Group Theory Methods in Physics
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Lecture – 43
Symplectic group

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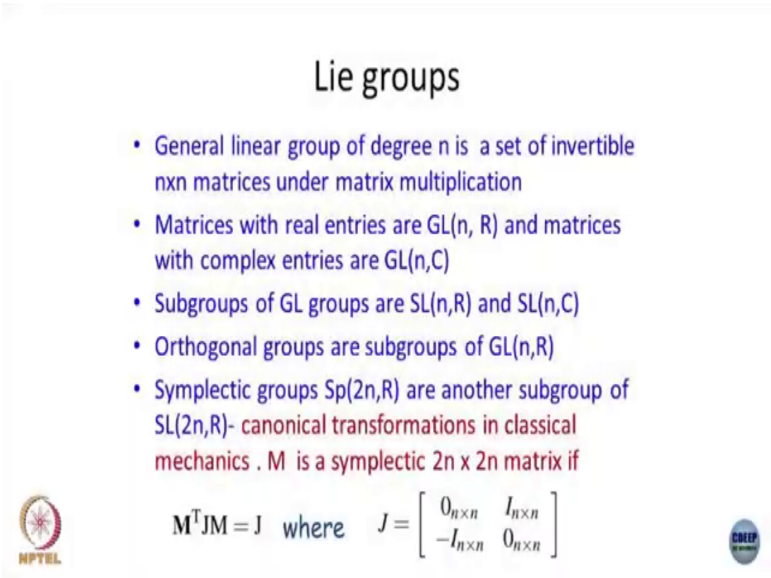

So, now, let me try and list out more as a summary of what all groups are there in the literature and what are the notations. Any questions ok? Lie groups, general linear group of degree n is a set of invertible n cross n matrices under matrix multiplication. Matrices with real entries are G L n, R and matrices with complex entries are G L n, C. Subgroups of G L n, sorry G L are S L n, R or S L n, C. Orthogonal groups or subgroups of G L n, R; Orthogonal groups have only real entries there.

Symplectic groups, I am sure you would have done symplectic groups in your classical mechanics group course; where you do not only work with position you also work with momentum, that is what we call it as a phase space, right. So, you have. So, let me take a simple word 2 dimensional phase space. So, we can have x and p x. So, that is a 2 dimensional phase space.

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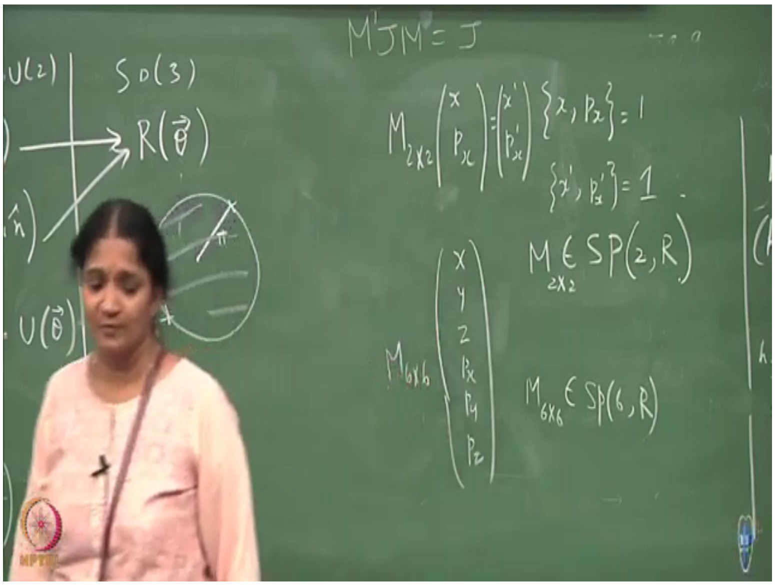

If you are in 3 dimension, it will become x y z and p x p y p z. So, it is a 6 dimensional phase space and then you are allowed to do; here it will be a 2 cross 2 matrix, here it will be a 6 cross 6 matrix. So, what are the matrix transmissions you can do such that your Poisson bracket is preserved, right. What is Poisson bracket? Is 1, x and x is 0, p x p x is 0, you know that ok.

So, this is what you need to preserve under any transmission. So, I can call this. So, this one gives me x prime p x prime. This Poisson bracket will go to x prime p x prime under such a transformation, such that this is going to be still 1; this is what you are learnt. So, what are that set of 2 cross 2 matrices, that set of 2 cross 2 matrices belong to a symplectic group, ok.

So, these M's are elements of symplectic group 2 and if they are real entries S P 2, R. What about here? M will be elements of S P 6 comma R. So, this Poisson bracket being preserved, what does it imply on these matrices; what does it imply on these matrices? M J M transpose have to be J or it is other way around. So, M transpose, anyway it does not really matter, what I call as M, you can call it M transpose.

What is J? If you remember if I generalized orthogonal transformation, this J was replaced by some diagonal matrix with some of them positive, some of them negative, right remember.

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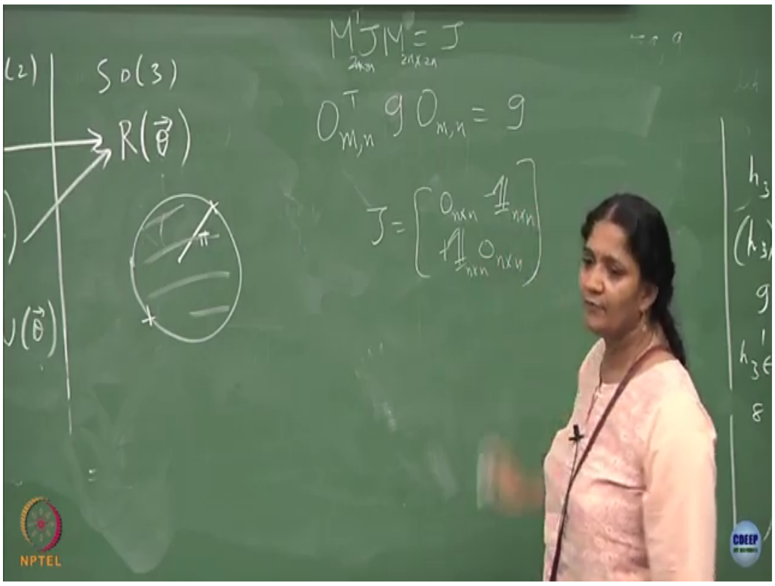

Generalized Orthogonal matrices I called it as a matrix g, right. By m comma n I mean that, this one will be, m of them will be of one signature; n of them will be of other signature. Similarly here, these have to be even dimensional. So, let me right it, it is always even dimension; phase space is even dimensional, it does not work in odd dimension because every position you will have a momentum, ok. So, this is 2 n. What is this J, such that your Poisson bracket is preserved? I am sure you would have done it, anybody knows; is it diagonal or off diagonal, off diagonal.

So, J turns out to be 0 0 which is n cross n, n cross n it is 0, ok. All the entries in this n cross n matrix, do not take it to be orthogonal, it is 0. And this one is minus identity and plus identity, that is also n cross n. If you have not done this, Goldstein has a chapter on canonical transformations please go and take a look at it, ok.

I am going to confined myself only to unitary groups in the rest of the lectures; even though I will give you a formal aspects, I am not going to look at other groups in detail, ok. But anybody is interested; this is the natural canonical transformations which you do in phase space in your classical mechanics course. The set of matrices belong to a group which is called symplectic group; element which belongs to the sympathetic group will always satisfy this condition.

So, let me stop on this Lie group business, is that fine. So, some of these listing is just for completeness on the slide which I am showing; that symplectic groups S p 2 n, R are another subgroups of S L, special linear groups. They are nothing, but classical transformation, canonical transformation in classical mechanics; M is called as a symplectic 2 n cross 2 n matrix, symplectic matrix, if it satisfies this condition, ok. I have summarized whatever I did on board on the slide.

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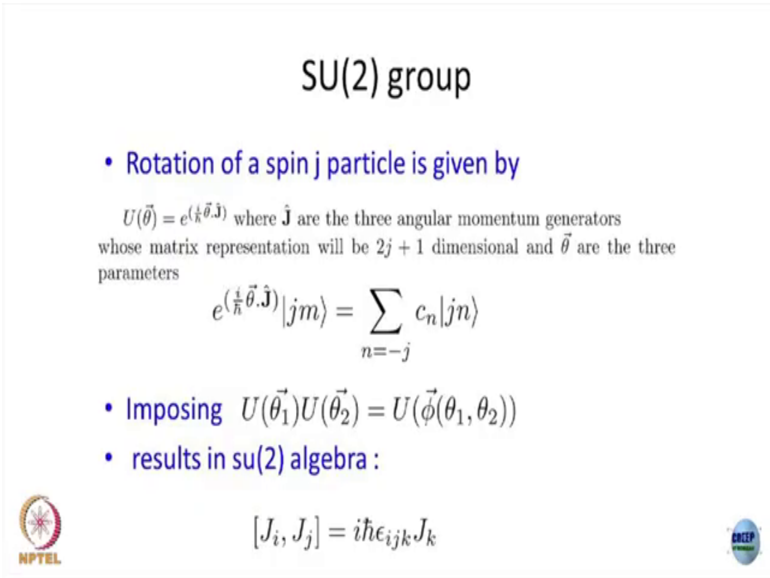

So, this also I have already given you a flavor that, rotation of any spin J particle, the corresponding matrix will be what; J hat is a formal linear operator I have written. Number of parameters is 3, number of generators is 3, no change in that; but the dimension of this matrix will depend on what vector space it is going to act on. If the spin of the particle is J; how many states it allows? It allows from plus J to minus J in steps of one, decreasing steps of one.

So, the dimensionality of the vector space is 2 j plus 1; for spin half it is plus half and minus half j is half. So, 2 j plus 1 is 2. But in general it is a 2 j plus 1 vector space and the corresponding j operators have to be 2 j plus 1 cross 2 j plus 1 matrices which are irreducible representations acting on that vector space. But they still are S U 2 belongs to the S U 2 Lie algebra.

If I give you a 3 cross 3 matrix and I will give a Lie algebra; if I ask what is the group, then you should know it is S U 2, but it is a 3 dimensional irreducible higher dimensional representation of S U 2 Lie algebra,. So, what is the meaning of saying that it is an irreducible vector space; the group elements when it acts on this state, any arbitrary state it should only mix amongst the 2 j plus 1 states; should not take you out of that state, that is the meaning of saying that this vector spaces irreducible vector space, ok.

Student: 2 3 times back we have some vector space it cannot go out of it in it is space. So, (Refer Slide Time: 10:23).

I will give you an example.

Student: Usual increasing dimensions square root of so.

So, no I will give you an example now, I will give you one example and then you will see what is happening. So, a simple example is that take 2 spin half particles.

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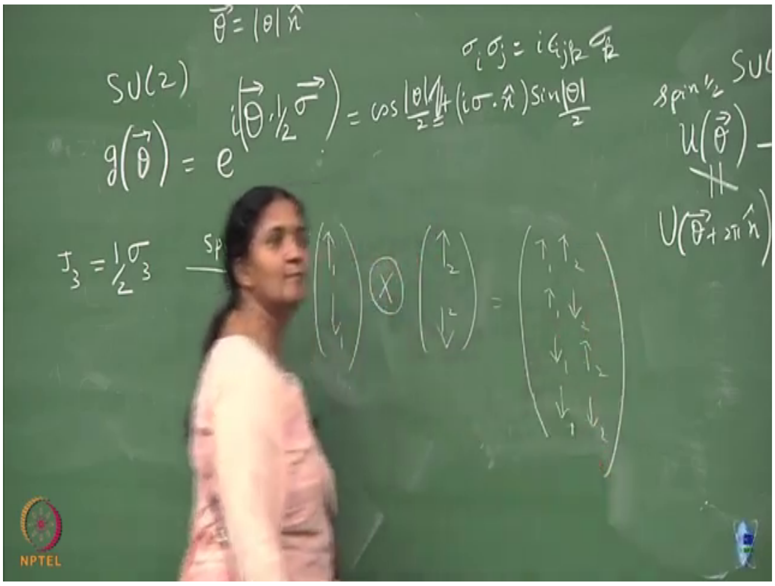

So, suppose I take up and down as one vector space and let me take another one another up and down; let me put a subscript 1, 1 and 2, 2 to remember that it is particle 1 and particle 2. Just like we took momentum and position, taking particle 1 and particle 2; for these you know these are the matrices which are irreducible representation, for this you know the same matrices are the irreducible represent. Now, I want to take a tensor product of these 2. So, what will happen? I am sure you all know, is that right.

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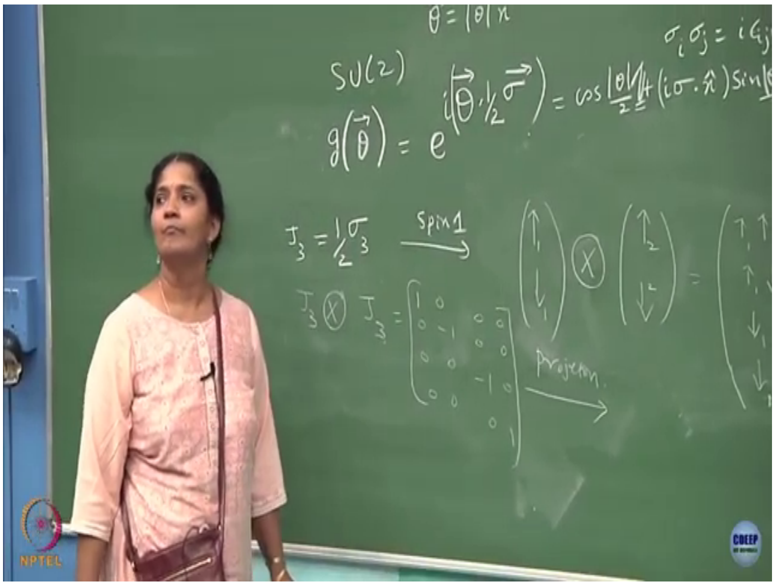

What will happen to the matrices on it? You take this J 3 and take a product of J 3, right. What will this be, minus 1; am I right, we are done it correctly, I think I have done it correctly. Is this a reducible representation? It is a reducible representation and not an irreducible representation. What you can do is, now we can do a projection a projection to break this into a 3 cross 3 and a 1 cross 1.

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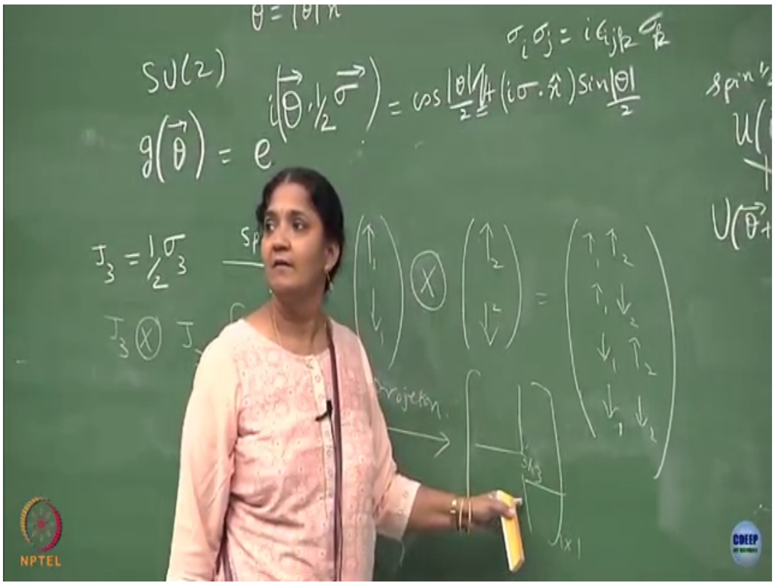

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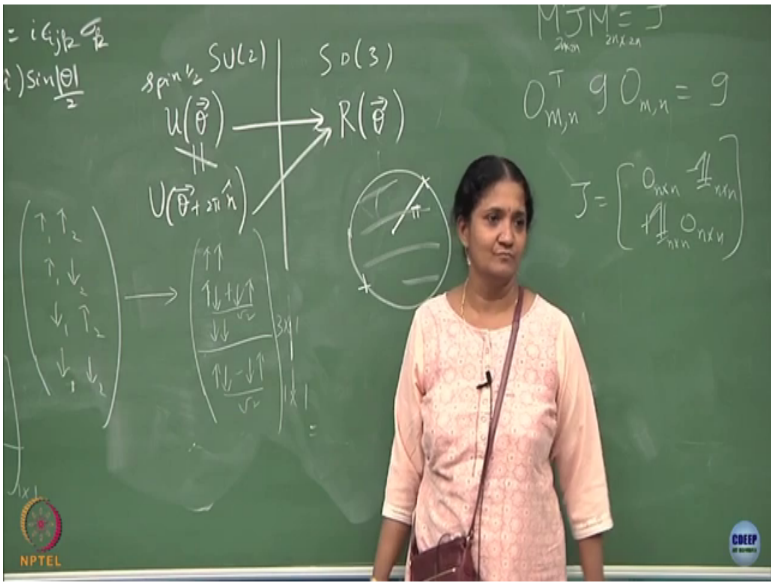

What is the vector space corresponding to this 3 cross 3? This will get reduced, the basis state also will get reduced to someone. So, it is a 3 cross 1 and a 1 cross 1; this is a binary basis which is obtained by taking tensor product and doing a projection to find the basis for the 3 dimensional irreducible representation of S U 2 algebra and 1 dimensional representation of the S U 2 algebra, right.

Another way of saying is this is like, singlet it better be a 1 dimensional representation with the binary basis, you can start doing this for tertiary basis and so on ok. So, I am connecting up what you did in the discreet ropes and so. Only thing here is that I am working with the Lie algebra, generators of the Lie algebra acting on the vector space.

But if you want to look at the group elements it is not going to do anything new; because exponentiation will only again follow the same. So, whatever group operation I am going to do on this, if it is an irreducible vector space it is only makes only this, ok. So, that will not be the situation for your in general.

So, what I am trying to say is that, you cannot go from here to here; these 2 are non-talkative space. So, this is a reducible space which obtained by tensor product of 2 spin half particle. And then what is the projector is what you will ask; what is the analogue formula for the projector and that I will, that is the clips cord and coefficients which I will try and give you some flavor.