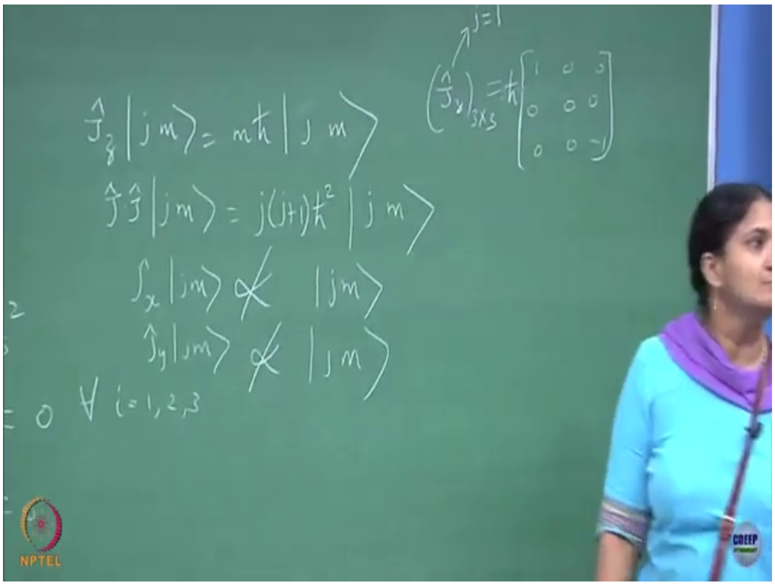
(Refer Slide Time: 07:49)



And J dot J you can see with any of the J i’s it is 0, for all i’s, ok. So, this tells you that you can write a state which is a simultaneous eigen state of J dot J as well as any one of the J’s; the reason why any one of the J’s is because, amongst the J 1, J 2, J 3 the commutator is non-zero. So, you cannot simultaneously diagonalize J 1 as well as J 2 or J 3; one of them can be simultaneously diagonalized with this one.

In all the quantum mechanics textbooks we use a choice which we call it as J z, which is this is just a choice. You can write a new book or a new you know set of notations where you can call J x to be your axis where this is happening; but usually you are following even in spherical coordinates z to be the axis which and we choose the J z eigen values to be m, right

(Refer Slide Time: 09:29)



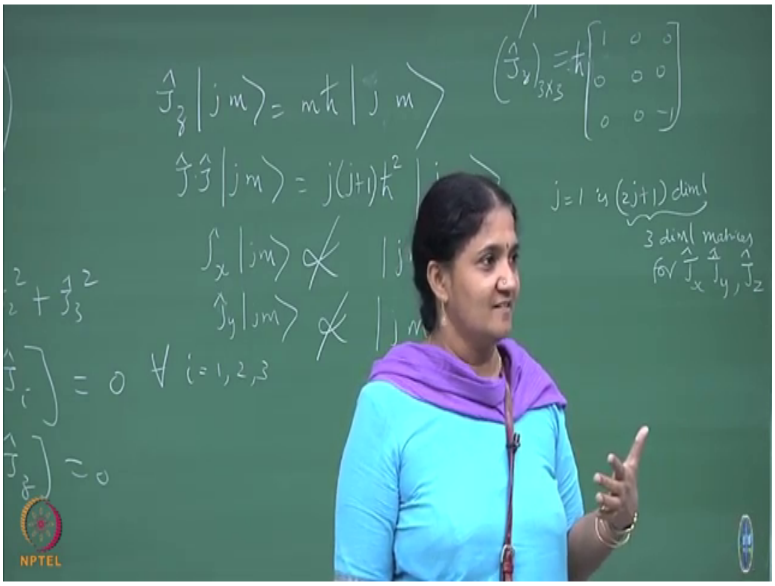
What I mean by this is that, J z on j m will be m h cross j m ok, is that right. And J dot J on j m, you can find the number; but if you find it explicitly it turns out to be this, both on eigenvalue equations. But if you do Jth J x on j m, it is not proportional to j m; similarly J y is not proportional to j m, is that clear.

Because of this property and my choice that you are going to take J’s to J z to be simultaneously commuting with J dot J. So, this convention which we write j m is a simultaneous eigen state of a Casimir operator, this is what we call it as a Casimir operator. Casimir operator is one which commutes with all the generators of the group, I am confining to s u 2 group; s u 2 group have J 1, J 2, J 3 as the generators. Casimir operator will commute with all the. And Casimir is constructed using bilinears of the generators.

And then we make a choice and write those states which are simultaneous eigen value equation. So, simultaneous eigen states of this as well as this, ok. In fact using this you can also see the matrix representation of it for every j, this I constructed it for j equal to half by showing that it is 1 and minus 1. If you take j equal to 1, you can construct what is the matrix for it ok.

So, if you want to find J z which is a 3 cross 3 matrix, then you can rewrite it as, right; because it is an eigen value equation. If m is j it will give you plus 1; if of course, I would suppress the h cross, if you want I can put the h cross and if m is 0. So, j m is running from minus j to plus j. So, if m is j it is 1; m is 0 it is 0; m is minus j it gives you a minus 1 for j equal to this corresponds to j equal to 1, ok.

(Refer Slide Time: 12:45)



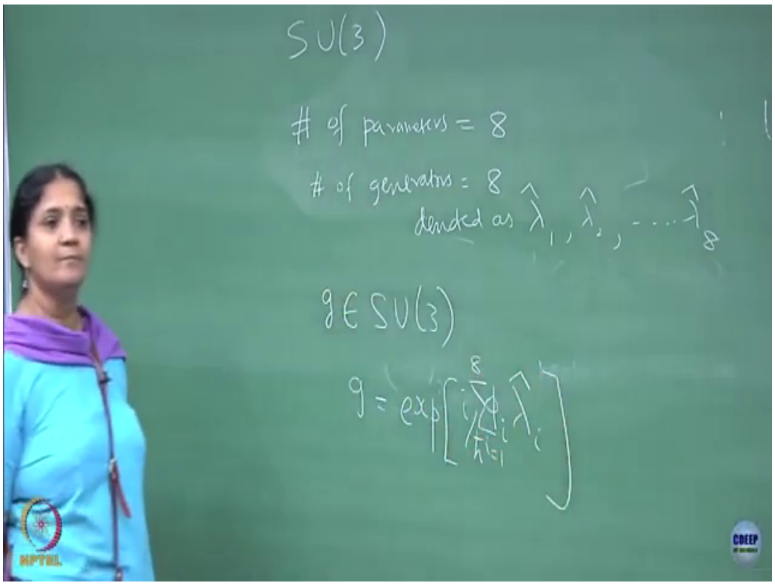
So, some of the notations j equal to 1 is 2 j plus 1 dimensional matrices. So, so the degree of the matrix is 2 j plus 1. So, j equal to 1 this will be 3 dimensional matrices for J x, J y and J z, is that clear. Similarly using these you can see it is off diagonal and you can work out what are those off diagonal matrices for every representation. You do not need to, every representation I mean; different j's corresponds to different representations, just like an irreps 2 dimensional irrep, 3 dimensional irreps.

So, each one is 2 j plus 1 dimensional irrep and for different representations you can use these equations to construct the matrix representation, is that clear. So, bilinear in generators; bilinear means product of two, two linear operators, take a product of two linear operators quadratic yeah, this is the quadratic concept, yeah is that clear. For example, J dot J.

In every group you will have a set of matrices, matrix representations for the generators which will be diagonal, ok. So, if you go to S U 3 for example, I want to construct the generators with certain properties for unitary group, special unitary group. What is the properties? It has to be traceless and Hermitian; this is traceless and diagonal also.

If I go to S U 3; so for S U 2 if I am going to do, I am going to do only with the lowest non-trivial dimension. For S U 3 the lowest non trivial dimension is 3 cross 3 matrix. So, there you can allow one more diagonal matrix which is traceless, is that right. So, if I write the diagonal matrices, so S U 3 how many generators are there, S U 3?

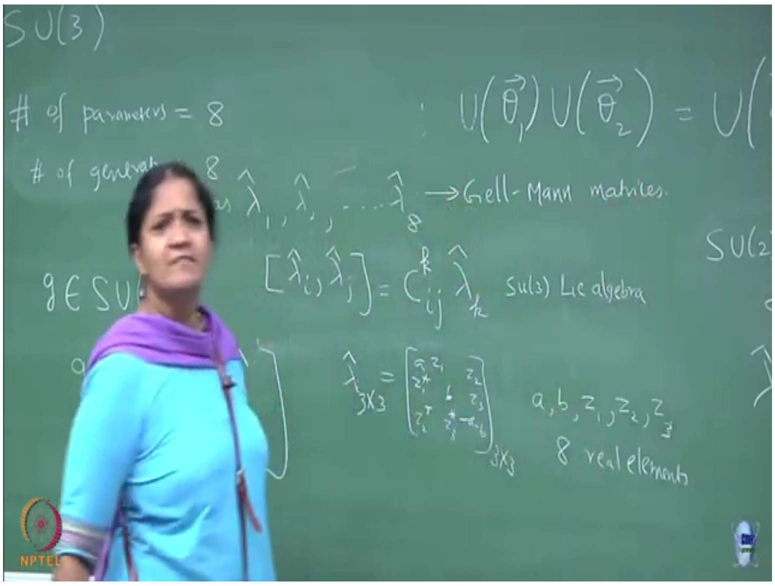
(Refer Slide Time: 15:29)



Student: 8.

Number of parameters, real parameters is 8; number of generators is 8 and I am going to denote it as lambda 1, lambda 2 up to lambda 8. Just like J 1, J 2, J 3 you will have 8 such matrices; any element g belonging to S U 3 will be rewritable as exponential of i by h cross phi which is 8 of them ok. So, I am going to do phi i lambda i hat summation over I going from 1 to 8, are you all looking that is the one. These are the S U 3 matrices, which involves 8 generator and there will an algebra between the 8 generators.

(Refer Slide Time: 16:53)



What is the algebra be? It will be some lambda i with lambda j, it should be give me some structure constant, ok. So, this is the S U 3 algebra, ok. This is the group element which is an exponentials of the generators and there are 8 parameters and 8 generators and the lowest non-trivial dimension has to be dictated by this, ok. So, you can write lambda hat the lowest one; just like we wrote for S U 2 as Pauli matrices, here you have a set of matrices which are 3 cross 3 matrices, ok.

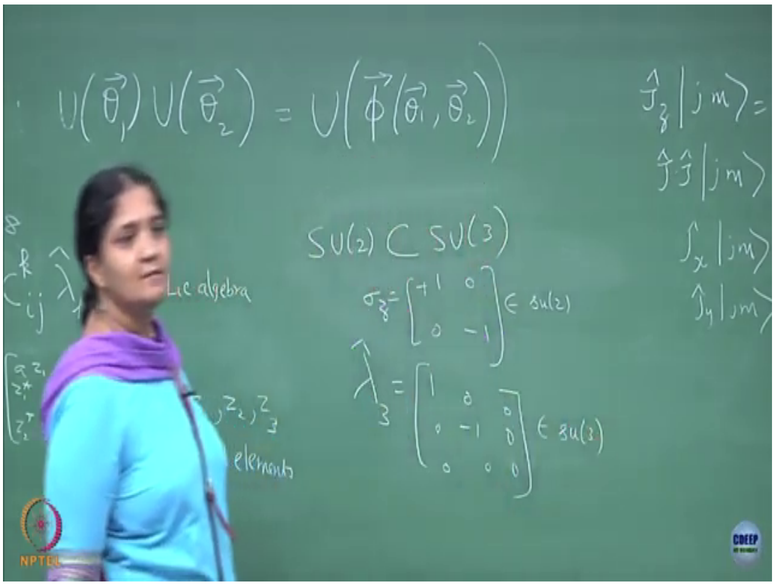
Lambda i's are 3 cross 3 and the aim is to find traceless Hermitian matrices. 3 cross 3 which is traceless, whatever you write here, you understand what I am saying, right. The generators have to be traceless and Hermitian and from there you found that the number of parameters will be 9; and then you traceless and Hermitian will have number of parameters 8, right. So, it is one two which are real, then these are complex conjugates, so this is 2, 4, 6, 8.

Complex numbers have 2; but this element is indeed it is not independent. So, 2 real's a, b, z 1, z 2, z 3 are the independent parameters required to specify a traceless Hermitian matrix. So, there are 8 number of real elements, these are complex. So, 6, 7, 8 is that clear. So, 8 real elements that is why 8 parameters.

Student: (Refer Time: 19:38).

Now I want to construct the analog of J z which I did for S U 2, the Pauli matrix sigma z was diagonal. How many diagonal matrices I can construct for the lowest non-trivial representation of S U 3?

(Refer Slide Time: 20:11)



So, tell me how many are possible; first of all inside S U 3 S U 2 will be a subgroup, ok. That is something which you should know that; any element of S U 3, you could write the elements which is, so S U 2 is definitely a subgroup of S U 3, ok

So, the conventional generator which you have for sigma z, which was plus 1, 0, 0, minus 1; this should sit inside, a S U 3 generator has one diagonal generator. So, let me call the generator for conventional reason just like we call that as J 3. So, lambda 3 should be a 3 cross 3 matrix for S U 3. So, this is an element of S U 2 algebra. So, this 3 cross 3 will be 1, ok.

So, this one is an element of S U 3; this is traceless and Hermitian. Can we construct one more is the next question.

(Refer Slide Time: 21:31)

So, it turns out that for 3 cross 3, you have one more diagonal generator which you could construct. Because of this 3 cross 3 freedom ok; otherwise you would not have been able to do, it is 1, 0, 0, 0, 1, 0, 0, 0, minus 2 ok. And then you go to S U 4, now tell me what is the procedure by induction. These two should sit inside S U 4, we will definitely have rank 2 test; sorry you will definitely have two diagonal matrices and you could also construct one more, ok.

So, that is the chain process by induction, where I have told you that there is one diagonal generator for S U 2 and that is sitting inside this and then you can construct one more. In the literature we call those diagonal generators with the subscript 3 and subscript 8 for these lambda matrices.

Incidentally these like Pauli matrices the name which is given to this, this was initiated by Gell Mann; when he was looking at the particle data, ok. So, just like your periodic table, there is also in particle physics you can call it as a particle zoo; protons neutrons are simple ones, but there are other particles. He will just looked at their charges, looked at their angular momentums and he saw their masses and he grouped them; and he said these groups are looking like irreducible representations, I should find a theory to explain it.

So, he started working on this extension of this unitary S U 2 group to S U 3. And he exactly fitted every particle with whatever nature has told us in a beautiful fashion. We will see this to appreciate how much group theory actually speaks physics. So, in that sense, since he started doing this I think this these are the matrices which now is well known as Gell Mann matrices, just like Pauli matrices these are called Gell Mann matrices.

Recently he passed away, he was the one who got the Nobel prize for this quark model; saying that protons neutrons are all made of 3 quarks. So, 3 quarks in our group theory language should be seen as tensor product of three primary basis. And, look at the irreducible representations over tensor product of three fundamental, three initial representations; three irreps will give you the composition and then you break it up and you will see them. We will do this, just to see how the clarity on S U 3 particle physics is.